Restrictions of unitary representations: Examples and applications to automorphic forms

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$\mathfrak{g}$ Lie algebra,

K max compact subgroup

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H reductive subgroup of G with max compact subgroup $K_H = H \cap K$

$\mathfrak{h}$ Lie algebra of H

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Applications to the cohomology of discrete groups and automorphic forms.

Remarks about the restriction of unitary representations.

First case: If as a unitary representation on a Hilbert space

$$\pi_{|H} = \oplus \pi_H$$

for irreducible unitary representations $\pi_H \in \hat{H}$ then we call π H-admissible case:

Theorem (Kobayashi)
Suppose that π is H-admissible for a symmetric subgroup H. Then the underlying $(\mathfrak{g}, K)$ module is a direct sum of irreducible $(\mathfrak{h}, K_H)$ -modules (i.e π is infinitesimally H-admissible).

If an irreducible $(\mathfrak{h}, K \cap H)$ module U is a direct summand of a H-admissible representation π , we say that it is a H-type of π.

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Let G=SL(4, $\mathbb{R}$). There are 2 conjugacy classes of symplectic subgroups. Let H_1 and H_2 be symplectic groups in different conjugacy classes.

There exists an unitary representation π of G which is H_1 admissible but not H_2 admissible .

Some representations and their $(\mathfrak{g}, K)$-modules

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$ and let T be a maximal torus in K. Then $x_0 \in T$ defines a θ–stable parabolic subalgebra

$$\mathfrak{q}_{\mathbb{C}} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_{\mathbb{C}}$$

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of $\mathfrak{g}_{\mathbb{C}}$.

For a θ–stable parabolic $\mathfrak{q}$ and an integral and sufficiently regular character λ of $\mathfrak{q}$ we can construct a family of representations $A_{\mathfrak{q}}(\lambda)$.

These representations $A_{\mathfrak{q}}(\lambda)$ were constructed by Parthasarathy using the Dirac operator and also independently using homological algebra by G. Zuckerman in 1978. Write $A_{\mathfrak{q}} := A_{\mathfrak{q}}(0)$

Consider G= U(p,q), K=U(p)$\times$U(q) with $p, q > 1$.
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If $A_{\mathfrak{q}}$ is not holomorphic or anti holomorphic it is not not H_3-admissible.

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Similar results are also true for the connect component of G =S0(p,q), H_1 connected component of SO(p,q-1) and H_3 connected component of SO(p-1,q).

Applications to the cohomology of discrete groups

$\Gamma \subset \mathbf{G}(\mathbb{Q})$ a torsion-free congruence subgroup.

$S(\Gamma) := \Gamma\backslash X = \Gamma\backslash G/K$ is a locally symmetric space.

$S(\Gamma)$ has finite volume under a G-invariant volume form inherited from X.

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Consider

$$H^*(\Gamma, \mathbb{C}) = H^*_{deRahm}(S(\Gamma), \mathbb{C}).$$

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If π be a representation of G we can also define $H^*(\mathfrak{g}, K, \pi \otimes E)$. For an irreducible unitary representation π

$$H^*(\mathfrak{g}, K, \pi \otimes E) = \mathrm{Hom}_K(\wedge^* p, \pi \otimes E).$$

Here $\mathfrak{g} = \mathfrak{k} \oplus p$ is the Cartan deposition of $\mathfrak{g}$.

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If Γ cocompact $L^2(\Gamma\backslash G) = \oplus m(\pi, \Gamma)\pi$, and

$$H^*(\mathfrak{g}, K, C^\infty(\Gamma\backslash G)) = \oplus_{\pi \in \widehat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K, \pi).$$

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Vanishing theorems for $H^*(\Gamma\backslash X, \tilde{E})$ by G. Zuckerman in 1978 and later by Vogan-Zuckerman 1982, nonvanishing theorems by Li using representation theory and classification of irreducible representations with nontrivial $(\mathfrak{g}, K)$-cohomology.

Back to the example $G = U(p,q)$ and the representation $A_{\mathfrak{q}}$.

Proposition

If π is a H_1-type of $A_{\mathfrak{q}}$ then there exists a finite dimensional representation F of H_1 so that

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Write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and $s = \dim\, \mathfrak{u} \cap p$. Then s is the smallest degree so that

$$H^s(\mathfrak{g}, K, A_{\mathfrak{q}}) = \mathrm{Hom}_K(\wedge^s p, A_{\mathfrak{q}}) \neq 0.$$

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Let $\mathfrak{g} = \mathfrak{h} \oplus q$ and $q_1 = p \cap q$.

Let $A_{\mathfrak{q}_{H_1}}$ the H_1 type of $A_{\mathfrak{q}}$ generated by the minimal K-type of $A_{\mathfrak{q}}$ and $s_1 = \dim\ \mathfrak{u} \cap p \cap \mathfrak{h}_1$.

There is canonical identification of

$$\mathrm{Hom}_K(\wedge^s p, A_{\mathfrak{q}})$$

and

$$\mathrm{Hom}_{K\cap H_1}(\wedge^{s_1}(p \cap \mathfrak{h}_1)^*, A_{\mathfrak{q}_{H_1}} \otimes \wedge^{s-s_1} q_1)$$

Theorem 2.

$$H^{s_1}(\mathfrak{h}, K \cap H, A_{\mathfrak{q}_{H_1}} \otimes \wedge^{s-s_1} q) \neq 0$$

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Remark 1: Under our assumptions: $s_1 < s$.

Remark 2:
This result combined with Matsushima Murakami and "Oda restriction" of differential forms allows an maps from cohomology of $X\backslash\Gamma$ to a locally symmetric space for H_1..

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Remark 3:
I conjecture that in the restriction of $A_{\mathfrak{q}}$ to H_3 there is always a direct summand $A_{\mathfrak{q}_{H_3}}$ whose lowest nontrivial cohomology class is in degree s. Special case of this conjecture was proved by Li and Harris.

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Theorem 3. *(Kobayashi)*
Let π be an irreducible unitary representations of G and suppose that U is an irreducible direct summand of π. If the intersection of the underlying $(\mathfrak{h}, K \cap H)$–module of U with the underlying $(\mathfrak{g}, K)$–module of π is nontrivial then the representation π is H–admissible.

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Consider G=SO(n,1), L= SO(2r) $\times$ SO(n-2r,1) , $2r \neq n$ and H= SO(n-1,1) $\times$ SO(1). The representation $A_{\mathfrak{q}}$ is not H-admissible, so Kobayashi's theorem implies that finding direct summands is an analysis problem and not an algebra problem.

Warning: There exists a unitary representation π of SL(2,$\mathbb{C}$) whose restriction to SL(2,$\mathbb{R}$) contains a direct summand σ but σ doesn't contain any smooth vectors of π. (joint with Venkataramana)

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$\widehat{\pi}$ nonspherical principal series representation of SL(2,$\mathbb{C}$) with infinitesimal character ρ. The restriction to SL(2,$\mathbb{R}$) has the discrete series $D^+ \oplus D^-$ with infinitesimal character ρ_H as direct summand, but

$$(D^+ \oplus D^-) \cap \widehat{\pi}^\infty$$

Proof uses concrete models of the representations. J. Vargas recently proved more general case.

Restriction of complementary series representations.

Let $G = SL(2,\mathbb{C})$, $B(\mathbb{C})$ the Borel subgroup of upper triangular matrices in G,
and

$$\rho(\begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix}) = \mid a \mid^2 .$$

For $u \in \mathbb{C}$

$$\pi_u = \{f \in C^\infty(G) | \; f(bg) = \rho(b)^{1+u} f(g)$$

for all $b \in B(\mathbb{C})$ and all $g \in G(\mathbb{C})$ and in addition are $SU(2)$-finite. For $0 < u < 1$ completion to the unitary complementary series rep $\widehat{\pi}_u$ with respect to an inner product $< \, , \, >_{\pi_u}$.

Similiar define the complementary series $\widehat{\sigma}_t$ of H=Sl(2,$\mathbb{R}$) for $0 < t < 1$.

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Theorem 4. *(Mucunda 74) Let* $\frac{1}{2} < u < 1$ *and* $t = 2u - 1$*. The complementary series representation* $\widehat{\sigma}_t$ *of* $SL(2, \mathbb{R})$ *is a direct summand of the restriction of the complementary series representation* $\widehat{\pi}_u$ *of* $SL(2, \mathbb{C})$*.*

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Theorem 4. *(Mucunda 74) Let $\frac{1}{2} < u < 1$ and $t = 2u - 1$. The complementary series representation $\hat{\sigma}_t$ of $SL(2, \mathbb{R})$ is a direct summand of the restriction of the complementary series representation $\hat{\pi}_u$ of $SL(2, \mathbb{C})$.*

Idea of a different proof jointly with Venkataramana: Consider the geometric restriction res of functions on G to functions on H. We show that

$$res : \pi_{-u} \to \sigma_{-t}$$

is continuous with respect to the inner products $< \, , \, >_{\pi_u}$ and $< \, , \, >_{\sigma_t}$

More precisely we prove

Theorem 5. *(joint with Venkataramana)*
There exists a constant C such that for all $\psi \in \pi_{-u}$, the estimate

$$C \parallel \psi \parallel^2_{\pi_{-u}} \quad \geq \quad \parallel res(\psi) \parallel^2_{\sigma_{-(2u-1)}} .$$

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holds.

We conjecture that similar estimates hold for the geometric restriction map of groups G of rank one of the subgroups H of the same type.

Generalization to the restriction of complementary series representations of G=SO(n,1) to H=SO(n-1,1).

τ_n standard representation of SO(n-2).

For $0 < s < 1 - \frac{2i}{n-1}$ we have a unitary complementary series representation

$$R(n, \wedge^i \tau_n, s) = Ind_P^G \wedge^i \tau_n \otimes \rho^{1-s}$$

with the $(\mathfrak{g}, K)$–modules

$$r(n, \wedge^i \tau_n, s) = ind_P^G \wedge^i \tau_n \otimes \rho^{1-s}$$

Theorem 6. *(joint with Venkataramana)*
If

$$\frac{1}{n-1} < s < \frac{2i}{n-i} \text{ and } \; i \leq [n/2] - 1,$$

then

$$R(n-1, \wedge^i \tau_{n-1}, \frac{(n-1)s-1}{n-2})$$

occurs discretely in the restriction of the complementary series representation $R(n, \wedge^i \tau_n, s)$ *to SO(n-1,1).*

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As s tends to the limit $\frac{2i}{n-i}$ the representation $R(n, \wedge^i \tau_n, s)$ tends to a representation $A^n_{\mathfrak{q}_i}$ in the Fell topology.

Theorem 7. *(joint with Venkataramana)*
The representation $A_{\mathfrak{q}_i}^{n-1}$ of SO(n-1,1) occurs discretely in the restriction of the representation $A_{\mathfrak{q}_i}^{n}$ of SO(n,1).

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Applications to Automorphic forms

The representation $A_{\mathfrak{q}_i}^{n}$ of SO(n,1) is the unique representation of SO(n,1) with nontrivial $(\mathfrak{g}, K)$- cohomology in degree i.

It is tempered for i=[n/2].

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Theorem 8. *(joint with Venkataramana)*
If for all n, the tempered representation $A^n_{\mathfrak{q}_i}$ (i.e. when $i =$ [n/2]) is not a limit of complementary series in the automorphic dual of SO(n, 1), then for all integers n, and for $i < [n/2]$, the cohomological representation $A^n_{\mathfrak{q}_i}$ is isolated (in the Fell topology) in the automorphic dual of SO(n, 1).

Conjecture (Bergeron)
Let X be the real hyperbolic n-space and $\Gamma \subset$ SO(n, 1) a congruence arithmetic subgroup. Then non-zero eigenvalues λ of the Laplacian acting on the space $\Omega^i(X)$ of differential forms of degree i satisfy:

$$\lambda > \epsilon$$

for some $\epsilon > 0$ independent of the congruence subgroup Γ, provided i is strictly less than the middle" dimension (i.e. i ¡ [n/2]).

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Evidence for this conjecture
For n=2 Selberg proved that Eigen values λ of the Laplacian on function satisfy $\lambda > 3/16$ and more generally Clozel showed there exists a lower bound on the eigenvalues of the Laplacian on functions independent of Γ.

A consequence of the previous theorem:

Corrollary(Joint with Venkataramana)
If the above conjecture holds true in the middle degree for all even integers n, then the conjecture holds for arbitrary degrees of the differential forms

Happy Birthday, Gregg