COMPANION TO NOTES FOR ATLAS WORKSHOP

ABSTRACT. This is a companion to the notes written by Prof. Jeff Adams for the Atlas workshop. We have worked out all of the exercises suggested in those notes.

CONTENTS

1. REPRESENTATIONS OF $SL(2,\mathbb{R})$

Exercise 1.0.2. Let

$$E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Show that H, E and F satisfy the familiar identities $[H,E] = 2E$, $[H,F] = -2F$ and $[E,F] = H$. Also, iH is a basis of $\mathfrak{k}_\circ \subset \mathfrak{g}_\circ$.

Solution: We only check $[E,F] = H$ the others are similar:

$$\begin{aligned} [E,F] &= EF - FE \\ &= \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \\ &= \frac{1}{4}\left\{\begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix} - \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}\right\} \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = H. \end{aligned}$$

To show that iH forms a basis for $\mathfrak{k}_\circ$, note that $\mathfrak{k}_\circ \cong i\mathbb{R}$ by differentiating the following isomorphism of $SO(2)$ and S^1

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mapsto e^{i\theta}.$$

Therefore $\mathfrak{k}_o$ has i as its basis element. Now,

$$iH = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Therefore, under the isomorphism above, $iH \mapsto i$, therefore, $\{iH\}$ forms a basis for $\mathfrak{k}_o$.

Exercise 1.0.5. Verify that $V(\lambda, \epsilon)$ is a $(\mathfrak{g}, K)$ - module.

Solution: We need to verify the following

(1) This is a representation of $\mathfrak{g}$ and K.
(2) Every vector is K- finite : $\dim\langle \pi(k)v | k \in K\rangle < \infty$ for all $v \in V$.
(3) The representation of $\mathfrak{g}$, restricted to $\mathfrak{k}_o$, is the differential of the representation of K.

Recall that $V(\lambda, \epsilon)$ is defined as $\text{Span}\{v_j | j \in \epsilon + 2\mathbb{Z}\}$ with the action of K given by

$$\pi(t(e^{i\theta}))(v_k) = e^{ik\theta} v_k \tag{1}$$

and the action of $\mathfrak{g}$ given by

$$\pi(H)v_j = jv_j \tag{2}$$

$$\pi(E)v_j = \frac{1}{2}(\lambda + (j+1))v_{j+2} \tag{3}$$

$$\pi(F)v_j = \frac{1}{2}(\lambda - (j-1))v_{j-2} \tag{4}$$

Therefore to check (a), we first check that π is a representation of $\mathfrak{g}$. It amounts to show that

$$[\pi(X), \pi(Y)] = \pi([X, Y])$$

for all $X, Y \in \mathfrak{g}$. We then only need to show that the above equation holds true for the basis $\{H, E, F\}$ of $\mathfrak{g}$. We work out one of the commutation relations, i.e. we show that

$$[\pi(E), \pi(F)]v_j = \pi([E, F]) = \pi(H)v_j \quad \text{for all } j$$

$$\begin{aligned}
[\pi(E),\pi(F)]v_j &= \pi(E)\pi(F)v_j - \pi(F)\pi(E)v_j \\
&= \frac{1}{2}(\lambda-(j-1))\pi(E)v_{j-2} - \frac{1}{2}(\lambda+(j+1))\pi(F)v_{j+2} \\
&= \frac{1}{4}(\lambda-(j-1))(\lambda+((j-2)+1))v_j - (\lambda+(j+1))(\lambda+((j+2)-1))v_j \\
&= \frac{1}{4}\{(\lambda-(j-1))(\lambda+(j-1)) - (\lambda+(j+1))(\lambda-(j+1))\}v_j \\
&= \frac{1}{4}\{\lambda^2-(j-1)^2-\lambda^2+(j+1)^2\}v_j \\
&= jv_j.
\end{aligned}$$

The other relations are similar.
To show that π defines a representation of K we only need to show the following

$$\begin{aligned}
\pi(t(e^{i\theta})\pi(t(e^{i\phi})v_k &= e^{ik\phi}\pi(t(e^{i\theta})v_k = e^{ik\phi}e^{ik\theta}v_k \\
&= e^{ik(\phi+\theta)}v_k \\
&= \pi(t(e^{i(\theta+\phi)}))v_k
\end{aligned}$$

To prove (b), we note that given any $v \in V$, $v = \sum_{j\in I} a_j v_j$ where I is a finite subset of $\epsilon + 2\mathbb{Z}$. Then we have $\pi(t(e^{i\theta}))v = \sum_{j\in I} e^{ij\theta}a_j v_j$. Therefore, if $W = \mathrm{Span}\{v_j | j \in I\}$, W is finite dimensional and $\langle \pi(k)v | k \in K\rangle \subset W$. Therefore, proving that $\langle \pi(k)v|k \in K\rangle$ is finite dimensional.

To prove (c), note that the differential of the action $\pi(t(e^{i\theta}))v_k = e^{ik\theta}v_k$ is precisely,

$$\frac{d}{d\theta}\pi(t(e^{i\theta}))|_{\theta=0}v_k = \frac{d}{d\theta}e^{ik\theta}|_{\theta=0}v_k = ikv_k = \pi(iH)v_k.$$

Exercise 1.0.10. (1) $V(\lambda,\epsilon)$ is reducible if and only if

$$\lambda \in \epsilon + 2\mathbb{Z} + 1.$$

(2) Suppose $\lambda = n \in \epsilon + 2\mathbb{Z} + 1$ with $n \geq 0$. Then $V(\lambda,\epsilon)$ has two infinite dimensional sub-representations

$$\begin{aligned}
V_+(\lambda,\epsilon) &= \langle v_{n+1}, v_{n+3}, \dots\rangle \\
V_-(\lambda,\epsilon) &= \langle v_{-n-1}, v_{-n-3}, \dots\rangle
\end{aligned}$$

Furthermore $\frac{V}{(V_+\oplus V_-)}$ is an irreducible finite dimensional representation $V_0(\lambda,\epsilon)$ of dimension n with basis $\{v_{-n+1}, v_{-n+3}, \dots, v_{n-3}, v_{n-1}\}$ (the image of these in the quotient).
In other words, there is an exact sequence of $(\mathfrak{g}, K)$- modules:

$$0 \to V_+(\lambda,\epsilon) \oplus V_-(\lambda,\epsilon) \to V(\lambda,\epsilon) \to V_0(\lambda,\epsilon) \to 0.$$

Furthermore,

$$\begin{aligned} V_{\pm}(\lambda,\epsilon) &\cong V_{\pm}(-\lambda,\epsilon) \\ V_0(\lambda,\epsilon) &\cong V_0(-\lambda,\epsilon) \end{aligned}$$

An important spcial case is $\epsilon = 1, \lambda = 0$, in which case $V = V_+ \oplus V_-$ and V_0 vanish.

(3) Suppose $\lambda = \epsilon + 2\mathbb{Z} + 1$, and $n \leq 0$. Then $V(\lambda,\epsilon)$ has a finite dimensional sub-representation $V_0(\lambda,\epsilon) = \langle v_{n+1}, v_{n+3}, \dots, v_{-n-1}\rangle$ of dimension n (vanishing if $n = 0$), and $\frac{V}{V_0}$ is the direct sum of two irreducible, infinite dimensional representations:

$$\begin{aligned} V_+(\lambda,\epsilon) &= \langle v_{-n+1}, v_{-n+3}, \dots\rangle \\ V_-(\lambda,\epsilon) &= \langle v_{n-1}, v_{n-3}, \dots\rangle \end{aligned}$$

In other words, there is an exact sequence

$$0 \to V_0(\lambda,\epsilon) \to V(\lambda,\epsilon) \to V_+(\lambda,\epsilon) \oplus V_-(\lambda,\epsilon) \to 0.$$

Solution:

(1) We first show that $V(\lambda,\epsilon)$ is irreducible if and only if $\pi(E)v_j \neq 0$ and $\pi(F)v_j \neq 0$ for all j.
Suppose $\pi(E)v_j = 0$ would mean that $\langle v_j - 2k | k > 0\rangle$ is an invariant subspace of $V(\lambda,\epsilon)$, therefore $V(\lambda,\epsilon)$ is reducible. Conversely, if $\pi(E)v_j \neq 0$ for all j then we see that $\langle \pi(E)v_j | j \in \epsilon + \mathbb{Z}\rangle = \langle v_j | j \in \epsilon + \mathbb{Z}\rangle = V(\lambda,\epsilon)$.
Now,

$$\begin{aligned} V(\lambda,\epsilon)\text{is reducible} &\Leftrightarrow \pi(E)v_j = 0 \text{ or } \pi(F)v_j = 0, \text{ for some } j \in \epsilon + 2\mathbb{Z} \\ &\Leftrightarrow \frac{1}{2}(\lambda + (j+1))v_{j+2} = 0 \text{ or } \frac{1}{2}(\lambda - (j-1))v_{j-2} = 0 \\ &\Leftrightarrow \frac{1}{2}(\lambda + (j+1)) = 0 \text{ or } \frac{1}{2}(\lambda - (j-1)) = 0 \text{ since } v_j \neq 0 \forall j \\ &\Leftrightarrow \lambda = j \pm 1, \text{ i.e. } \lambda \in \epsilon + 2\mathbb{Z} + 1. \end{aligned}$$

(2) Without loss of generality we may assume $\pi(E)v_j = \frac{1}{2}(\lambda + (j+1))v_{j+2} = 0$. Therefore, $\lambda = -j-1 = n$ and since $n \geq 0$ we must have $j \leq 0$. Also, $\pi(F)v_{-j} = \frac{1}{2}(\lambda - (-j-1)v_{-j+2} = 0$. Therefore,

$$\begin{aligned} V_+(\lambda,\epsilon) &= \langle \pi(E)v_{-j+2k} | k \geq 0\rangle = \langle v_{n+1}, v_{n+3}, \dots\rangle \\ V_-(\lambda,\epsilon) &= \langle \pi(F)v_{j-2k} | k \geq 0\rangle = \langle v_{-n-1}, v_{-n-3}, \dots\rangle \end{aligned}$$

The irreducibility of the above sub-representations is clear. Also, it is easy to see that $V_0(\lambda,\epsilon) = \frac{V}{(V_+ \oplus V_-)}$ is an irreducible finite dimensional representation of dimension n with basis $\{v_{-n+1}, v_{-n+3}, \ldots, v_{n-3}, v_{n-1}\}$. To prove the isomorphisms at the end of (2) we need to first prove (3).

(3) Without loss of generality we may assume $\pi(E)v_j = \frac{1}{2}(\lambda + (j+1))v_{j+2} = 0$. Therefore, $\lambda = -j-1 = n$, i.e. $-j = n+1$, and since $n \leq 0$ we must have $j \geq 0$. Also, $\pi(F)v_{-j} = \frac{1}{2}(\lambda - (-j-1)v_{-j+2} = 0$. Therefore,

$$V_0(\lambda,\epsilon) = \langle v_{-j}, v_{-j+2}, \ldots, v_{j-2}, v_j\rangle = \langle v_{n+1}, v_{n+3}, \ldots, v_{-n-1}\rangle$$

is an irreducible invariant subspace of $V(\lambda,\epsilon)$. The rest of the problem follows from this assertion.

Exercise 1.0.13. Let C be defined as $C = H^2 + 2EF + 2FE + 1$. Then show that $\pi(C)\pi(X) = \pi(X)\pi(C)$ for all $X \in \mathfrak{g}$. and that given λ, ϵ,

$$\pi(C)v = (\lambda^2 - 1)v, \quad \text{for all } v \in V(\lambda,\epsilon).$$

Solution:We only need to check the commutativity on the basis elements $\{H, E, F\}$: For example, we show that

$$\pi(C)\pi(E) = \pi(E)\pi(C).$$

In fact one can show that

$$CX = XC$$

for all $X \in \mathfrak{g}$.

$$\begin{aligned} CE - EC &= (H^2 + 2EF + 2FE + 1)E - E(H^2 + 2EF + 2FE + 1) \\ &= H(HE) + 2EFE + 2(FE)E - (EH)H - 2E(EF) - 2EFE \\ &= H([H,E] - EH) + 2([F,E] - EF)E - ([E,H] - HE)H - 2E([E,F] - FE) \\ &= H(2E - EH) + 2(-H - EFE)E - (-2E - HE)H - 2E(H - FE) \\ &= 0 \end{aligned}$$

We now compute

$$\begin{aligned} \pi(C)v_k &= \pi(H)^2 v_k + 2\pi(E)\pi(F)v_k + 2\pi(F)\pi(E)v_k + v_k \\ &= (k^2 + \frac{1}{2}(\lambda^2 - (k-1)^2) + \frac{1}{2}(\lambda^2 - (k+1)^2))v_k + v_k \\ &= (\lambda^2 - 1)v_k + v_k \\ &= \lambda^2 v_k \end{aligned}$$

Since the $\{v_k\}$ form a basis for $V(\lambda,\epsilon)$, we are done.

Exercise 1.0.15. Show that V^* is a $(\mathfrak{g}, K)$ - module. What is $V(\lambda, \epsilon)^*$? Include the cases when this module is reducible. What about $V_0(\lambda, \epsilon)^*$ and $V_\pm(\lambda, \epsilon)^*$?

Solution: The fact that $V(\lambda, \epsilon)^*$ is a $\mathfrak{g}$ and K representation is easy to check from the definition of the actions. What remains to check is the K finite condition, which follows from the definition of V^* (find elements in $\mathrm{Hom}(V, \mathbb{C})$ which are not K - finite).

To compute $V(\lambda, \epsilon)^*$ we do the following: Let $\{f_j | j \in \epsilon + 2\mathbb{Z}\}$ be the dual basis to $\{v_j | j \in \epsilon + 2\mathbb{Z}\}$.

$$\begin{aligned}
\pi^*(t(e^{i\theta}))(f_k)(v_j) &= f_k(\pi(t(e^{i\theta})^{-1})v_j) = e^{-ik\theta}\delta_{k,j} \\
\pi^*(H)(f_k)(v_j) &= f_k(-\pi(H)v_j) = -j\delta_{k,j} \\
\pi^*(E)(f_k)(v_j) &= f_k(-\pi(E)v_j) = f_k(-\frac{1}{2}(\lambda + (j+1))v_{j+2}) = \frac{1}{2}(-\lambda - (j+1))\delta_{k,j+2} \\
\pi^*(F)(f_k)(v_j) &= f_k(-\pi(F)v_j) = f_k(-\frac{1}{2}(\lambda - (j-1))v_{j+2}) = \frac{1}{2}(-\lambda + (j-1))\delta_{k,j-2}.
\end{aligned}$$

So that

$$\begin{aligned}
\pi^*(t(e^{i\theta}))(f_k) &= e^{-ik\theta}f_k \\
\pi^*(H)(f_k) &= -kf_k \\
\pi^*(E)(f_k) &= \frac{1}{2}(-\lambda - (k-1))f_{k-2} \\
\pi^*(F)(f_k) &= \frac{1}{2}(-\lambda + (k+1))f_{k+2}.
\end{aligned}$$

Let $\tau : (\mathfrak{g}, K) \longrightarrow (\mathfrak{g}, K)$ be the map defined by

$$\tau(k) = k^{-1} \quad \forall k \in K$$

$$\tau(H) = -H, \quad \tau(E) = F, \quad \tau(F) = E$$

We can now define an isomorphism $\Phi : (\pi, V(-\lambda, \epsilon)) \longrightarrow V(\lambda, \epsilon)^*$ as follows:

$$\Phi(v_k) = f_{-k} \quad \forall k \in \epsilon + 2\mathbb{Z}.$$

We only need to show that this is an intertwining operator.
Now,

$$\begin{aligned}
\pi^*(t(e^{i\theta}))(\Phi(v_k)) &= \pi^*(t(e^{i\theta}))(f_{-k}) = e^{ik\theta}f_{-k} = e^{ik\theta}\Phi(v_k) = \Phi(e^{ik\theta}v_k) \\
&= \Phi(\pi(t(e^{i\theta}))v_k)
\end{aligned}$$

Therefore, Φ intertwines the action on K.

Also,

$$\begin{aligned}\pi^*(H)(\Phi(v_k)) &= \pi^*(H)(f_{-k}) = kf_{-k} = k\phi(v_k) \\ &= \phi(\pi(H)v_k)\end{aligned}$$

$$\begin{aligned}\pi^*(E)(\Phi(v_k)) &= \pi^*(E)(f_{-k}) = \frac{1}{2}(-\lambda-(-k-1))f_{-k-2} = \frac{1}{2}(-\lambda+(k+1))f_{-(k+2)} \\ &= \frac{1}{2}(-\lambda+(k+1))\Phi(v_{k+2}) = \Phi(\pi(E)v_k)\end{aligned}$$

Similarly we have,

$$\begin{aligned}\pi^*(F)(\Phi(v_k)) &= \pi^*(F)(f_{-k}) = \frac{1}{2}(-\lambda+(-k+1))f_{-k+2} = \frac{1}{2}(-\lambda-(k-1))f_{-(k-2)} \\ &= \frac{1}{2}(-\lambda-(k-1))\Phi(v_{k-2}) = \Phi(\pi(F)v_k)\end{aligned}$$

We have hence shown that Φ intertwines the $\mathfrak{g}$ action too. We are done.

2. Hermitian Forms

Exercise 2.0.3. Show that the invariance condition is equivalent to

$$\begin{aligned}(\pi(k)v, \pi(k)w) &= (v, w) \quad \forall k \in K \\ (\pi(E)v, w) &= -(v, \pi(F)w) \\ (\pi(F)v, w) &= -(v, \pi(E)w) \\ (\pi(H)v, w) &= (v, \pi(H)w)\end{aligned} \tag{5}$$

Solution: We need to show (5) and

$$\begin{gathered}(\pi(k)v, \pi(k)w) = (v, w) \quad \forall k \in K \\ (\pi(X)v, w) + (v, \pi(\sigma(X)w) = 0 \quad \forall X \in \mathfrak{g}\end{gathered} \tag{6}$$

are equivalent. But this follows from noting that σ is the complex conjugation map, and when applied to E, F and H gives the following identities:

$$\sigma(E) = F, \quad \sigma(F) = E, \quad \sigma(H) = -H.$$

Exercise 2.0.5. Suppose $(,)$ is an invariant Hermitian form on $V(\lambda,\epsilon)$. Then $(v_j,v_k)=0$ for $j\neq k$.

Solution: We have

$$\begin{aligned}(v_j,v_k) &= \frac{1}{j}(\pi(H)v_j,v_k)=\frac{1}{j}(v_j,\pi(H)v_k)\\ &= \frac{k}{j}(v_j,v_k)\end{aligned}$$

which shows that if $(v_j,v_k)\neq 0$ we must have $j=k$.

Exercise 2.0.6. Suppose $V(\lambda,\epsilon)$ is irreducible and $V(\lambda,\epsilon)$ has an invariant form. Show that we must have

$$(v_{j+2},v_{j+2})=\frac{(-\overline{\lambda}-(j+1))}{(\lambda+(j+1))}(v_j,v_j)\quad\forall j.$$

Solution: For $j\in\epsilon+2\mathbb{Z}$,

$$\begin{aligned}(v_{j+2},v_{j+2}) &= \left(\frac{2}{\lambda+j+1}\pi(E)v_j,v_{j+2}\right)=\frac{-2}{(\lambda+j+1)}(v_j,\pi(F)v_{j+2})\\ &= \frac{-2}{(\lambda+j+1)}\left(v_j,\frac{\lambda-(j+1)}{2}v_j\right)\\ &= \frac{(-\overline{\lambda}+(j+1))}{(\lambda+(j+1))}(v_j,v_j).\end{aligned}$$

Note that since $V(\lambda,\epsilon)$ is irreducible, $\lambda\notin 1+\epsilon+2\mathbb{Z}$ and hence $\lambda\neq -j-1$ for $j\in\epsilon+2\mathbb{Z}$. Therefore we are done.

Exercise 2.0.9. Suppose $V(\lambda,\epsilon)$ is irreducible, and $V(\lambda,\epsilon)$ has a c-invariant form. Show that for all j:

$$(v_{j+2},v_{j+2})_c=\frac{(\overline{\lambda}-(j+1))}{(\lambda+(j+1))}(v_j,v_j)_c. \tag{7}$$

Conclude that if $\lambda\in\mathbb{R}$ then $V(\lambda,\epsilon)$ supports a unique c-invariant Hermitian form normalized so that $(v_0,v_0)=1\quad(\epsilon=0)$ or $(v_{-1},v_{-1})_c=1\quad(\epsilon=1)$.

Solution: Note that

$$\sigma_c(H)=-H\quad\sigma_c(E)=-F\quad\sigma_c(F)=-E.$$

Thus we have a similar version of invariance condition for the c-invariant

Hermitian form as in Exercise (2.0.3):

$$
\begin{aligned}
(\pi(E)v, w)_c &= (v, \pi(F)w)_c \\
(\pi(F)v, w)_c &= (v, \pi(E)w)_c \\
(\pi(H)v, w)_c &= -(v, \pi(H)w)_c
\end{aligned} \tag{8}
$$

for all $v, w \in V$. So that,

$$
\begin{aligned}
(v_{j+2}, v_{j+2})_c &= \left(\frac{2}{\lambda + j + 1}\pi(E)v_j, v_{j+2}\right)_c = \frac{2}{(\lambda + j + 1)}(v_j, \pi(F)v_{j+2})_c \\
&= \frac{2}{(\lambda + j + 1)}\left(v_j, \frac{\lambda - (j+1)}{2}v_j\right)_c \\
&= \frac{(\overline{\lambda} - (j+1))}{(\lambda + (j+1))}(v_j, v_j)_c.
\end{aligned}
$$

Note that the denominator is undefined when $\lambda = -(j+1) \in 1+\epsilon+2\mathbb{Z}$, which case is not allowed since $V(\lambda, \epsilon)$ is irreducible. Now suppose $\lambda \notin \mathbb{Z}$. If $\lambda \in \mathbb{R}$ we have $\overline{\lambda} = \lambda$ and we can define

$$
(v_0, v_0)_c = 1 \quad \text{if } \epsilon = 0, \quad (v_{-1}, v_{-1})_c = 1 \quad \text{if } \epsilon = 1.
$$

and using Equation (7) we can inductively define $(v_j, v_j)_c \in \mathbb{R}$ assuming $(v_0, v_0)_c = 1$ or $(v_{-1}, v_{-1})_c = 1$ depending whether $\epsilon = 0$ or 1, and $(v_i, v_j) = 0$ if $i \neq j$. What remains to be shown is that the form defined this way is invariant. It is enough to show on the basis elements. We work out one of the cases, the others are similar.

$$
\begin{aligned}
(\pi(E)v_j, v_j)_c + (v_j, \pi(\sigma_c(E))v_j)_c &= (\pi(E)v_j, v_j)_c + (v_j, \pi(-F)v_j)_c \\
&= (\pi(E)v_j, v_j)_c - (v_j, \pi(F)v_j)_c \\
&= \frac{1}{2}(\lambda + (j+1))(v_{j+2}, v_j)_c + \frac{1}{2}(-\lambda + (j+1))(v_j, v_{j-2})_c \\
&= 0.
\end{aligned}
$$

Exercise 2.0.11. Show that the n-dimensional irreducible representation has a positive definite c-invariant Hermitian form. It supports an invariant Hermitian form, which is not positive definite unless $n = 1$.

Solution: We recall that any finite dimensional irreducible representation is of the form $V_0(-n, \epsilon)$, $n > 0$, with basis $\{v_{-n+1}, v_{-n+3}, \ldots, v_{n-3}, v_{n-1}\}$. From previous exercise, we know that if $\epsilon = 0$, we can set $(v_0, v_0)_c = 1$ and we have $\dfrac{-n-j-1}{-n+j+1} > 0$ if and only if $-n+1 \leq j < n-1$ and hence $(v_j, v_j)_c > 0$ for all j. Therefore, we have a positive definite c-invariant

Hermitian form. Similar argument holds for the case $\epsilon = 1$ by setting $(v_1, v_1)_c = 1$. However, from the formula for invariant Hermitian form, we must have

$$(v_{j+2}, v_{j+2}) = \frac{n-(j+1)}{-n+j+1}(v_j, v_j).$$

Note that $\frac{n-(j+1)}{-n+j+1} = \frac{n-j-1}{-n+j+1} = -1$ for all $j \neq -n-1$. Therefore, such Hermitian invariant form cannot be positive definite unless $n = 1$.

Exercise 2.0.12. Suppose $\lambda > 0$ and $\lambda \notin \mathbb{Z}$,

(1) Show that $V(\lambda, 1)$ has an invariant Hermitian form, which is not positive definite.
(2) Show that $V(\lambda, 0)$ has an invariant Hermitian form, which is positive definite if and only if $\lambda < 1$.

The representations in (2) are the complimentary series for $SL(2, \mathbb{R})$.

Solution:

(1) If $\lambda \in \mathbb{R}^+$ and $\lambda \notin \mathbb{Z}$ then $V(\lambda, 1)$ has an invariant Hermitian form because we can set $(v_1, v_1) = 1$ and use the formula

$$(v_{j+2}, v_{j+2}) = \frac{-\lambda+j+1}{\lambda+j+1}(v_j, v_j)$$

to define the form for all j. We now have an invariant Hermitian form, but is not positive definite since $(v_{-1}, v_{-1}) = -1$.

(2) If we set $(v_0, v_0) = 1$ then

$$(v_{-2}, v_{-2}) = \frac{\lambda-1}{-\lambda-1}(v_0, v_0) = \frac{\lambda-1}{-\lambda-1}.$$

Note that $\frac{\lambda-1}{-\lambda-1} > 0$ if and only if $-1 < \lambda < 1$, but from assumption only positive λ is allowed, so we conclude that $V(\lambda, 0)$ has a positive invariant Hermitian form if and only if $\lambda < 1$.

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Exercise 3.0.2. Try proving this result. It is equivalent to proving that any involution in $GL(2, \mathbb{Z})$ is conjugate to a matrix with diagonal entries 1, -1 or 2×2 matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Exercise 3.0.5. Prove that there is a natural isomorphism

$$\widehat{T(\mathbb{R})} \cong X^*/(1-\theta)X^*.$$

Solution: Define the map $\Phi : X^*(\mathbb{C}) \longrightarrow \widehat{H(\mathbb{C})^\theta}$ by,

$$\Phi(\alpha) = \alpha|_{H(\mathbb{C})^\theta} \quad \alpha \in X^*.$$

We show that $\mathrm{Ker}(\Phi) = (1-\theta)X^*$.
For, $\alpha \in (1-\theta)X^*$ we have $\alpha = (1-\theta)\beta$ for some $\beta \in X^*$. Therefore, for $h \in H(\mathbb{C})^\theta$,

$$\Phi(\alpha)(h) = (1-\theta)\beta(h) = \beta((1-\theta)(h)) = \beta(h - \theta(h)) = 0.$$

Hence, $(1-\theta)X^* \subset \mathrm{Ker}(\Phi)$.
Conversely, suppose $\alpha \in \mathrm{Ker}(\Phi)$. Any $h \in H(\mathbb{C})^\theta$ is of the form

$$h = (z_1, \dots, z_a, \pm 1, \dots, \pm 1, u_1, u_1, \dots, u_c, u_c),$$

and since α is trivial on $H(\mathbb{C})^\theta$ we must have $\alpha = (0, \dots, 0, 2k_1, \dots, 2k_b, p_1, -p_1, \dots, p_c, -p_c)$. We need to show that

$$\alpha = (1-\theta)\beta, \tag{9}$$

for some $\beta \in X^*$. Any element of $H(\mathbb{C})$ is of the form

$$h = (z_1, \dots, z_a, w_1, \dots, w_b, v_1, u_1, \dots, v_c, u_c),$$

and β is determined as

$$\beta(h) = (z_1^{n_1}, \dots, z_a^{n_a}, w_1^{r_1}, \dots, w_b^{r_b}, v_1^{s_1}, u_1^{t_1}, \dots, v_c^{s_c}, u_c^{t_c}),$$

so that

$$\beta = (n_1, \dots, n_a, r_1, \dots, r_b, s_1, t_1, \dots, s_c, t_c).$$

where $n_i, r_i, s_i, t_i \in \mathbb{Z}$ are integers to be determined. We compute the following:

$$\begin{aligned}
(1-\theta)\beta(h) &= \beta((1-\theta)h) \\
&= \beta(z_1 z_1^{-1}, \dots, z_a z_a^{-1}, w_1 w_1, \dots, w_b w_b, v_1 u_1^{-1}, u_1 v_1^{-1}, \dots, v_c u_c^{-1}, u_c v_c^{-1}) \\
&= \left(1, \dots, 1, w_1^{2r_1}, \dots, w_b^{2r_b}, \left(\frac{v_1}{u_1}\right)^{s_1}, \left(\frac{u_1}{v_1}\right)^{t_1}, \dots, \left(\frac{v_c}{u_c}\right)^{s_c}, \left(\frac{u_c}{v_c}\right)^{t_c}\right) \\
&= \left(1, \dots, 1, w_1^{2r_1}, \dots, w_b^{2r_b}, \left(\frac{v_1}{u_1}\right)^{s_1}, \left(\frac{v_1}{u_1}\right)^{-t_1}, \dots, \left(\frac{v_c}{u_c}\right)^{s_c}, \left(\frac{v_c}{u_c}\right)^{-t_c}\right),
\end{aligned}$$

so that

$$(1-\theta)\beta = (0, \dots, 0, 2r_1, \dots, 2r_b, s_1, -t_1, \dots, s_c, -t_c).$$

Therefore, to get $\alpha = (1-\theta)\beta$ we can take $r_i = k_i$, $i = 1, \dots, b$ and $s_j = t_j = p_j$, $j = 1, \dots, c$.
Hence $\mathrm{Ker}(\Phi) \subset (1-\theta)X^*$, completing the proof that

$$\mathrm{Ker}(\Phi) = (1-\theta)X^*.$$

Since we are allowed to assume that any algebraic character of $H(\mathbb{C})^\theta$ is the restriction of a character of $H(\mathbb{C})$, we note that Φ is surjective. Therefore, by the second isomorphism theorem we have the result of the exercise.

Exercise 3.0.7. Show that the irreducible $(\mathfrak{h}, H(\mathbb{C})^\theta)$ modules are parameterized by pairs (λ, κ) satisfying:

(1) $\lambda \in X^* \otimes \mathbb{C} \cong \mathfrak{h}^*$;
(2) $\kappa \in \frac{X^*}{(1-\theta)X^*}$;
(3) $(1+\theta)\lambda = (1+\theta)\kappa$.

The corresponding character Λ of $H(\mathbb{R})$ is the differential λ and the restriction of Λ to $H(\mathbb{R})^\theta$ is the restriction of κ.

Solution: The irreducible $(\mathfrak{h}, H(\mathbb{C})^\theta)$- modules are given as a pair $(\Lambda, \mathbb{C})$ such that

(a) Λ is a $\mathfrak{h}$ representation, we denote its action on $\mathfrak{h}$ by λ.
(b) Λ is a $H(\mathbb{C})^\theta$ - representation, we denote its action on $H(\mathbb{C})^\theta$ by κ.
(c) $d\kappa = \lambda$ on $\mathrm{Lie}(H(\mathbb{C})^\theta)$.

From above we have $\lambda \in \mathfrak{h}^*$ and $\kappa \in \widehat{H(\mathbb{C})^\theta} \cong \frac{X^*}{(1-\theta)X^*}$. We have therefore found a choice of (λ, κ) which satisfies conditions (1) and (2) of the exercise. We only need to show that (λ, κ) gives rise to a $(\mathfrak{h}, H(\mathbb{C})^\theta)$ module if and only if condition (3) of the exercise holds. That is we need to show that

$$d\kappa = \lambda \text{ on } \mathrm{Lie}(H(\mathbb{C})^\theta) \Leftrightarrow (1+\theta)\lambda = (1+\theta)\kappa \text{ on } \mathfrak{h}.$$

We first note that

$$\mathfrak{h}^\theta = (1+\theta)\mathfrak{h}. \tag{10}$$

If $h = (1+\theta)h_1 = h_1 + \theta(h_1)$, then $\theta(h) = \theta(h_1) + \theta^2(h_1) = \theta(h_1) + h_1 = h$. Hence, $(1+\theta)\mathfrak{h} \subset \mathfrak{h}^\theta$.
Conversely, if $h \in \mathfrak{h}^\theta$, we can take $h_1 = \frac{h}{2}$, so that $h = (1+\theta)(h_1)$. Therefore, $\mathfrak{h}^\theta \subset (1+\theta)\mathfrak{h}$, proving the equality in (10).

Now,

$$\begin{aligned} d\kappa = \lambda \text{ on } \mathrm{Lie}(H(\mathbb{C})^\theta) &\Leftrightarrow d\kappa(h) = \lambda(h) \quad \forall h \in \mathfrak{h}^\theta = (1+\theta)\mathfrak{h}, \\ &\Leftrightarrow d\kappa(h_1 + \theta(h_1)) = \lambda(h_1 + \theta(h_1)) \quad h_1 \in \mathfrak{h}, \\ &\Leftrightarrow d\kappa((1+\theta)(h_1)) = \lambda((1+\theta)(h_1)) \quad h_1 \in \mathfrak{h}, \\ &\Leftrightarrow (1+\theta)\lambda(h_1) = (1+\theta)\kappa(h_1) \quad h_1 \in \mathfrak{h}, \\ &\Leftrightarrow (1+\theta)\lambda = (1+\theta)\kappa \text{ on } \mathfrak{h}. \end{aligned}$$

This completes the solution.

Exercise 3.0.8. Work out the case of $H(\mathbb{R}) \cong \mathbb{C}^*$. In this case $H(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$, and $\sigma(z,w) = (\overline{w}^{-1}, \overline{z}^{-1})$, and $\theta(z,w) = (w,z)$.

Solution: We have $H(\mathbb{R}) = \mathbb{C}^*$, hence $H(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$. We have $H(\mathbb{C})^\theta = \{(z,z) | z \in \mathbb{C}^*\} \cong C^*$ and $H(\mathbb{C})^\sigma = \{(z, \frac{1}{z}) | z \neq 0\} \cong \mathbb{C}^*$. Also, $\mathfrak{h}^* \cong \mathbb{C} \times \mathbb{C}$.

We find a parameter set for $\frac{X^*}{(1-\theta)X^*}$. For $(n_1, m_1), (n_2, m_2) \in X^*$, we have

$$\begin{aligned}
(n_1,m_1) \sim (n_2,m_2) \quad &\Leftrightarrow \quad (n_1 - n_2, m_1 - m_2) \in (1-\theta)X^* \\
&\Leftrightarrow \quad (n_1 - n_2, m_1 - m_2) = (1-\theta)(n_3, m_3) \\
&\Leftrightarrow \quad (n_1 - n_2, m_1 - m_2) = (n_3 - m_3, m_3 - n_3) \\
&\Leftrightarrow \quad (n_1 - n_2, m_1 - m_2) = (l, -l) \text{ for some } l \in \mathbb{Z}
\end{aligned}$$

Therefore,

$$(n,m) \equiv (n+l, m-l) \quad \text{for any } l \in \mathbb{Z}.$$

Therefore a set of representatives for $\frac{X^*}{(1-\theta)X^*}$ is given by

$$\{(n,0) | n \in \mathbb{Z}\}.$$

is a set of representatives of $\frac{X^*}{(1-\theta)X^*}$. Therefore, for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C} \times \mathbb{C}$ and $\kappa \in \{(n,1) | n \in \mathbb{Z}\}$ we compute:

$$\begin{aligned}
(1+\theta)\lambda(z,w) &= \lambda((1+\theta)(z,w)) = \lambda((z,w) + (w,z)) \\
&= \lambda_1(z+w) + \lambda_2(z+w) = (\lambda_1 + \lambda_2)(z+w).
\end{aligned}$$

Also,

$$\begin{aligned}
(1+\theta)\kappa(z,w) &= \kappa(z+w, z+w) \\
&= n(z+w)
\end{aligned}$$

Therefore, characters are parameterized by (λ, κ), such that $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C} \times \mathbb{C}$ and $\kappa \in \{(n,0) | n \in \mathbb{Z}\}$ such that

$$\lambda_1 + \lambda_2 = n.$$

Note: This parameterization agrees with the familiar parameterization of $\widehat{\mathbb{C}^*}$ given by (ν, k) where

$$\chi_{\nu,k}(re^{i\theta}) = r^\nu e^{ik\theta}.$$

Our choice of parameter $((\lambda_1,\lambda_2),k)$ gives $\chi_{\nu,k}$ by defining

$$\begin{aligned}\psi_{((\lambda_1,\lambda_2),k)}(re^{i\theta}) &= (re^{i\theta})^{\lambda_1}(\overline{re^{i\theta}})^{-\lambda_2} = r^{(\lambda_1-\lambda_2)}e^{i(\lambda_1+\lambda_2)\theta}\\ &= r^{\nu}e^{ik\theta},\end{aligned}$$

where, $\lambda_1-\lambda_2=\nu$ and $\lambda_1+\lambda_2=k$.

3.1. **Covers of Tori.**

Exercise 3.1.4. The genuine characters of $H(\mathbb{R})_\gamma$ are canonically parameterized by the set of pairs (λ,κ) with $\lambda\in\mathfrak{h}^*$ and $\kappa\in\gamma+\frac{X^*}{(1-\theta)X^*}$, and satisfying $(1+\theta)\lambda=(1+\theta)\kappa$.

Solution: By Exercise (3.0.7) we know that $\widehat{H(\mathbb{R})_\gamma}$ is parameterized by (λ,κ) where $\lambda\in\mathfrak{h}^*$ and $\kappa\in\frac{X^*(H_\gamma)}{(1-\theta)X^*(H_\gamma)}$ satisfying $(1+\theta)\lambda=(1+\theta)\kappa$. Therefore, starting with a genuine character Λ of $H(\mathbb{R})_\gamma$ we get (λ,κ) as above, where κ is a genuine character of $T(\mathbb{R})_\gamma=H(\mathbb{R})^\theta_\gamma$. The goal is to show that $\gamma+\frac{X^*}{(1-\theta)X^*}$ parametrizes the genuine characters of $T(\mathbb{R})_\gamma$. Let $\zeta=(1,-1)$ be the nontrivial element in H_γ First note that for $\gamma+\kappa'\in\gamma+\frac{X^*}{(1-\theta)X^*}$, with κ' a character of $T(\mathbb{R})$, $(\gamma+\kappa')(\zeta)=\gamma(\zeta)\cdot 1=-1$ since γ is a genuine character of H_γ. Thus, $\gamma+\kappa'$ is a genuine character of $T(\mathbb{R})_\gamma$. Conversely, we need to show that every genuine character κ of $T(\mathbb{R})_\gamma$ is in $\gamma+\frac{X^*}{(1-\theta)X^*}$. But that is equivalent to showing that $\kappa-\gamma\in\frac{X^*}{(1-\theta)X^*}$, i.e. just need to claim that $\kappa-\gamma$ is a nongenuine character of $T(\mathbb{R})_\gamma$ and hence it factors through $T(\mathbb{R})$. Recall that $\kappa=\Lambda|_{T(\mathbb{R})_\gamma}$ and hence $(\kappa-\gamma)(\zeta)=(\Lambda)(\zeta)(\gamma(\zeta))^{-1}=(-1)\cdot(-1)=1$, since both Λ and γ are genuine characters of H_γ. So we can conclude that $\kappa-\gamma\in\widehat{T(\mathbb{R})}=\frac{X^*}{(1-\theta)X^*}$. Therefore, $\kappa\in\gamma+\frac{X^*}{(1-\theta)X^*}$.

Exercise 3.1.5. Think through Example 3.2.5.

Solution: Let $G=SL(2,\mathbb{R})$.
First consider $H(\mathbb{R})=\left\{\begin{pmatrix}x&0\\0&1/x\end{pmatrix}:x\in\mathbb{R}\right\}$, the split Cartan subgroup of G. Note that for every $g=\begin{pmatrix}x&0\\0&1/x\end{pmatrix}\in H(\mathbb{R})$, $2\rho(g)=x^2>0$ and therefore $\rho(g)=\sqrt{x^2}$ is a well-defined character of $H(\mathbb{R})$. So we have that $H(\mathbb{R})_\rho\simeq H(\mathbb{R})\times\mathbb{Z}/2\mathbb{Z}$.

Now let $H(\mathbb{R})$ be the compact Cartan subgroup of G, then $H(\mathbb{R})\simeq\left\{\begin{pmatrix}e^{i\theta}&0\\0&-e^{i\theta}\end{pmatrix}:\theta\in[0,2\pi]\right\}\simeq\{e^{i\theta}\}=S^1$. Write every $g\in H(\mathbb{R})$ as

$g = e^{i\theta}$. Then $2\rho(e^{i\theta}) = e^{2i\theta}$ and we can define ρ such that $\rho(e^{i\theta}) = e^{i\theta}$ for all θ or $\rho(e^{i\theta}) = -e^{i\theta}$ for all θ, either of which is a well-defined character of $H(\mathbb{R}) \simeq S^1$. So $H(\mathbb{R})_\rho \simeq H(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$.

Next let $G = PSL(2,\mathbb{R}) = [SL(2,\mathbb{C})/\pm I]^\sigma$, where $\sigma(g) = \overline{g}$.
Note that the complex Cartan $H = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \Big/ \pm I : z \in \mathbb{C} \right\}$

First consider the anti-holomorphic involution $\sigma(g) = \overline{g}^{-1}$ of H. Then $H(\mathbb{R}) = H^\sigma = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & -e^{i\theta} \end{pmatrix} \Big/ \pm I \right\} \simeq \{e^{i\theta} : \theta \in [0,\pi)\} \simeq S^1$
$H(\mathbb{R})_\rho \simeq \{(e^{i\theta}, \pm e^{i\theta}) : \theta \in [0,\pi)\} = \{(e^{i\theta}, e^{i\theta}) : \theta \in [0,\pi)\} \cup \{(e^{i\theta}, e^{i(\theta+\pi)}) : \theta \in [0,\pi)\} \simeq \{e^{i\theta} : \theta \in [0,2\pi)\} = S^1$.
Thus, we can view $H(\mathbb{R})_\rho$ is a 2-fold cover of $H(\mathbb{R})$ via the covering map $z \to z^2$.

Next consider the anti-holomorphic involution $\sigma(g) = \overline{g}$ of H, then it's easy to check that
$$H(\mathbb{R}) = H^\sigma = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \Big/ \pm I : x \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix^{-1} \end{pmatrix} \Big/ \pm I : x \in \mathbb{R} \right\}$$

Therefore, we can write an typical element of $H(\mathbb{R})$ to be
$g = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$, $x > 0$, or $g = \begin{pmatrix} ix & 0 \\ 0 & -ix^{-1} \end{pmatrix}$, $x > 0$. So $H(\mathbb{R})$ is isomorphic to $\mathbb{R}^*$ by sending $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ to x and sending $\begin{pmatrix} ix & 0 \\ 0 & -ix^{-1} \end{pmatrix}$ to $-x$.
Since $2\rho \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = x^2$ and $2\rho \begin{pmatrix} ix & 0 \\ 0 & ix^{-1} \end{pmatrix} = -x^2$, we have $H(\mathbb{R})_\rho =$
$\left\{ \left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \pm x \right) : x > 0 \right\} \cup \left\{ \left(\begin{pmatrix} ix & 0 \\ 0 & -ix^{-1} \end{pmatrix}, \pm ix \right) : x > 0 \right\}$,
which is clearly isomorphic to $\mathbb{R}^* \cup i\mathbb{R}^*$ and $\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, i \right)$ is an element of order 4.

On the other hand, $H_\rho = \left\{ \left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \Big/ \pm I, \pm z \right) : z \in \mathbb{C} \right\}$ and hence $H_\rho(\mathbb{R}) = \left\{ \left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \Big/ \pm I, \pm z \right) : z \in \mathbb{C} \right\}^\sigma$
$$= \left\{ (\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \Big/ \pm I, x) : x \in \mathbb{R} \right\} \cup \left\{ \left(\begin{pmatrix} ix & 0 \\ 0 & -ix^{-1} \end{pmatrix} \Big/ \pm I, ix \right) : x \in \mathbb{R} \right\}$$

since $\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \Big/ \pm I, \pm x\right)$ define the same element in H_ρ^σ, and also $\left(\begin{pmatrix} ix & 0 \\ 0 & -ix^{-1} \end{pmatrix} \Big/ \pm I, \pm ix\right)$ define the same element in H_ρ^σ. Therefore we can write
$H_\rho(\mathbb{R}) = \left\{\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, x\right) : x > 0\right\} \cup \left\{\left(\begin{pmatrix} ix & 0 \\ 0 & -ix^{-1} \end{pmatrix}, ix\right) : x > 0\right\}$, which is clearly isomorphic to $\mathbb{R}^*$ and is a subgroup of index 2 in $H(\mathbb{R})_\rho$.

4. Cartan Subgroups

Exercise 4.0.6. Show that every semi simple element of $GL(2,\mathbb{R})$ is conjugate to either $\operatorname{diag}(x,y)$ or $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Conclude that $GL(2,\mathbb{R})$ has two Cartan subgroups (upto conjugacy), one $\mathbb{R}^*$ and the other $\mathbb{C}^*$.

Solution: Let $A \in GL(2,\mathbb{R})$ be a semi simple element. Since complex zeros of the characteristic polynomial occur in conjugate pairs, A either has both real or both complex eigen values.
In the case when there is one (and hence both) real eigen-value, A is conjugate (over $\mathbb{R}$) $\operatorname{diag}(x,y)$ for some $x,y \in \mathbb{R}^*$. If A has a complex eigen-value $a+ib$ with $b \neq 0$, we know that there is a complex eigen vector $(v+iw, u+it) \in \mathbb{C}^2$ with $wt \neq 0$. In this case, we see that

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} v+iw \\ u+it \end{pmatrix} = (a+ib)\begin{pmatrix} v+iw \\ u+it \end{pmatrix} = \begin{pmatrix} (a+ib)(v+iw) \\ (a+ib)(u+it) \end{pmatrix}$$

$$\text{i.e} \quad \begin{pmatrix} (pv+qu)+i(pw+qt) \\ (rv+su)+i(wr+st) \end{pmatrix} = \begin{pmatrix} (av-bw)+i(aw+bv) \\ (au-bt)+i(at+bu) \end{pmatrix}$$

Comparing real and imaginary parts we get

$$\begin{aligned} pv+qu &= av-bw, \\ pw+qt &= aw+bv, \\ rv+su &= au-bt, \\ wr+st &= at+bu, \end{aligned}$$

The above equations can be combined into the following matrix equation:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} v & w \\ u & t \end{pmatrix} = \begin{pmatrix} v & w \\ u & t \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

so that if $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $T = \begin{pmatrix} v & w \\ u & t \end{pmatrix}$, we have

$$\begin{pmatrix} v & w \\ u & t \end{pmatrix}^{-1} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} v & w \\ u & t \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Exercise 4.0.7. Find representatives of all conjugacy classes of Cartan subgroups in $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$.

Solution: Let $g \in GL(n,\mathbb{R})$ be an semisimple element with eigenvalues $x_1, \cdots, x_m, z_1, \overline{z_1}, \cdots, z_l, \overline{z_l}$, where $x_j \in \mathbb{R}^*$ and $z_j = a_j + ib_j \in \mathbb{C}^*$ with $a_j \neq 0$, and $n = m+2l$. Then g is conjugate to $\mathrm{diag}(x_1, \cdots, x_m, z_1, \overline{z_1}, \cdots, z_l, \overline{z_l})$ under $GL(n,\mathbb{C})$. By the previous Exercise,

$$g \sim \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & -b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_l & -b_l \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_l & a_l \end{pmatrix}$$

(conjugation over $GL(n,\mathbb{R})$). Therefore, such elements are the representatives of the Cartan subgroup $\mathbb{R}^{*m} \times \mathbb{C}^{*l}$, with $n = m + 2l$.

Next consider $SL(n,\mathbb{R})$.
First observe $SL(2,\mathbb{R})$. Let $g \in SL(2,\mathbb{R})$.
Suppose g has 2 real eigenvalues, then $g = P \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} P^{-1}$ where $P \in GL(2,R)$ and $x \in \mathbb{R}$. If $\det P > 0$ then take $P_1 = \frac{P}{\sqrt{\det P}} \in SL(2,\mathbb{R})$ and hence $g = P_1 \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} P_1^{-1}$. If $\det P < 0$. then interchanging two columns of P to get P_1 with $\det P_1 > 0$ and set $P_2 = \frac{P_1}{\sqrt{\det P_1}} \in SL(2,\mathbb{R})$, and thus $g = P_2 \begin{pmatrix} 1/x & 0 \\ 0 & x \end{pmatrix} P_2^{-1}$. Therefore $\{\begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}, x \in \mathbb{R}\}$ form the set of representatives of the Cartan subgroups $\mathbb{R}^*$.

Suppose g has two complex eigenvalues $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$. Pick a complex eigenvecor $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ corresponding to $\cos\theta +$

$i\sin\theta$, then $g = P\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} P^{-1}$, where $P = \begin{pmatrix} v_1 & -w_1 \\ v_2 & -w_2 \end{pmatrix} \in GL(2,\mathbb{R})$. If $\det P > 0$, then $g \sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ under $SL(2,\mathbb{R})$ by dividing P by $\sqrt{\det P}$. On the other hand, if $\det P < 0$, let $P_1 = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$, then $g = P_1\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} P_1^{-1}$ and hence $g \sim \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ under $SL(2,\mathbb{R})$.

First suppose n is odd. Every $g \in SL(n,\mathbb{R})$ is conjugate under $SL(n,\mathbb{R})$ to
$$\begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & -b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_l & -b_l \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_l & a_l \end{pmatrix},$$
where $x_1 = [x_2 \cdots x_m (a_1^2 + b_1^2) \cdots (a_l^2 + b_l^2)]^{-1}$, and such elements are representatives in the Cartan subgroup $\mathbb{R}^{*m-1} \times \mathbb{C}^{*l}$, where $m > 0$ and $m + 2l = n$.

Now suppose n is even. If $g \in SL(n,\mathbb{R})$ has at least a real eigenvalue, then g is conjugate to an element described as above, and they form the representatives of the Cartan subgroups $\mathbb{R}^{*m-1} \times \mathbb{C}^{*l}$, where $m > 0$ and $m + 2l = n$. If g has no real eigenvalues, then g is conjugate to
$$\begin{pmatrix} z_1 & 0 & 0 & 0 & 0 \\ 0 & \overline{z_1} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & z_l & 0 \\ 0 & 0 & 0 & 0 & \overline{z_l} \end{pmatrix}$$
under $GL(n,\mathbb{C})$, where $l = n/2$, with $\prod_j (z_j \overline{z_j}) = 1$.

Note that the set $\{(z_1, \overline{z_1}, \cdots, z_l, \overline{z_l}) : z_j \in \mathbb{C}, \prod_j (z_j \overline{z_j}) = 1\}$ is isomorphic to $\mathbb{C}^{*l-1} \times S^1$ as groups via the map $(z_1, \overline{z_1}, \cdots, z_l, \overline{z_l}) \rightarrow$

$(z_1, \cdots, z_{l-1}, z_1 z_2 \cdots z_l)$, and hence $g \sim \begin{pmatrix} a_1 & -b_1 & 0 & 0 & 0 \\ b_1 & a_1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & a_l & -b_l \\ 0 & 0 & 0 & b_l & a_l \end{pmatrix}$, where $z_j = a_j + ib_j$ and $\prod(z_j \overline{z_j}) = 1$, are the representatives for the Cartan subgroup $\mathbb{C}^{*l-1} \times S^1$.