Maxwell, Casimir, Zuckerman

Maxwell's equations for an electromagnetic field in a vacuum can be formulated naturally in terms of differential forms on Minkowski space $\mathbf{R}^{3+1}$, that is, Cartesian four-dimensional space equipped with the indefinite metric $x^2 + y^2 + z^2 - t^2$. More specifically, if d is the standard exterior derivative operator on differential forms on $\mathbf{R}^{3+1}$, and if d^* is its dual with respect to the Minkowski metric, then Maxwell's equations can be interpreted as the conditions

$$dF = 0 = d^*F,$$

where

$$F = E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dx \wedge dt + B_1 dy \wedge dz - B_2 dx \wedge dz + B_3 dx \wedge dy,$$

is a two-form incorporating the pair of three-vectors

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

used in the traditional description of the electric field and the magnetic field.

It is well known that the Lorentz group $O(3,1)$ of isometries of the Minkowski metric act as symmetries of the space of solutions to Maxwell's equations. That is, the natural action of $O(3,1)$ on the space of differential forms on $\mathbf{R}^{3+1}$ commutes with the operators d and d^*, and therefore preserves their kernel, which is the space of solutions to Maxwell's equations.

Less well-known is that the operators d and d^* fit inside a larger system of operators, isomorphic to the Lie superalgebra $osp(2,2)$. All the operators of $osp(2,2)$ commute with the action of $O(3,1)$. Furthermore, the operators of the superalgebra $osp(2,2)$ constitute in some sense a maximal family commuting with the action of $O(3,1)$. This suggests that it could be profitable to analyze the space of differential forms on $\mathbf{R}^{3+1}$ from the point of view of the joint action of $O(3,1)$ and $osp(2.2)$. This, or rather the more general situation, of analyzing the action of the indefinite orthogonal group $O(p,q)$ on the differential forms on $\mathbf{R}^{p+q}$, and the commuting action of the Lie superalgebra $osp(2.2)$, is the main subject of this talk.

The pair $(O(p,q), osp(2,2))$ is an extension to differential forms of the pair $(O(p,q), sl_2)$ that acts on scalar-valued functions on $\mathbf{R}^{p+q}$. The action of the pair $(O(p,q), sl_2)$ has applications to representation theory, to special functions, to differential equations, to mathematical physics and to the theory of automorphic forms. Similarly, study of the pair $(O(p,q), osp(2,2))$ sheds light on Maxwell's equations and on representation theory.

Key to the analysis of the action of $(O(p,q), osp(2,2))$ are study of the Casimir operators of $O(p,q)$ and of $osp(2,2)$, and use of the Zuckerman translation functors to compare the action on forms with the action on scalar-valued functions. Remarkably, the two Casimir operators both give rise to the same operator on the space of differential forms.

This talk is based on the Ph.D. thesis of Dan Lu.