

Second workshop on the Atlas of Lie Groups
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ABSTRACT. The goal of the Atlas of Lie Groups project is to create a computer library of data on the structure theory and representations, admissible or unitary, of a real reductive Lie group G . This overview attempts to provide the reader with the general background and motivation for this task.

1. About This Overview

In many areas of pure and applied Mathematics one is interested in the study of a measure space X on which the Lie group G acts in a nice way so that it makes sense to talk about the space $L^2(X)$ on which the group G also acts via unitary operators. Spaces on which the group acts unitarily are called unitary representations.

To understand this space it is often useful to try to understand how this space breaks up into a direct integral (or direct sum) of simpler subspaces of functions invariant under the action of G . In other words

$$(1.1) \quad L^2(X) = \int_{\hat{G}_u} m_X(\pi) \pi d\mu(\pi)$$

where

- π is any subspace of $L^2(X)$ on which G acts via unitary operators
- $m_X(\pi)$ is the multiplicity of the subrepresentation π occurring in X (in most interesting cases $0 \leq m_X(X) \leq \infty$);
- $\hat{G}_u$ is the set of inequivalent unitary irreducible representations of G .

The point of this is to be able to transfer knowledge about the individual representation π into knowledge about the space X .

Therefore, finding ways to parametrize or describe the set $\hat{G}_u$ is not only an interesting problem in itself. Moreover, finding a systematic way to describe it so that it can be calculated via a computer algorithm will be extremely useful for those mathematicians in any fields seeking answers to such questions.

2. History

A special case of G acting on a space is when $X = G$. Then the above problem becomes studying the space $L^2(G)$ as a unitary representation of G itself. Harish Chandra was interested in this question. He defined the tempered representations (we will call this class $\hat{G}_t$) as the subset of $\hat{G}_u$ whose elements are components (discrete or continuous) in the decomposition of $L^2(G)$ in 1.1.

Harish-Chandra realized that unitary representations form part of a larger class of representations sharing some of the nice properties of irreducible unitary representations (like Schur's lemma). He called this larger set *admissible*. We will call this class $\hat{G}_a$. In fact, if we call $\hat{G}_h$ the class of irreducible inequivalent representations possessing a G invariant Hermitian form we have the following inclusions

$$(2.1) \quad \hat{G}_t \subseteq \hat{G}_u \subseteq \hat{G}_h \subseteq \hat{G}_a$$

Now, Harish-Chandra was only interested in the Harmonic analysis question and did not work on the classification problem. Langlands used Harish-Chandra's work to classify all admissible irreducible representations of G .

Roughly, Langlands gave a mapping from $\hat{G}_a$ to a set of irreducible admissible representations of all Cartan subgroups of G . Modulo conjugation by G , this correspondence is a bijection.

That is, given a Cartan subgroup H of G and a representation $\xi \in \hat{H}_a$ (we can think of ξ as a one-dimensional character since usually H is abelian), he constructs a representation $\pi(\xi)$ of G . Then

$$(2.2) \quad \hat{G}_a = \coprod \left(\begin{array}{c} H \text{ Cartan subgroup} \\ \xi \in \hat{H}_a \end{array} \right) / \text{conj by } G \quad \pi(\xi)$$

This framework gives us a road map for describing $\hat{G}_u$.

- (1) List all real reductive Lie groups G .
- (2) List all Cartan subgroups H for each G .
- (3) List all characters of $\hat{H}$ ($= \hat{H}_a$) for all Cartan subgroups H for each G .
- (4) Decide when each $\pi(\xi)$ is unitary for each ξ in the list.

Problems 1, 2 and 3 are solved. However there is no systematic way to deal with problem 4 for all cases. In principle there are algorithms to deal with problems 1, 2 and 3 and for some cases of problem 4.

However there is still a considerable amount of computational complexity

3. Computational Simplification

To simplify the problem a bit there are some reduction arguments that we can use. Suppose that G is a reductive Lie group and let us denote its identity component by G_0 .

PROPOSITION 1. *Let G as above and $\pi \in \hat{G}_a$ and denote by $\pi|_{G_0}$ the representation π restricted to G_0 .*

Then π is unitary if and only if some irreducible constituent of $\pi|_{G_0}$ is unitary. And this happens if and only if all irreducible constituents of $\pi|_{G_0}$ are unitary.

3.1. Example of an application: Suppose that the group G_0 is a real split group and let MAN be a minimal parabolic subgroup of G , let $\delta \in \hat{M}$, $\nu \in \hat{A}$ (note $\delta \otimes \nu$ is a character of the Cartan subgroup MA of G). Denote by $\pi(\delta \otimes \nu)$ the representation of G induced from the representation $\delta \otimes \nu$ of MAN and by $\bar{\pi}(\delta \otimes \nu)$ the corresponding irreducible representation of G via Langlands parametrization.

PROPOSITION 2. *Let $W(G, A) = N_G(A)/Z_G(A)$ and denote by $W_\delta = \text{stabilizer of } \delta \text{ in } W(G, A)$. Then,*

$\bar{\pi}(\delta \otimes \nu) \in \hat{G}_h$ if and only if there is $w \in W_\delta$ satisfying $w\nu = -\bar{\nu}$.

This restricts the set $\{(\xi, H) | H, \text{ cartan subgroup of } G, \xi \in \hat{H}_a\}$ that we need to work with in this case.

Now, when can we say something about unitarity of the representation in this case?

PROPOSITION 3. *Let G , G_0 , W and W_δ as above and denote by W_δ^o , the subgroup of W generated by the ‘good’ roots. (see notes on ‘Non spherical principal series’). Suppose there is $w \in W_\delta$ satisfying $w\nu = -\bar{\nu}$. But there is no $w \in W_\delta^o$ satisfying $w\nu = -\bar{\nu}$. Then $\bar{\pi}(\delta \otimes \nu)$ is not unitary.*

PROOF. Choose $G' \supseteq G$ as disconnected as possible (covering of $G_{\text{adj}(\mathbb{R})}$) so that extending δ to a representation δ' of G' gives that $W_{\delta'} = W_\delta^o$. By Proposition 2, this means that $\bar{\pi}_{G'}(\delta \otimes \nu)$ is not Hermitian, so it is not unitary.

By Proposition 1, $\bar{\pi}_G(\delta \otimes \nu)$ is not unitary either. $\square$

3.1.1. Example of the example: Let $G = SL(2, \mathbb{R})$ and $\delta \neq \text{trivial}$. Then $G' = SL^\pm(2, \mathbb{R}) = 2 \times 2$ real matrices with $\det = \pm 1$.

Let ν be anything non zero and real. Then the representation $\bar{\pi}_G(\delta \otimes \nu)$ is Hermitian but the representation $\bar{\pi}_{G'}(\delta' \otimes \nu)$ is not.

Recall that our goal is to identify the set $\widehat{G}_u \subseteq \widehat{G}_a$ inside the parametrization

$$(3.1) \quad \widehat{G}_a = \coprod \pi(\xi) \left(\begin{array}{c} H \text{ cartan subgroup} \\ \xi \in \widehat{H}_a \end{array} \right) / \text{conj by } G$$

Note that

$$(3.2) \quad \hat{G}_t \leftrightarrow \coprod_{(\xi \in \widehat{H}_a \text{ unitary})} \pi_i(\xi)$$

REMARK 1. If (π, V) is an irreducible representation of G on a reflexive topological vector space. Denote by (π^h, V^h) , the Hermitian dual of (π, V) . This is also an irreducible representation of G . Then we have the following

DEFINITION 1. We say that (π, V) is hermitian if and only if $\pi \cong \pi^h$ (this is Harish-Chandra's infinitesimal equivalence)

REMARK 2. A warning is in order: A hermitian topological representation might not have a continuous invariant hermitian form

So to decide if $\pi(\xi)$ is hermitian we need to see if $\pi(\xi) \cong \pi(\xi)^h$.

PROPOSITION 4. With notation as above, $\pi(\xi)^h \cong \pi(\xi^h)$

Here

$$(3.3) \quad \xi : H \rightarrow \mathbb{C}^\times$$

$$(3.4) \quad \xi^h : H \rightarrow \mathbb{C}^\times$$

$$(3.5) \quad \xi^h(t) = \overline{\xi(t)}^{-1}$$

Note ξ is unitary $\iff \xi = \xi^h$. So we have the following

PROPOSITION 5. $\pi(\xi)$ is hermitian $\iff$ there is a $w \in W(G, H)$ such that

$$(3.6) \quad w\xi = \xi^h$$

Recall that $W(G, H) = N_G(H)/H = N_G(H)/H$, if H is abelian.

Now the question then becomes:

Given $H \subset G$ cartan subgroup, $\xi : H \rightarrow \mathbb{C}^\times$ and $w \in W(G, H)$ such that $w\xi = \xi^h$ (so that $\pi(\xi)$ is hermitian), is the form positive definite? (is π unitary?)

4. A more explicit picture of H .

Suppose $G = G_{\mathbb{C}}(\mathbb{R})$ is the group of real points of a connected reductive algebraic group $G_{\mathbb{C}}$ defined over $\mathbb{R}$. Recall that $H = H_{\mathbb{C}}(\mathbb{R})$, where $H_{\mathbb{C}}$ is a Cartan subgroup of the algebraic group $G_{\mathbb{C}}$, and if X_* is the lattice of one-parameter subgroups of $H_{\mathbb{C}}$ or co-roots, we can write

$$(4.1) \quad H_{\mathbb{C}} = \mathbb{C}^\times \otimes_{\mathbb{Z}} X_*$$

Moreover, X_* also has a real structure $\sigma : X_* \rightarrow X_*$, an automorphism of order 2. so we can write

$$(4.2) \quad X_* = L^+ \oplus M \oplus M \oplus L^-$$

a direct sum of lattices, where the automorphism σ acts by $+1$ on L^+ and by -1 on L^- and σ permutes the two copies of M .

σ acts on $H_{\mathbb{C}}$ via $\sigma(z \otimes x) = \bar{z} \otimes \sigma x$

So H can be written as

$$(4.3) \quad H = (\mathbb{R}^\times)^{\text{rank of } L^+} \times (\mathbb{C}^\times)^{\text{rank of } M} \times (S^1)^{\text{rank of } L^-}$$

Moreover, the dimension of the $+1$ (resp. -1) eigenspace of σ is equal to the rank of L^+ (resp. L^-) plus the rank of M and

$$(4.4) \quad H/H_0 = (\mathbb{Z}/2\mathbb{Z})^{rk L^+}$$

Now, the roots $\Delta(\mathfrak{g}, \mathfrak{h})$ of H in $\mathfrak{g}$, are contained in $X^* = \text{dual lattice to } X_*$ and $\check{\Delta}(\mathfrak{g}, \mathfrak{h}) \subset X_* = \text{co-roots}$. We say that a root α is real $\iff \sigma\check{\alpha} = \check{\alpha}$. The map $\alpha : \mathbb{C}^\times \rightarrow H_{\mathbb{C}}$ is defined over $\mathbb{R}$. So $\check{\alpha} : \mathbb{R}^\times \rightarrow H$ defines an element $m_\alpha = \check{\alpha}(-1)$ of order two in H .

Then $H \cap G_0 = \langle H_0, m_\alpha \rangle$ for all α real.

Now, if

$$(4.5) \quad H = (\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b \times (S^1)^c$$

H contains the vector group

$$(4.6) \quad A = (\mathbb{R}_+)^a \times (\mathbb{R}_+)^b$$

and the compact group

$$(4.7) \quad T = (\pm 1)^a \times (S^1)^b \times (S^1)^c$$

and so that $H = TA$.

So, if $\xi \in \hat{H}$, $\xi = \xi_T \cdot \xi_A$, with $\xi_T \in \hat{T}$ and $\xi_A \in \hat{A}$ corresponding to $\nu_\xi \in \mathfrak{a}^*$ ($\mathfrak{a} = +1$ eigenspace of H), then

$$(4.8) \quad \xi^h = \xi_T \cdot \bar{\xi}_A^{-1}$$

Recall that before we had $w \cdot \xi = \xi^h$ means that $w \cdot \xi_T = \xi_t$ and $w \cdot \nu_\xi = -\overline{\nu_\xi}$

Put $L = Z_G(d\xi_T)$. this is a Levi subgroup of G (not of a real parabolic subgroup). Also, L is θ stable if H is and it is still in our class of real reductive groups: L is the real points of a connected reductive algebraic group over $\mathbb{C}$.

Moreover, H is a cartan subgroup of L , so, it makes sense to define the Weyl group

$$(4.9) \quad W(L, H) = N_L(H) / H.$$

Then, since $w \cdot \xi_T = \xi_t$ and $w \cdot (d\xi_T) = d\xi_t$, $w \in W(L, H)$.

So it makes sense also to ask whether the representation $\pi_L(\xi)$ (associated to the same character of H) as well as $\pi_G(\xi)$ are Hermitian or unitary.

5. Certificate of non unitarity

One would wish to be able to say that $\pi_G(\xi)$ is unitary $\iff \pi_L(\xi)$ is unitary. However this is often true but not always. Here $\pi_G(\xi)$ is constructed using Vogan's classification Levi subgroup L via cohomological parabolic induction.

Therefore, an algorithm that would list all unitary representations needs to know when it doesn't happen.

- (1) The right-left direction ($\Leftarrow$) is very close to true. It is true if $\pi_L(\xi)$ is 'weakly unipotent'

Notes:

- (a) In Barbasch and Ciubotaro's classification of spherical unitary representations not all of them are weakly unipotent
 - (b) In the classical case and under some condition there is an algorithm to detect if a representation is weakly unipotent
- (2) The left-right direction ($\Rightarrow$) is not known completely. Let's think about the contrapositive: $\pi_L(\xi)$ not unitary $\implies \pi_G(\xi)$ not unitary.

What happens is that if $\pi_L(\xi)$ is not unitary it will have a 'non unitarity certificate': a representation μ_L for $L \cap K$ where the form is indefinite and such that this provides a certificate for G . that is, a representation μ_G for K where the form (for the representation $\pi_G(\xi)$) will also be indefinite. roughly it all boils down to writing the proposed highest weight for the K type μ_G associated to μ_L , if it is dominant then we win.

5.1. An example when non unitary implies unitary (via cohomological induction). Let $G = SO_e(4, 1)$. then $K = SO(4) \cong SU(2) \times SU(2)$,

$$(5.1) \quad \hat{K} \cong \{(m, n) \in \mathbb{Z} \times \mathbb{Z} | n \geq 0, m \geq 0, n + m \text{ even}\}.$$

These are the integral points (with even sum) on and above the diagonal lines in the figure below.

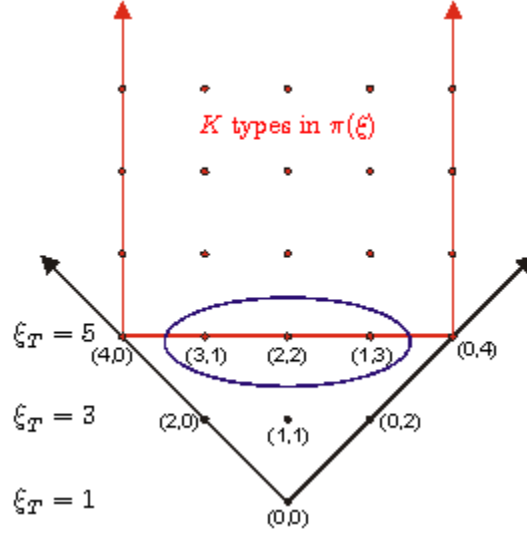
Let H be the maximally split Cartan subgroup of G ,

$$(5.2) \quad H = SO(2) \times SO_e(1, 1)$$

$$(5.3) \quad \cong \text{circle} \times \text{line}$$

$$(5.4) \quad = T \times A$$

Then the characters on T , $\xi_T \leftrightarrow$ odd integers (always non zero due to some ρ shifts), For example, in the figure below the first (bottom) row consisting only of the K type $(0, 0)$, corresponds to the character $\xi_T = 1$ the second row, consisting of K types $(2, 0)$, $(1, 1)$, $(0, 2)$ correspond to the character $\xi_T = 3$, etc.



Now let $L = SO(2) \times SO_e(2, 1)$. A representation of G cohomologically induced from a representation of this subgroup with parameter $\xi = \xi_T \cdot \xi_A$ will have its K types included in a vertical region like the one in the figure below. In this case, $\xi_T = 5$.

Then the hermitian form on the ‘lower line’ of ‘ ξ_T ’ K types ($\xi_T = 5$ here) corresponds to the hermitian form on the smallest ‘ ξ_T ’ of K types in the spherical representation of $SO_e(2, 1)$.

Here $\xi = \xi_T \cdot \nu$, where $\nu \cong \mathbb{C}$ is the character of a representation of $SO_e(1, 1)$

So $\pi_L(\xi)$ is unitary $\iff \nu$ is imaginary or real with $|\nu| \leq 1$.

Now in this case, the $L \cap K$ types endowed with the non unitarity certificate are the bottom 3 $L \cap K$ types $(3, 1)$, $(2, 2)$ and $(1, 3)$.

Then, we have if $\xi_T \geq 3$ then $\pi_L(\xi)$ is unitary $\iff \pi_G(\xi)$.

(1)

However, if $\xi_T = 1$ then $\pi_G(\xi_T \cdot \nu)$ is unitary also when ν is real and $|\nu| \leq 2$.