E8 Theory

$$
\begin{aligned}
S_D &= \int \utilde{d^4x}\,|e|\left\{\bar{\psi}\,\gamma^\mu (e_\mu)^i \left(\partial_i + \tfrac{1}{4}\omega_i{}^{\mu\nu}\gamma_{\mu\nu} + G_i{}^B T_B\right)\underset{\cdot}{\psi} + \bar{\psi}\,\phi\,\underset{\cdot}{\psi}\right\} \\
&= \int \utilde{d^4x}\,|e|\left\{\bar{\psi}\,\gamma^\mu (e_\mu)^i \left(\partial_i + \tfrac{1}{4}\omega_i{}^{\mu\nu}\gamma_{\mu\nu} + \tfrac{1}{2}G_i{}^{\psi\chi}\gamma_{\psi\chi} + \tfrac{1}{4}(e_i)^\mu \phi^\psi \gamma_{\mu\psi}\right)\underset{\cdot}{\psi}\right\} \\
&= \int \utilde{e}\left\{\bar{\psi}\,\vec{e}\left(\underline{d} + \underline{H}\right)\underset{\cdot}{\psi}\right\} = \int \tfrac{1}{4!}\bar{\psi}\,\underline{e}\,\underline{e}\,\underline{e}\,\epsilon\,\underline{D}\underset{\cdot}{\psi} = \int \utilde{e}\,\bar{\psi}\,\not{D}\,\underset{\cdot}{\psi}
\end{aligned}
$$

Gravity: $\underline{\omega} = \tfrac{1}{2}\underline{dx}^i\,\omega_i{}^{\mu\nu}\gamma_{\mu\nu} \in Cl(3,1)^2 = spin(3,1)$ $\quad\underline{e} = \underline{dx}^i (e_i)^\mu \gamma_\mu \in Cl(3,1)^1 = 4$

GUT: $\underline{G} \in su(2)_L + u(1)_Y + su(3)$
$\subset su(2)_L + su(2)_R + su(4) = spin(4) + spin(6) \subset spin(10)$

Fermions: $\underset{\cdot}{\psi} \in 2\times(2_L{+}2_R)\times(1+3) = 32^{\mathbb{C}} = 64^{\mathbb{R}} \quad (\times 3)$

Higgs: $\phi = \phi^\psi \gamma_\psi \in Cl(4)^1 = 4 = \mathbb{C}^2$ or $Cl(N)^1 = N$

Connection: $\underline{H} = \tfrac{1}{2}\underline{\omega} + \tfrac{1}{4}\underline{e}\,\phi + \underline{G} \in spin(3,1) + 4\times 10 + spin(10) \subset spin(3,11)$

Curvature: $\underline{\underline{F}} = \underline{d}\,\underline{H} + \underline{H}\,\underline{H} = \tfrac{1}{2}(\underline{\underline{R}} - \tfrac{1}{8}\underline{e}\,\underline{e}\,\phi^2) + \tfrac{1}{4}(\underline{T}\phi - \underline{e}\,\underline{D}\phi) + \underline{\underline{F}}^G$

Superconnection: $\underset{\cdot}{\underline{A}} = \underline{H} + \underset{\cdot}{\psi} \in spin(3,11) + 64^{+\mathbb{R}}_S$
$\subset spin(4,12) + 128^{+\mathbb{R}}_S = E8(-24)$

Supercurvature: $\underset{\cdot}{\underline{\underline{F}}} = \underline{d}\,\underset{\cdot}{\underline{A}} + \underset{\cdot}{\underline{A}}\,\underset{\cdot}{\underline{A}} = \underline{\underline{F}} + \underline{D}\underset{\cdot}{\psi} + \underset{\cdot}{\psi}\underset{\cdot}{\psi}$

$$
S = \int \left\langle \underline{\underline{\bar{B}}}\,\underset{\cdot}{\underline{\underline{F}}} - \tfrac{\pi G}{4}\underline{\underline{B}}\,\epsilon\,\underline{\underline{B}} + g^2\,\underline{\underline{B}}' * \underline{\underline{B}}' \right\rangle \sim \int \left\langle \utilde{e}\,\bar{\psi}\,\not{D}\,\underset{\cdot}{\psi} + \tfrac{1}{16\pi G}\utilde{e}\left(R - \tfrac{3}{2}\phi^2\right)\phi^2 + \tfrac{1}{4g^2}\underline{D}\phi * \underline{\underline{\underline{D}}}\phi + \tfrac{1}{4g^2}\underline{\underline{F}}^G * \underline{\underline{F}}^G \right\rangle
$$

Structure of interactions

grav: $\underline{\omega} = d\underline{x}^k \omega_k{}^{\mu\nu} \tfrac{1}{2}\gamma_{\mu\nu} \in spin(3,1)$

weak: $\underline{W} = d\underline{x}^k W_k{}^{\pi} \tfrac{i}{2}\sigma_\pi \in su(2)_L$

hyper: $\underline{B} = d\underline{x}^k B_k i \in u(1)_Y$

strong: $\underline{g} = d\underline{x}^k g_k{}^{A} \tfrac{i}{2}\lambda_A \in su(3)$

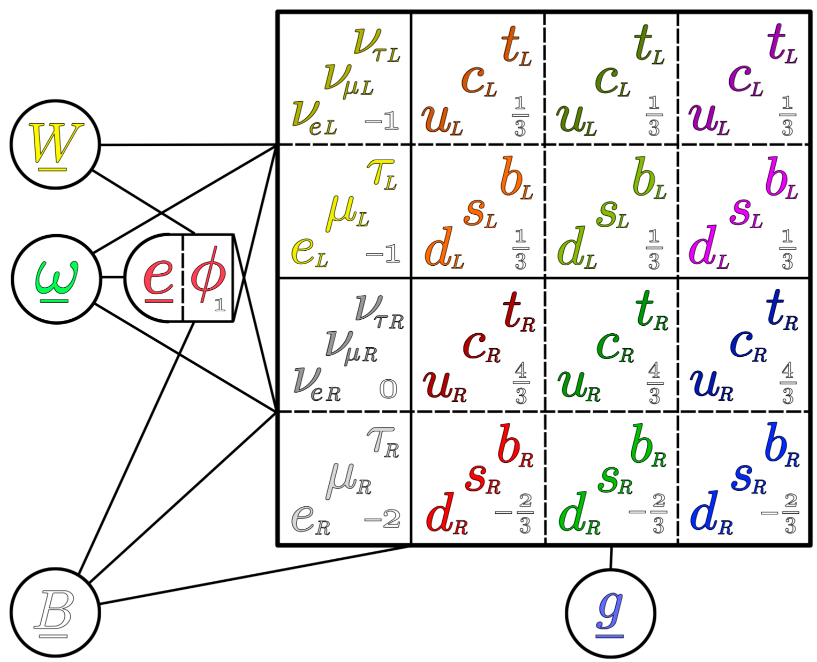
frame: $\underline{e} = d\underline{x}^k (e_k)^\mu \gamma_\mu \in 4$

Higgs: $\phi = \begin{bmatrix} \phi_+ \\ \phi_0 \end{bmatrix} \in 2^{\mathbb{C}}$

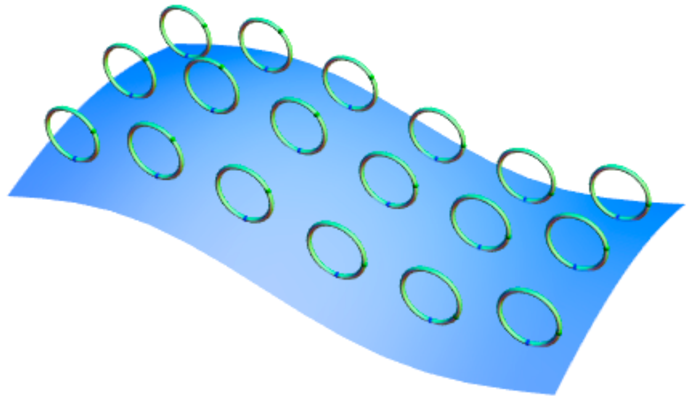
fermions:

$$\begin{bmatrix} u_L^{\wedge} \\ u_L^{\vee} \\ u_R^{\wedge} \\ u_R^{\vee} \end{bmatrix}, \begin{bmatrix} u_L \\ d_L \end{bmatrix}, \begin{bmatrix} u^r \\ u^g \\ u^b \end{bmatrix}$$

$$\in 4_S^{\mathbb{C}}, \quad 2^{\mathbb{C}}, \quad 3^{\mathbb{C}}$$



Fiber bundle



Principal bundle over 4D base.

Fermion, $\underset{\cdot}{\psi}$, Higgs, ϕ, and frame, $\underline{e}$, fibers in various representations of the structure group,

$$Spin(3,1) \times SU(2) \times U(1) \times SU(3)$$

Connection: $\underline{A} = d\underline{x}^k A_k{}^B T_B$

Ehresmann connection: $\vec{\underline{\mathcal{A}}}(x,y) = \underline{A}^B(x)\,\vec{\xi}_B(y) + \vec{\underline{\mathcal{I}}}$

Curvature: $\underline{\underline{F}} = \underline{d}\underline{A} + \frac{1}{2}[\underline{A}, \underline{A}]$

Frölicher-Nijenhuis: $\vec{\underline{\underline{\mathcal{F}}}} = -\frac{1}{2}[\vec{\underline{\mathcal{A}}}, \vec{\underline{\mathcal{A}}}] = -\vec{\underline{\mathcal{A}}}(\underline{\partial}\vec{\underline{\mathcal{A}}}) + \underline{\partial}(\vec{\underline{\mathcal{A}}}\vec{\underline{\mathcal{A}}})$

Action: $S(\underline{\underline{F}}, \underline{D}\underset{\cdot}{\psi}, \underline{A}, \underset{\cdot}{\psi}, \phi, \underline{e}, \ldots)$

Matrix representation

$$S_\psi = \int \underset{\sim}{e} \left\{ \bar{\psi} \gamma^\mu (e_\mu)^i \left(\partial_i + \tfrac{1}{2} \omega_i{}^{\nu\rho} \tfrac{1}{2} \gamma_{\nu\rho} + W_i{}^\pi T^W_\pi + B_i T^Y + g_i{}^A T^g_A \right) \underset{\cdot}{\psi} + \bar{\psi} \phi \underset{\cdot}{\psi} \right\}$$

$$\begin{aligned} \gamma_1 &= \sigma_2 \otimes \sigma_1 \\ \gamma_2 &= \sigma_2 \otimes \sigma_2 \\ \gamma_3 &= \sigma_2 \otimes \sigma_3 \\ \gamma_4 &= i\sigma_1 \otimes 1 \end{aligned}$$

$$Cl(3,1) \subset GL(4,\mathbb{C})$$

$$\gamma_{\mu\nu} = \gamma_\mu \gamma_\nu \in spin(3,1)$$

$$\epsilon = \gamma_1\gamma_2\gamma_3\gamma_4 = i\sigma_3 \otimes 1$$

$$P_{R/L} = \tfrac{1}{2}(1 \pm i\epsilon)$$

$$\Lambda_A = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_A \end{bmatrix} \in GL(4,\mathbb{C})$$

$$\begin{aligned} T^\omega_{\mu\nu} &= 1 \otimes 1 \otimes \gamma_{\mu\nu} \qquad \in GL(32,\mathbb{C}) \\ T^W_\pi &= 1 \otimes \tfrac{i}{2}\sigma_\pi \otimes P_L \\ T^g_A &= \tfrac{i}{2}\Lambda_A \otimes 1 \otimes 1 \\ T^Y &= 1 \otimes i\sigma_3 \otimes P_R \\ &\quad - i\,\mathrm{diag}(1, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}) \otimes 1 \otimes 1 \end{aligned}$$

$$\underset{\cdot}{\psi} = \begin{bmatrix} \nu \\ e \\ u^r \\ d^r \\ u^g \\ d^g \\ u^b \\ d^b \end{bmatrix} \in 32^{\mathbb{C}}$$

complex structure: $i \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -i\sigma_2 \;\Rightarrow\; T \in GL(64,\mathbb{R}),\; \underset{\cdot}{\psi} \in 64^{\mathbb{R}}$

Weights

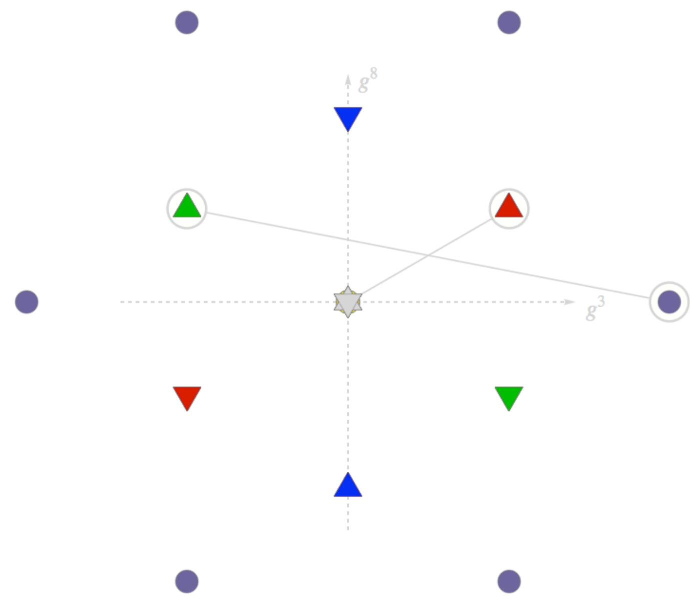
6D Cartan subalgebra: $C = \omega_S T^{\omega}_{12} + \omega_T T^{\omega}_{34} + W\, T^{W}_{3} + Y\, T^{Y} + g^3\, T^{g}_{3} + g^8\, T^{g}_{8}$

$$\begin{bmatrix} & -\frac{1}{2}g^3 - \frac{1}{2\sqrt{3}}g^8 & & & & \\ \frac{1}{2}g^3 + \frac{1}{2\sqrt{3}}g^8 & & & & & \\ & & & \frac{1}{2}g^3 - \frac{1}{2\sqrt{3}}g^8 & & \\ & & -\frac{1}{2}g^3 + \frac{1}{2\sqrt{3}}g^8 & & & \\ & & & & & \frac{1}{\sqrt{3}}g^8 \\ & & & & -\frac{1}{\sqrt{3}}g^8 & \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = i\left(\left(\frac{1}{2}\right) g^3 + \left(\frac{1}{2\sqrt{3}}\right) g^8 \right) \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

weight vectors
weights
$\longleftrightarrow$
states
quantum numbers

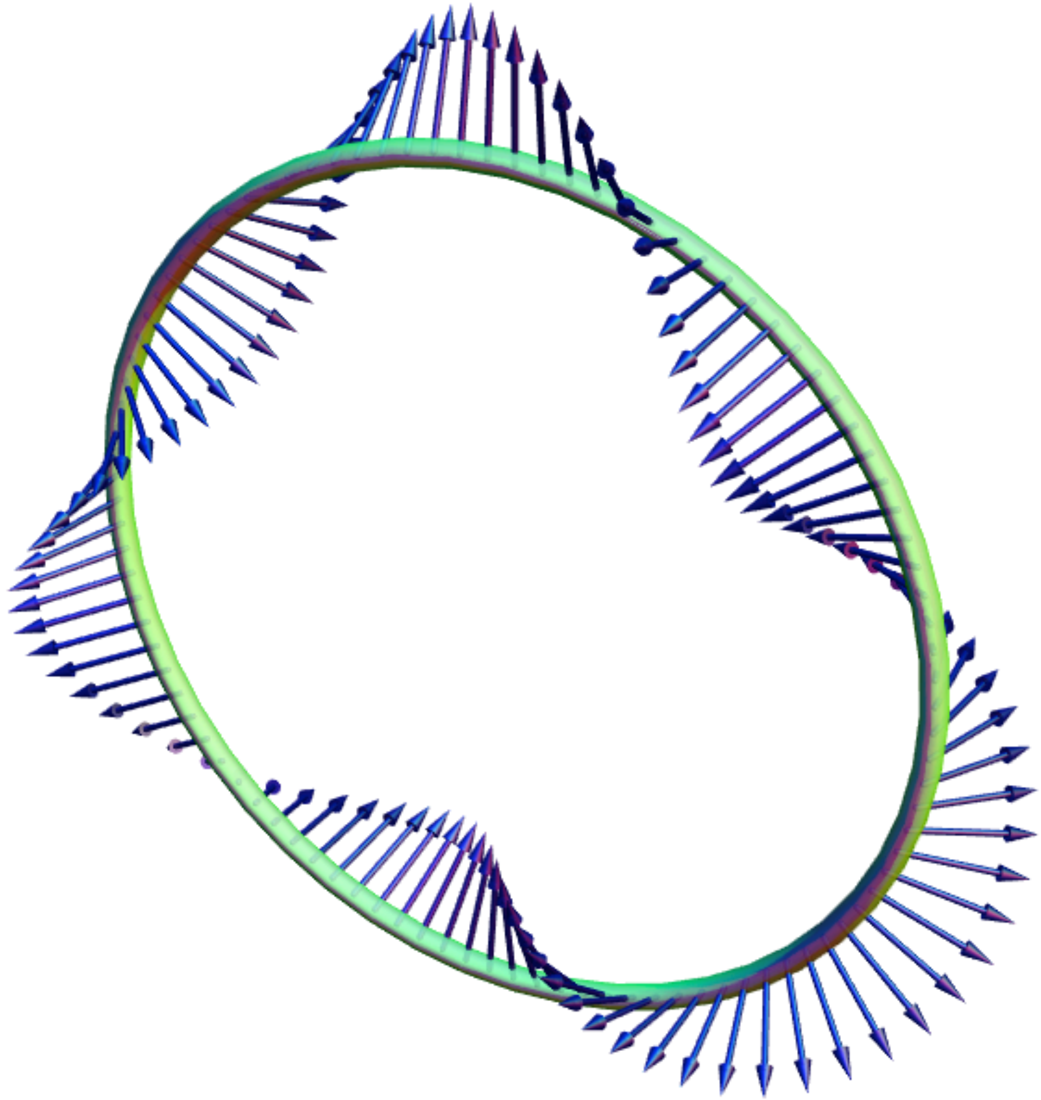
$\updownarrow$ $\updownarrow$

eigenvectors
eigenvalues
$\longleftrightarrow$
particles
charges



$su(3) + 3 + \bar{3}$

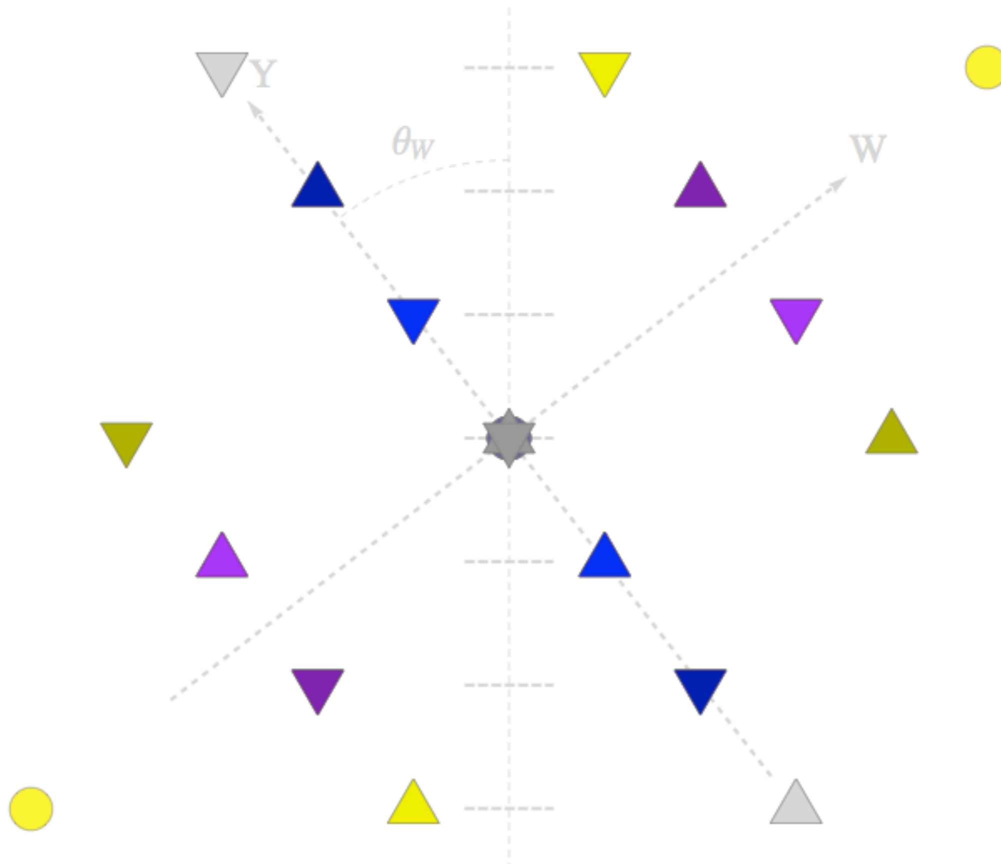
Twist



Electric charge

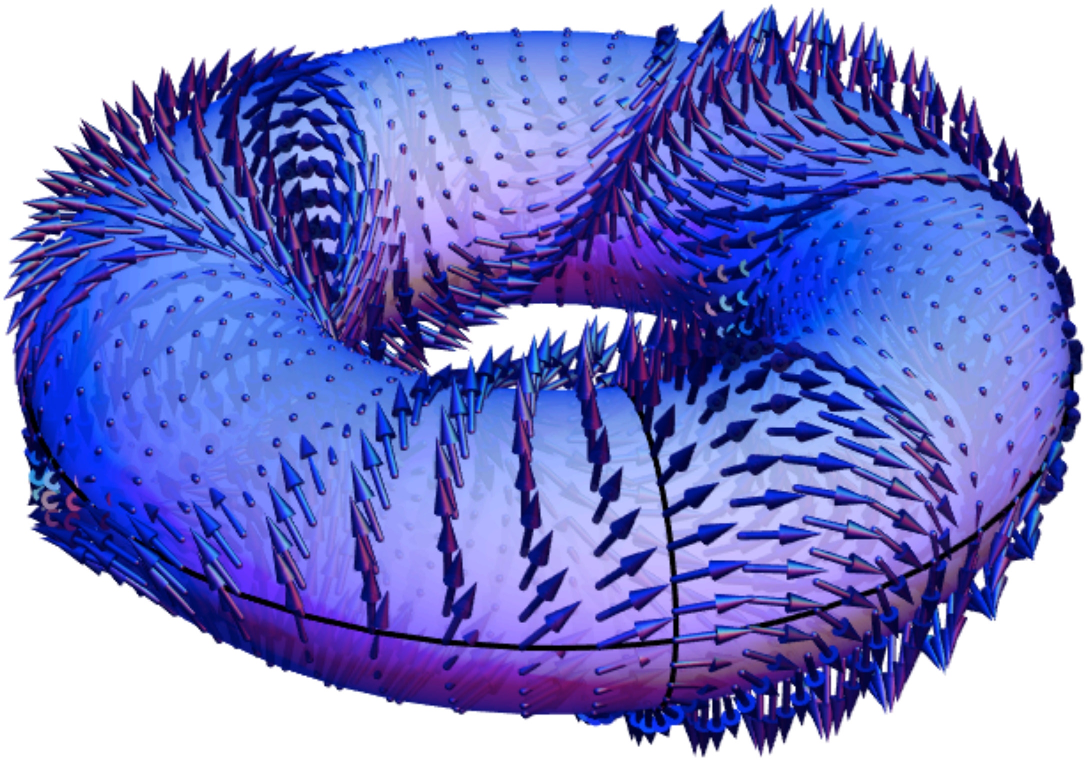
$$u(1)_Q + \ldots$$

Electroweak model

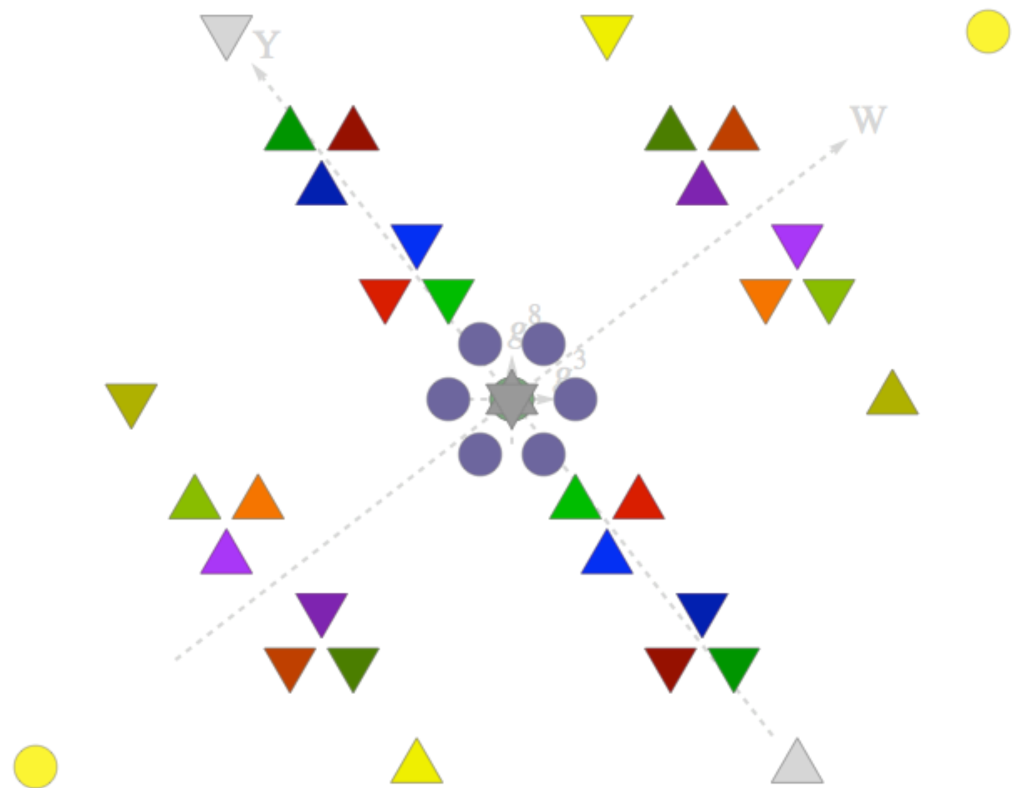


$$su(2)_L + u(1)_Y + \ (2_L{+}2_R)\times(1{+}1)$$

Torus twist



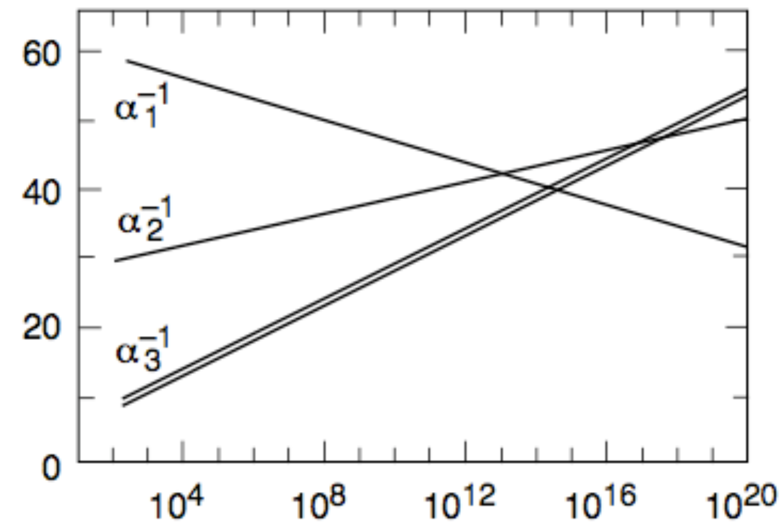
Standard Model



$$su(2)_L + u(1)_Y + su(3) \; + \; (2_L + 2_R) \times (1+3)$$

Grand Unified Theories

Embed the standard model gauge algebra and fermion representation space in the Lie algebra and representation space of a larger group.



Standard Model

$$G_{SM} = su(2)_L + u(1)_Y + su(3)$$

$$\psi_{SM} = (2_L + 2_R) \times (1+3)$$

Georgi-Glashow SU(5)

$$G_{SM} \subset G_{SU(5)} = su(5)$$

$$\psi_{SM} \supset \psi_{SU(5)} = \bar{5} + 10$$

Pati-Salam

$$G_{SM} \subset G_{PS} = su(2)_L + su(2)_R + su(4) = spin(4) + spin(6)$$

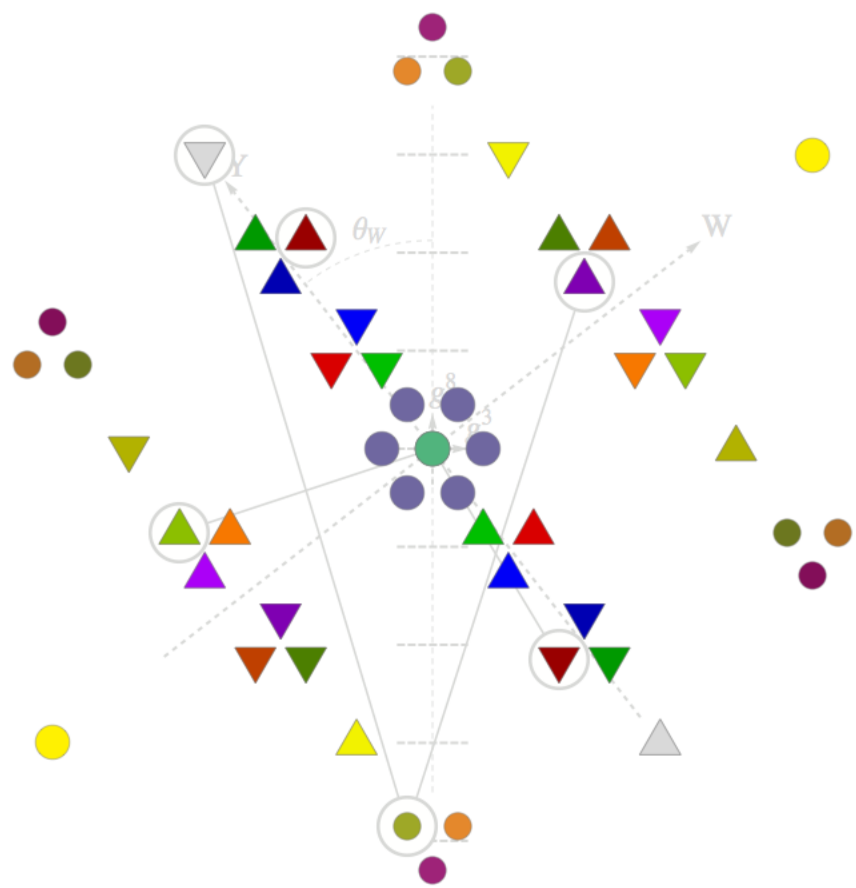
$$\psi_{SM} = \psi_{PS} = (2_L + 2_R) \times 4 = 4 \times 4$$

SO(10)

$$G_{PS} \subset G_{SO(10)} \qquad G_{SU(5)} \subset G_{SO(10)} \qquad G_{SO(10)} = spin(10)$$

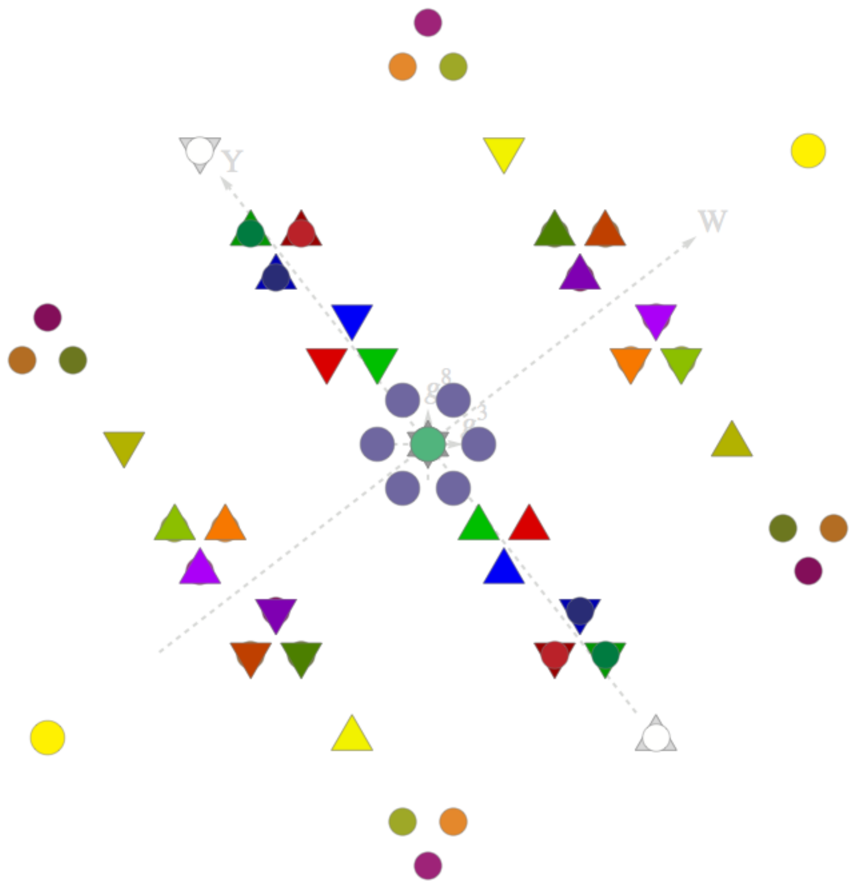
$$\psi_{SM} = \psi_{SO(10)} = 16_S^{+\mathbb{C}}$$

Georgi-Glashow SU(5)



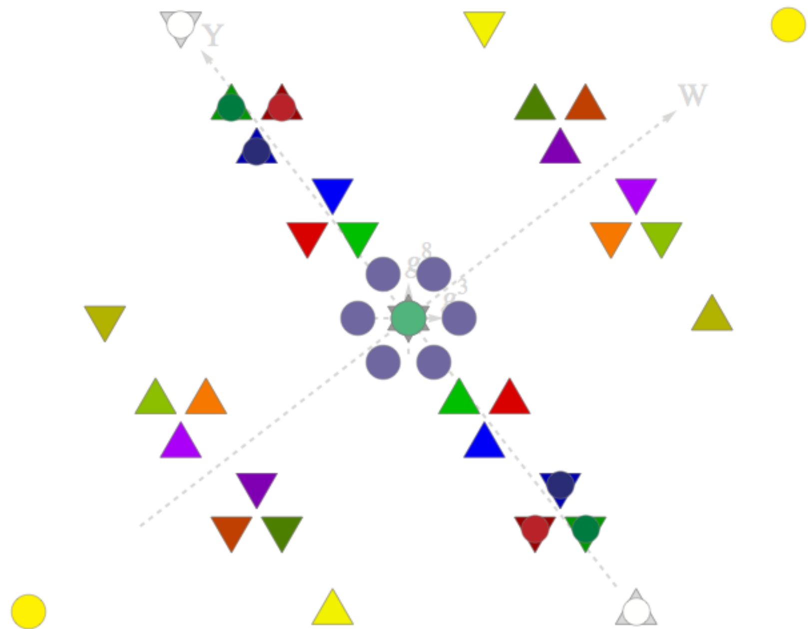
$su(5) + \bar{5} + 10$

SO(10)



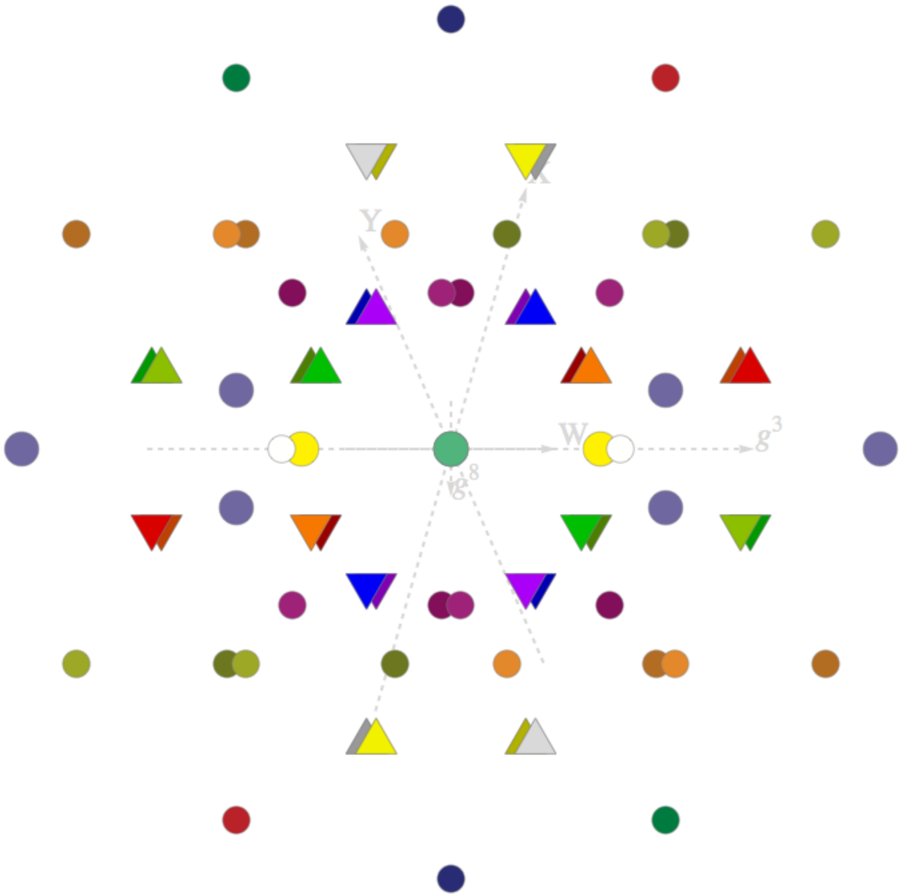
$spin(10) + 16_S^{+\mathbb{C}}$

Pati-Salam



$$spin(4) + spin(6) \; + \; 4 \times 4$$

E6



$$E6 = spin(10) + u(1)_{PQ} + 16_S^{+\mathbb{C}}$$

Gauge-gravity unification

$$S_\psi = \int \underset{\sim}{e} \left\{ \bar{\psi} \gamma^\mu (e_\mu)^i \left(\partial_i + \tfrac{1}{2} \underset{i}{\omega}{}^{\nu\rho} \tfrac{1}{2} \gamma_{\nu\rho} + \underset{i}{W}{}^{\pi} T^W_\pi + B_i T^Y + \underset{i}{g}{}^{A} T^g_A \right) \underset{\cdot}{\psi} + \bar{\psi} \phi \underset{\cdot}{\psi} \right\}$$

$$= \int \underset{\sim}{e} \left\{ \bar{\psi} \gamma^\mu (e_\mu)^i \left(\partial_i + \tfrac{1}{2} \underset{i}{\omega}{}^{\nu\rho} \tfrac{1}{2} \gamma_{\nu\rho} + \underset{i}{G}{}^{\alpha\beta} \tfrac{1}{2} \gamma_{\alpha\beta} + \tfrac{1}{4} (e_i)^\nu \phi^\alpha \gamma_\nu \gamma_\alpha \right) \underset{\cdot}{\psi} \right\}$$

$$= \int \underset{\sim}{e} \left\{ \bar{\psi} \vec{e} \left(\underline{d} + \underline{H} \right) \underset{\cdot}{\psi} \right\}$$

$$= \int \tfrac{1}{4!} \bar{\psi} \underline{e}\, \underline{e}\, \underline{e}\, \epsilon \underline{D} \underset{\cdot}{\psi}$$

Unified bosonic connection:

$$\underline{H} = \tfrac{1}{2} \underline{\omega} + \tfrac{1}{4} \underline{e} \phi + \underline{G} \quad \in spin(?)$$

With SO(10) GUT:

$$spin(3,1) + 4 \times 10 + spin(10) = spin(3,11) \text{ or } spin(13,1)$$

or with Pati-Salam GUT:

$$spin(3,1) + 4 \times 4 + spin(4) + spin(6) \subset spin(3,11), spin(13,1), spin(7,7) \text{ or } spin(9,5)$$

One generation of fermions:

$$64^{+\mathbb{R}}_S \text{ of } spin(3,11) \text{ or } spin(7,7)$$

Action

Bosonic connection: $\underline{H} = \frac{1}{2}\underline{\omega} + \frac{1}{4}\underline{e}\phi + \underline{G} \qquad \in spin(3,1) + 4\times 10 + spin(10) = spin(3,11)$

Curvature: $\underline{\underline{F}} = \underline{d}\,\underline{H} + \underline{H}\,\underline{H} = \frac{1}{2}(\underline{\underline{R}} - \frac{1}{8}\underline{e}\,\underline{e}\,\phi^2) + \frac{1}{4}(\underline{\underline{T}}\phi - \underline{e}\,\underline{D}\phi) + \underline{\underline{F}}^G$

Riemann: $\underline{\underline{R}} = \underline{d}\,\underline{\omega} + \frac{1}{2}\underline{\omega}\,\underline{\omega}$ Torsion: $\underline{\underline{T}} = \underline{d}\,\underline{e} + \frac{1}{2}\underline{\omega}\,\underline{e} + \frac{1}{2}\underline{e}\,\underline{\omega}$ Covariant: $\underline{D}\phi = \underline{d}\phi + \underline{G}\phi - \phi\underline{G}$

$spin(3,1)$ duality operator: $\epsilon = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$ Hodge: $\underline{\underline{\overrightarrow{\overrightarrow{\epsilon}}}} = \left\langle \underline{e}\,\underline{e}\,\epsilon\,\overrightarrow{e}\,\overrightarrow{e}\right\rangle$ Auxiliary: $\underline{\underline{B}} \in spin(3,11)$

Boson action: $S_H = \int \left\langle \underline{\underline{B}}\,\underline{\underline{F}} - \underline{\underline{B}}(\frac{\pi G}{4}\epsilon - g^2\underline{\underline{\overrightarrow{\overrightarrow{\epsilon}}}})\underline{\underline{B}}\right\rangle = \int \left\langle \frac{-1}{\pi G}\underline{\underline{F}}\,\epsilon\,\underline{\underline{F}} + \frac{1}{4g^2}\underline{\underline{F}}\,\underline{\underline{\overrightarrow{\overrightarrow{\epsilon}}}}\,\underline{\underline{F}}\right\rangle$

$$= \int \left\langle \frac{-1}{4\pi G}\underline{\underline{R}}\,\underline{\underline{R}}\,\epsilon + \frac{1}{16\pi G}\phi^2\underline{\underline{R}}\,\underline{e}\,\underline{e}\,\epsilon + \frac{3}{32\pi G}\phi^4\underset{\sim}{e} + \frac{1}{64g^2}\phi^2\underline{\underline{T}}\,\underline{\underline{\overrightarrow{\overrightarrow{\epsilon}}}}\,\underline{\underline{T}} + \frac{1}{64g^2}\underline{e}\,\underline{D}\phi\,\underline{\underline{\overrightarrow{\overrightarrow{\epsilon}}}}\,\underline{e}\,\underline{D}\phi + \frac{1}{4g^2}\underline{\underline{F}}^G\underline{\underline{\overrightarrow{\overrightarrow{\epsilon}}}}\,\underline{\underline{F}}^G\right\rangle$$

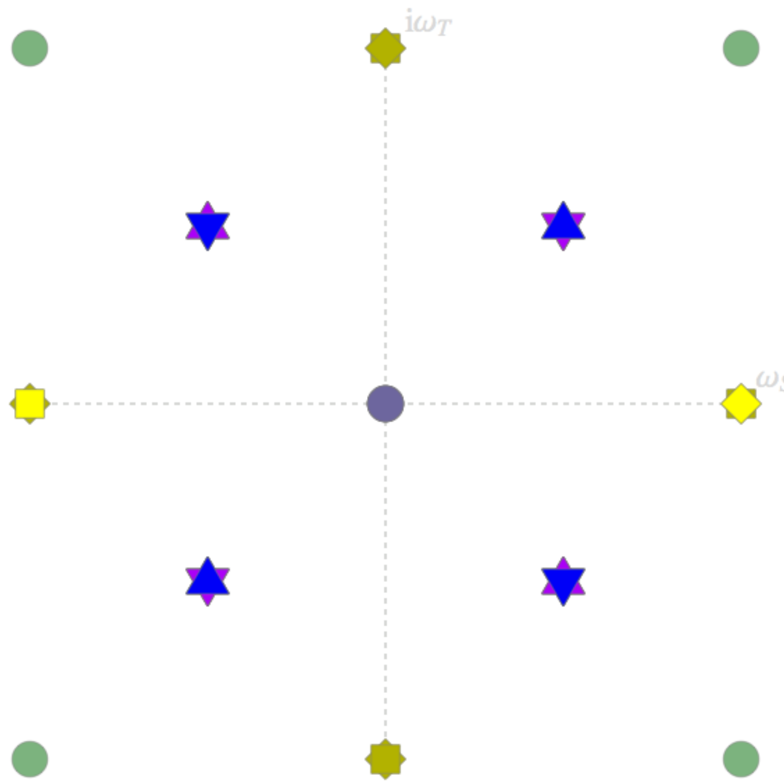
Good: Perturbed BF gives GR, Higgs, gauge action, and cosmological constant, $\Lambda = \frac{3}{4}\phi^2$.

Bad: Symmetry breaking by hand. Hodge seems contrived, with $\underline{e}$ pulled out of $\underline{H}$.

A possible $spin(3,11)$ invariant action: $S_H = \int \left\langle \underline{\underline{B}}\,\underline{\underline{F}} + \underline{\underline{B}}\,\Phi\,\underline{\underline{B}} - \frac{g}{2}\underline{\underline{B}}\left\langle \Phi^2\right\rangle \underline{\underline{B}}\right\rangle$

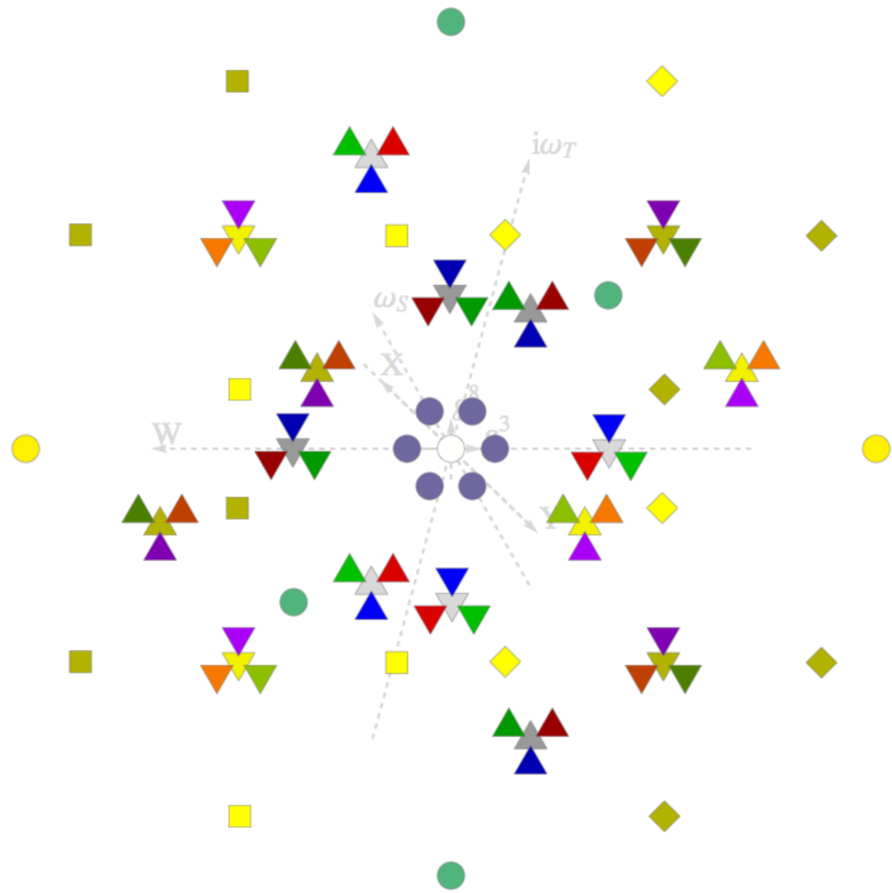
Fermion action: $S_\psi = \int \frac{1}{4!}\bar{\psi}\,\underline{e}\,\underline{e}\,\underline{e}\,\epsilon\,\underline{D}\underset{\cdot}{\psi} = \int \dot{\underline{\underline{\underline{B}}}}\,\underline{D}\underset{\cdot}{\psi}$

Spin



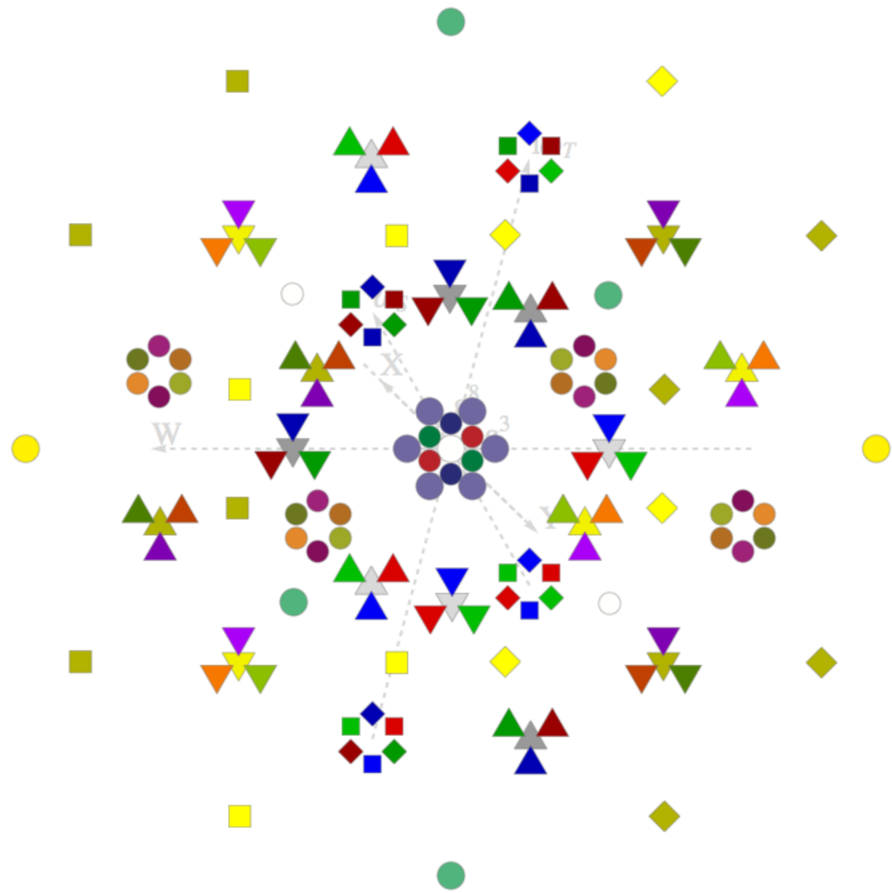
$$spin(3,1)\,+\,4_S^{\mathbb{C}}\,+\,4_V$$

Everything together



$$spin(3,1) + 4\times(2+\bar{2}) + su(2)_L + u(1)_Y + su(3) + 2\times(2_L+2_R)\times(1+3)$$

Spin(3 11) ToE



$$spin(3,11) + 64_S^{+\mathbb{R}}$$

Matrix representation of spin(3 11)

Real $Cl(3,11)$ basis elements in $GL(128,\mathbb{R})$

$$
\begin{aligned}
\Gamma_1 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1 \\
\Gamma_2 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\Gamma_3 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_3 \otimes 1 \\
\Gamma_4 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes 1 \otimes 1 \\
\Gamma_5 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \\
\Gamma_6 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \\
\Gamma_7 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1 \otimes 1 \\
\Gamma_8 &= i\sigma_1 \otimes 1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_9 &= i\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{10} &= i\sigma_1 \otimes 1 \otimes \sigma_2 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{11} &= i\sigma_1 \otimes \sigma_1 \otimes 1 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{12} &= i\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{13} &= i\sigma_1 \otimes \sigma_3 \otimes 1 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{14} &= i\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1
\end{aligned}
$$

$$
\begin{aligned}
\Gamma &= \Gamma_1 \Gamma_2 \ldots \Gamma_{14} \\
&= \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1
\end{aligned}
$$

$$P_\pm = \tfrac{1}{2}(1 \pm \Gamma)$$

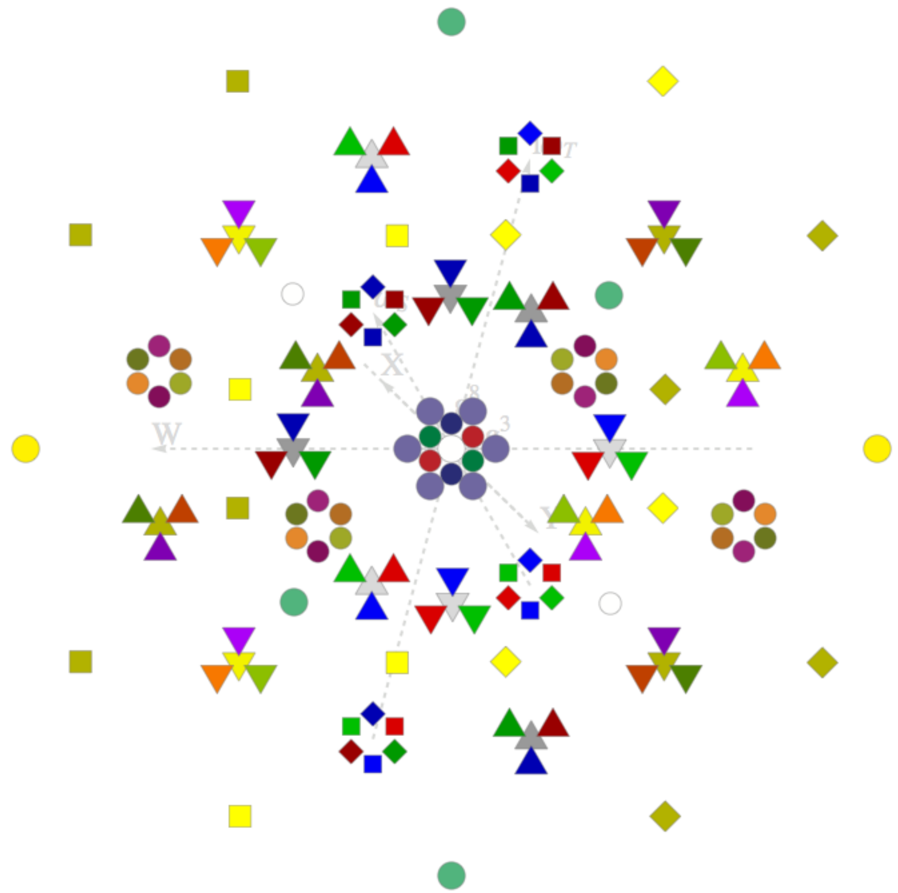
$$\Gamma^+_{\alpha\beta} = P_+ \Gamma_\alpha \Gamma_\beta \text{ in } GL(64,\mathbb{R})$$

18 Standard Model generators in $spin(3,11)$

$$
\begin{aligned}
T^\omega_{\mu\nu} &= \sqrt{2}\,\Gamma^+_{\mu\nu} \\
T^W_1 &= \Gamma^+_{5,8} - \Gamma^+_{6,7} \\
T^W_2 &= \Gamma^+_{5,7} + \Gamma^+_{6,8} \\
T^W_3 &= -\Gamma^+_{5,6} + \Gamma^+_{7,8} \\
T^g_1 &= \Gamma^+_{9,12} - \Gamma^+_{10,11} \\
T^g_2 &= \Gamma^+_{9,11} + \Gamma^+_{10,12} \\
T^g_3 &= -\Gamma^+_{9,10} + \Gamma^+_{11,12} \\
T^g_4 &= \Gamma^+_{9,14} - \Gamma^+_{10,13} \\
T^g_5 &= \Gamma^+_{9,13} + \Gamma^+_{10,14} \\
T^g_6 &= \Gamma^+_{11,14} - \Gamma^+_{12,13} \\
T^g_7 &= \Gamma^+_{11,13} + \Gamma^+_{12,14} \\
T^g_8 &= \tfrac{1}{\sqrt{3}}(-\Gamma^+_{9,10} - \Gamma^+_{11,12} + 2\,\Gamma^+_{13,14}) \\
T^Y &= \tfrac{\sqrt{3}}{\sqrt{5}}(\Gamma^+_{5,6} + \Gamma^+_{7,8}) \\
&\quad + \tfrac{2}{\sqrt{15}}(\Gamma^+_{9,10} + \Gamma^+_{11,12} + \Gamma^+_{13,14})
\end{aligned}
$$

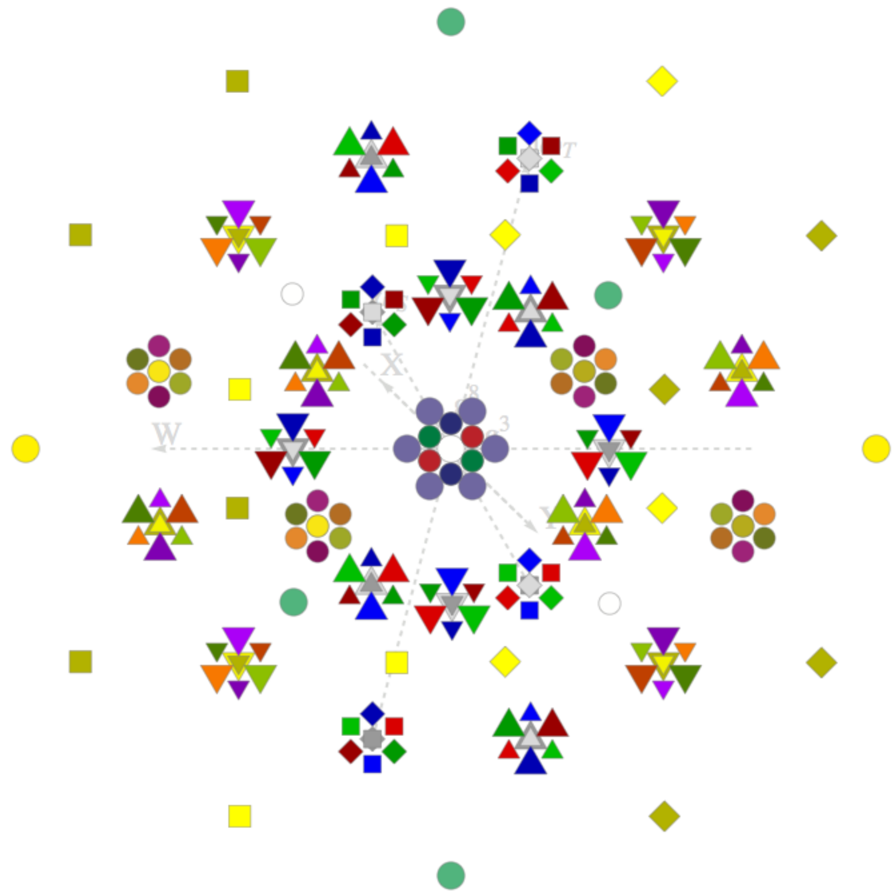
$$\psi \in 64^{+\mathbb{R}}_S$$

Spin(3 11) ToE



$$spin(3,11) + 64_S^{+\mathbb{R}}$$

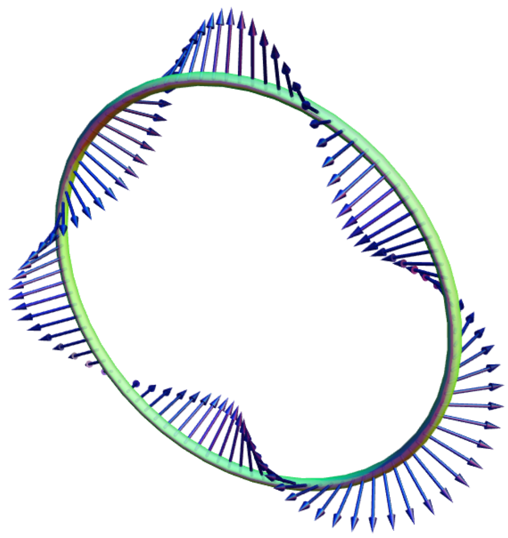
E8 ToE



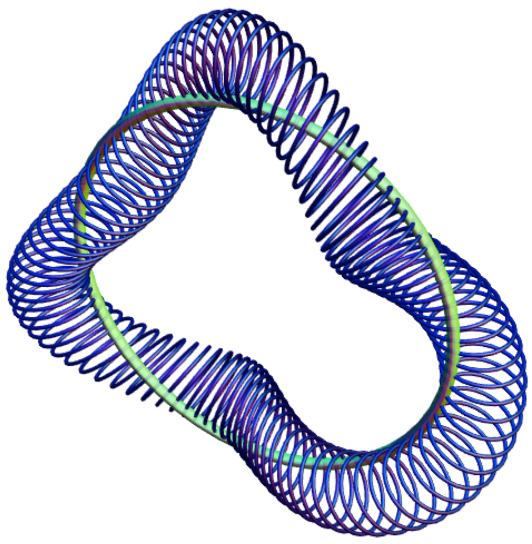
$$E8(-24) = spin(4,12) \; + \; 128_S^{+\mathbb{R}}$$

Superconnection

Geometric description of fermion fibers as Lie algebra valued anticommuting fields.



$\longleftrightarrow$



For a principal G-bundle, with H a reductive subalgebra in $G = H \oplus K$, define the **superconnection** to be the direct sum of an H connection and a K valued anticommuting field:

$$
\begin{array}{lll}
\underset{\cdot}{\underline{A}} = \underline{H} + \underset{\cdot}{\psi} & \in G & [H,H] \subset H \\
\underline{H} = \underline{dx}^i H_i{}^A T_A & \in H & [H,K] \subset K \\
\underset{\cdot}{\psi} = \underset{\cdot}{\psi}^\alpha T_\alpha & \in K & [K,K] \subset G
\end{array}
$$

Supercurvature:

$$
\underset{\cdot}{\underline{\underline{F}}} = \underline{d}\,\underset{\cdot}{\underline{A}} + \frac{1}{2}[\underset{\cdot}{\underline{A}}, \underset{\cdot}{\underline{A}}] = (\underline{d}\,\underline{H} + \frac{1}{2}[\underline{H}, \underline{H}]) + (\underline{d}\underset{\cdot}{\psi} + [\underline{H}, \underset{\cdot}{\psi}]) + \frac{1}{2}[\underset{\cdot}{\psi}, \underset{\cdot}{\psi}] = \underline{\underline{F}}_H + \underline{D}\underset{\cdot}{\psi} + \underset{\cdot}{\psi}\underset{\cdot}{\psi}
$$

BRST

The BRST technique accounts for gauge symmetries by introducing new fields with anticommuting coefficients and dynamics that fixes the original local gauge symmetry and introduces a new global (super) symmetry — the BRST transformation.

Connection: $\underline{A} = \underline{H} + \underline{K} \quad \in \; H + K = G$ with H reductive in G.

Action: $S = \int \left\langle \underline{\underline{B}}\,\underline{\underline{F}} + \underset{\sim}{V}\left(B^H\right) \right\rangle$ purely gauge (topological) in $\underline{K}$.

BRST transformation: (make gauge parameter, $\underset{\cdot}{\psi} = \psi^\alpha(x) T_\alpha \in K$, an anticommuting BRST field)

$$\underset{\cdot}{\delta}\underline{K} = -\,\underline{D}\underset{\cdot}{\psi} \qquad \underset{\cdot}{\delta}\underset{\cdot}{\psi} = -\,\tfrac{1}{2}\left[\underset{\cdot}{\psi}, \underset{\cdot}{\psi}\right]$$

$$\underset{\cdot}{\delta}\underline{\underline{B}} = \left[\underline{\underline{B}}, \underset{\cdot}{\psi}\right] \qquad \underset{\cdot}{\delta}\dot{\underline{\underline{B}}} = \underline{\underline{\underline{\lambda}}} \qquad \underset{\cdot}{\delta}\underline{\underline{\underline{\lambda}}} = 0 \qquad \Longrightarrow \quad \underset{\cdot}{\delta}S = 0, \quad \underset{\cdot}{\delta}\underset{\cdot}{\delta} = 0$$

Choose a **BRST potential**, $\dot{\Psi} = \int \left\langle \dot{\underline{\underline{B}}}\,\underline{K} \right\rangle$, and use it to make the BRST action,

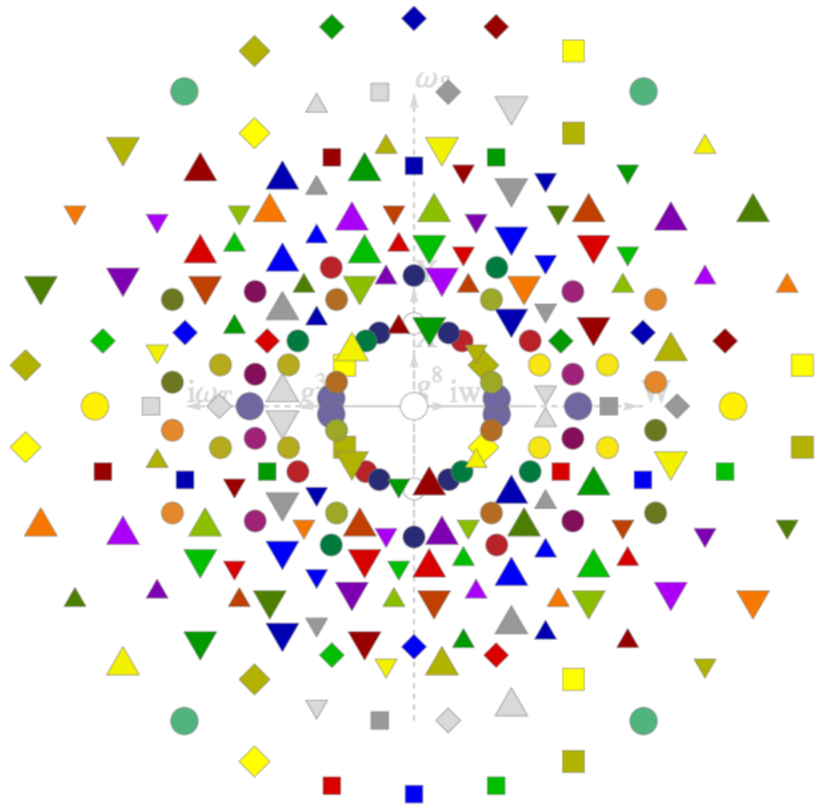
$$S' = \underset{\cdot}{\delta}\dot{\Psi} + S = \int \left\langle \underline{\underline{\underline{\lambda}}}\,\underline{K} + \dot{\underline{\underline{B}}}\,\underline{D}\underset{\cdot}{\psi} + \underline{\underline{B}}\,\underline{\underline{F}} + \underset{\sim}{V}\left(B^H\right) \right\rangle$$

Varying $\underline{\underline{\underline{\lambda}}}$ fixes the gauge to $\underline{K} = 0$, giving the effective action,

$$S^{\text{eff}} = \int \left\langle \dot{\underline{\underline{B}}}\,\underline{D}\underset{\cdot}{\psi} + \underline{\underline{B}}\,\underline{\underline{F}}^H + \underset{\sim}{V}\left(B^H\right) \right\rangle$$

which can be the Dirac action for a suitable algebra, and $\dot{\underline{\underline{B}}} = \frac{1}{4!}\bar{\psi}\,\underline{e}\,\underline{e}\,\underline{e}\,\epsilon$ The literature mentions the BRST superconnection, $\underset{\cdot}{\underline{A}} = \underline{H} + \underset{\cdot}{\psi}$, with $\underset{\cdot}{\psi}$ a "1-form in the space of connections," related to TQFT, and $\left(\underline{d} + \underset{\cdot}{\delta}\right)$ relating to BRST cohomology and anomalies.

E8 ToE (projection to the Coxeter plane)



Matrix representation of spin(4 12)

Real $Cl(4,12)$ basis elements in $GL(256,\mathbb{R})$

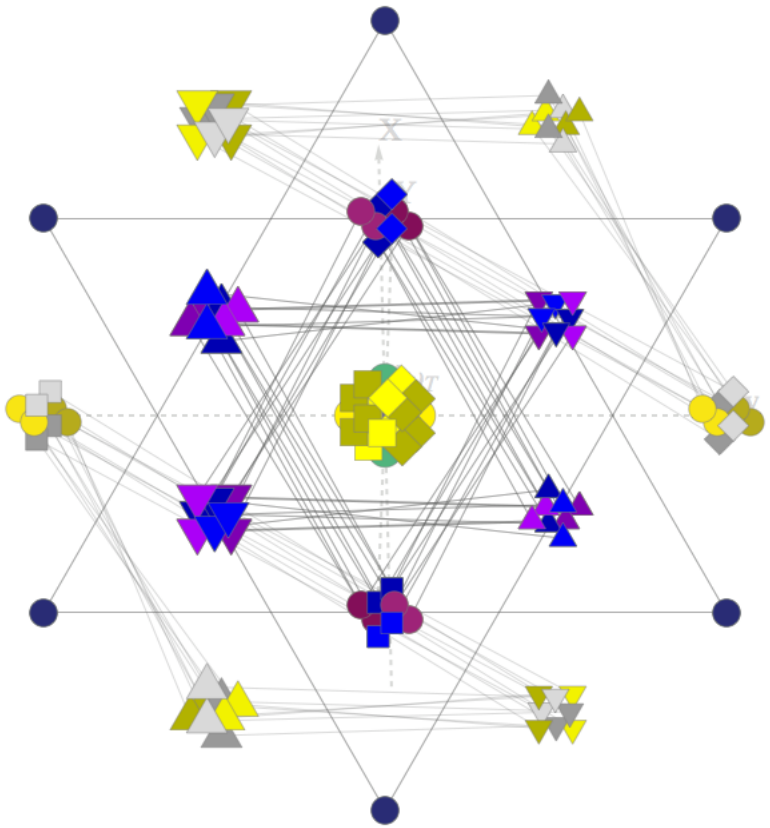
$$\begin{aligned}
\Gamma_1 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \otimes \sigma_1 \\
\Gamma_2 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\Gamma_3 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \otimes \sigma_3 \\
\Gamma_4 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \\
\Gamma_5 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes 1 \otimes 1 \\
\Gamma_6 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes 1 \otimes 1 \\
\Gamma_7 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1 \otimes 1 \\
\Gamma_8 &= i\sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \\
\Gamma_9 &= i\sigma_1 \otimes 1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{10} &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1 \otimes 1 \\
\Gamma_{11} &= i\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{12} &= i\sigma_1 \otimes 1 \otimes \sigma_2 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{13} &= i\sigma_1 \otimes \sigma_1 \otimes 1 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{14} &= i\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{15} &= i\sigma_1 \otimes \sigma_3 \otimes 1 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_{16} &= i\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1
\end{aligned}$$

120 generators in $spin(4,12)$
- 91 in $spin(3,11)$
  - 6 for ω in $spin(3,1)$
  - 45 in $spin(10)$
    - 12 for W, B, g
    - 3 for W', Z'
    - 30 for colored X bosons
  - 40 for $e\phi$ frame (4) × Higgs (10)
- 1 for Peccei-Quinn w in $spin(1,1)$
- 8 for $e\theta$ "axion" frame (4) × Higgs (2)
- 20 for more X bosons

128 generators in $128_S^{+\mathbb{R}}$ of $spin(4,12)$
- 64 for SM fermions in $64_S^{+\mathbb{R}}$ of $spin(3,11)$
- 64 for "mirror" fermions, with opposite w

Elementary Particle Explorer

E8 triality



Generations

Three generations of fermions in three copies of $64^{+\mathbb{R}}_S$ of $spin(3,11)$, differing only in mass.

Triality?

Maps between three blocks of 64 in E8.

No, these blocks have different quantum numbers. Doesn't seem viable.

Larger Lie group or supergroup?

Orthosymplectic, $D(7,3)$ or ?

Larger algebra?

E9. Possible relation to QFT.

Leech lattice. Three E8's as inner shell.

Kac-Moody algebras.

Axions?

Use Peccei-Quinn charge, w, in E8 (and E6) and scalars in E8.

Fermions of different generations as axion + fermion composites.

Used successfully in the past for solving the strong CP problem, and dealing with mirror fermions.

E8 appears to come with a nice Axion model building kit.

Something weirder?

E8 Theory summary

Superconnection:

$$\underset{=\cdot}{A} = \left(\tfrac{1}{2}\underset{-}{\omega} + \tfrac{1}{4}\underset{-}{e}\,\phi + \underset{-}{G}\right) + \underset{\cdot}{\psi}$$
$$\in H + K$$
$$\subset \left(spin(3,1) + 4\times(2+\bar{2}) + su(2)_L + u(1)_Y + su(3)\right) + 2\times(2_L+2_R)\times(1+3)$$
$$\subset \left(spin(3,1) + 4\times 10 + spin(10)\right) + 2\times 16_S^{+\mathbb{C}}$$
$$\subset spin(3,11) + 64_S^{+\mathbb{R}} \subset spin(4,12) + 128_S^{+\mathbb{R}} \subset E8$$

Curvature:

$$\underset{=\cdot}{F} = \underset{-}{d}\underset{-\cdot}{A} + \underset{-\cdot}{A}\underset{-\cdot}{A} = \underset{=}{F}^H + \underset{-}{D}\underset{\cdot}{\psi} + \underset{\cdot}{\psi}\underset{\cdot}{\psi}$$

Action:

$$S = \int \left\langle \left(\underset{\equiv}{\dot{B}} + \underset{=}{B}\right)\underset{=\cdot}{F} + \underset{\sim}{V}(B^H) \right\rangle$$

Generations? Axions? $spin(1,1)_{PQ}$, $\theta \underset{=}{F}\underset{=}{F}$, $\left\langle \bar{\psi}\,\underset{-}{e}\,\theta\,\underset{-}{e}\,\theta\,\underset{-}{e}\,\theta\,\epsilon\,\underset{-}{D}\underset{\cdot}{\psi} \right\rangle$?

Geometric interpretation of the superconnection? BRST? $\underset{\cdot}{\delta}\underset{-}{K} = -\underset{-}{D}\underset{\cdot}{\psi}$ TQFT?

Precise symmetry breaking mechanism? $\underset{\sim}{V}(B^H) = \underset{=}{B}\,\underset{=}{\overset{\rightrightarrows}{\Phi}}\,\underset{=}{B} + \ldots$?

Quantization? Asymptoticly safe R.G. flow of $\Lambda, G, g, \ldots$? Spinfoams?

Maximilian Tortoise