Unitary representations of reductive Lie groups

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Outline

$SL(2, \mathbb{R})$

What's a (unitary) dual look like?

Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$).
Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$.
Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

This week: **Step 3** for reductive Lie group G.

Example: $SL(2,\mathbb{R})$ on the upper half plane

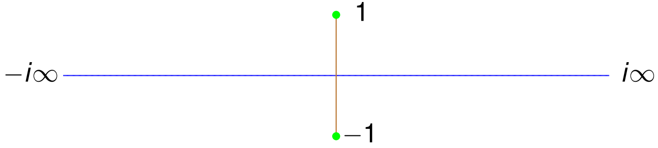
$SL(2,\mathbb{R})$ acts on upper half plane $\mathbb{H}$; $\Delta_{\mathbb{H}}$ = Laplacian.

$\rightsquigarrow$ repn $E(\nu)$ on $\nu^2 - 1$ eigenspace of Laplacian $\Delta_{\mathbb{H}}$

$\nu \in \mathbb{C}$ parametrizes line bdle on circle where bdry values live.

Most $E(\nu)$ irreducible; always unique irr subrep $J(\nu) \subset E(\nu)$.

Spherical reps for $SL(2,\mathbb{R}) \leftrightsquigarrow \mathbb{C}/\pm 1$



Spectrum of self-adjt $\Delta_{\mathbb{H}}$ on $L^2(\mathbb{H})$ is $(-\infty, -1]$. $\rightsquigarrow$
unitary principal series $\leftrightsquigarrow \{E(\nu) \mid \nu \in i\mathbb{R}\}$.

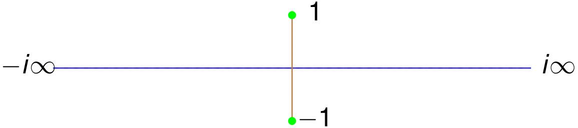
$E(\pm 1) =$ [harm fns on $\mathbb{H}$] $\supset$ [const fns on $\mathbb{H}$] $= J(\pm 1) =$ triv rep.

$J(\nu)$ is Herm. $\Leftrightarrow J(\nu) \simeq J(-\overline{\nu}) \Leftrightarrow \nu \in i\mathbb{R} \cup \mathbb{R}$.
By continuity, signature stays positive from 0 to ± 1.

complementary series reps $\leftrightsquigarrow \{E(t) \mid t \in (-1,1)\}$.

The moral[s] of the picture

Spherical unitary dual for $SL(2,\mathbb{R}) \leftrightsquigarrow \mathbb{C}/\pm 1$



$SL(2,\mathbb{R})$	$G(\mathbb{R})$
$E(\nu), \nu \in \mathbb{C}$	$I(\nu), \nu \in \mathfrak{a}^*_{\mathbb{C}}$
$E(\nu), \nu \in i\mathbb{R}$	$I(\nu), \nu \in i\mathfrak{a}^*_{\mathbb{R}}$
$J(\nu) \hookrightarrow E(\nu)$	$I(\nu) \twoheadrightarrow J(\nu)$
$[-1, 1]$	polytope in $\mathfrak{a}^*_{\mathbb{R}}$

Will deform Herm forms
unitary axis $i\mathfrak{a}^*_{\mathbb{R}} \rightsquigarrow$
real axis $\mathfrak{a}^*_{\mathbb{R}}$.

Deformed form pos $\rightsquigarrow$
unitary rep.

Reps appear in families, param by ν in cplx vec space $\mathfrak{a}^*$.

Pure imag params $\leftrightsquigarrow$ L^2 harm analysis $\leftrightsquigarrow$ unitary.

Each rep in family has distinguished irr piece $J(\nu)$.

Difficult unitary reps $\leftrightarrow$ deformation in real param

Principal series for $SL(2, \mathbb{R})$

Want to understand more explicitly analysis of repns $E(\nu)$ for $SL(2, \mathbb{R})$. Use different picture

$$I(\nu, \epsilon) = \{f \colon (\mathbb{R}^2 - 0) \to \mathbb{C} \mid f(tx) = |t|^{-\nu-1}\,\mathrm{sgn}(t)^\epsilon f(x)\},$$

functions homogeneous of degree $(-\nu - 1, \epsilon)$.

The -1 next to $-\nu$ makes later formulas simpler.

Lie algs easier than Lie gps $\rightsquigarrow$ write $\mathfrak{sl}(2, \mathbb{R})$ action, basis

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[D, E] = 2E, \quad [D, F] = -2F, \quad [E, F] = D.$$

action on functions on $\mathbb{R}^2$ is by

$$D = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \quad E = -x_2 \frac{\partial}{\partial x_1}, \quad F = -x_1 \frac{\partial}{\partial x_2}.$$

Now want to restrict to homogeneous functions...

Principal series for $SL(2,\mathbb{R})$ (continued)

Study homog fns on $\mathbb{R}^2 - 0$ by restr to $\{(\cos\theta, \sin\theta)\}$:

$I(\nu,\epsilon) \simeq \{w\colon S^1 \to \mathbb{C} \mid w(-s) = (-1)^\epsilon w(s)\},\ f(r,\theta) = r^{-\nu-1} w(\theta).$

Compute Lie algebra action in polar coords using

$$\frac{\partial}{\partial x_1} = -x_2\frac{\partial}{\partial\theta} + x_1\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial x_2} = x_1\frac{\partial}{\partial\theta} + x_2\frac{\partial}{\partial r},$$

$$\frac{\partial}{\partial r} = -\nu - 1, \qquad x_1 = \cos\theta, \qquad x_2 = \sin\theta.$$

Plug into formulas on preceding slide: get

$$\rho^\nu(D) = 2\sin\theta\cos\theta\frac{\partial}{\partial\theta} + (\cos^2\theta - \sin^2\theta)(\nu+1),$$

$$\rho^\nu(E) = \sin^2\theta\frac{\partial}{\partial\theta} + (\cos\theta\sin\theta)(\nu+1),$$

$$\rho^\nu(F) = -\cos^2\theta\frac{\partial}{\partial\theta} + (\cos\theta\sin\theta)(\nu+1).$$

Hard to make sense of. Clear: family of reps analytic (actually linear) in complex parameter ν.

Big idea: see how properties change as function of ν.

A more suitable basis

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2,\mathbb{R})$ defined on functions on S^1 of homogeneity (or parity) ϵ:

$$\rho^\nu(D) = 2\sin\theta\cos\theta\frac{\partial}{\partial\theta} + (\cos^2\theta - \sin^2\theta)(\nu+1),$$
$$\rho^\nu(E) = \sin^2\theta\frac{\partial}{\partial\theta} + (\cos\theta\sin\theta)(\nu+1),$$
$$\rho^\nu(F) = -\cos^2\theta\frac{\partial}{\partial\theta} + (\cos\theta\sin\theta)(\nu+1).$$

Problem: $\{D, E, F\}$ adapted to wt vectors for diagonal Cartan subalgebra; rep $\rho^{\nu,\epsilon}$ has no such wt vectors.

But rotation matrix $E - F$ acts simply by $\partial/\partial\theta$.

Suggests new basis of the complexified Lie algebra:

$$H = -i(E-F), \quad X = \frac{1}{2}(D + iE + iF), \quad Y = \frac{1}{2}(D - iE - iF).$$

Same commutation relations $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$, but cplx conj is different: $\overline{H} = -H$, $\overline{X} = Y$.

$$\rho^\nu(H) = \frac{1}{i}\frac{\partial}{\partial\theta},$$

$$\rho^\nu(X) = \frac{e^{2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right), \qquad \rho^\nu(Y) = \frac{-e^{-2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} - i(\nu+1)\right).$$

Matrices for principal series

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2,\mathbb{R})$ defined on functions on S^1 of homogeneity (or parity) ϵ:

$$\rho^\nu(H) = \frac{1}{i}\frac{\partial}{\partial\theta},$$

$$\rho^\nu(X) = \frac{e^{2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right), \quad \rho^\nu(Y) = \frac{-e^{-2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} - i(\nu+1)\right).$$

These ops act simply on basis $w_m(\cos\theta, \sin\theta) = e^{im\theta}$:

$$\rho^\nu(H)w_m = mw_m,$$

$$\rho^\nu(X)w_m = \frac{1}{2}(m+\nu+1)w_{m+2}, \quad \rho^\nu(Y)w_m = \frac{1}{2}(-m+\nu+1)w_{m-2}.$$

Suggests reasonable function space to consider:

$$\begin{aligned} I(\nu,\epsilon)^K &= \text{fns homog of deg } (\nu,\epsilon), \text{ finite under rotation} \\ &\simeq \text{trig polys on } S^1 \text{ of parity } \epsilon \\ &= \text{span}(\{w_m \mid m \equiv \epsilon \pmod 2\}). \end{aligned}$$

Space $I(\nu,\epsilon)^K$ has beautiful rep of $\mathfrak{g}$: irr for most ν, easy submods otherwise. Not preserved by rep of $G = SL(2,\mathbb{R})$.

Invariant forms on principal series by hand

Write $I(\nu) = I(\nu, 0) =$ even fns homog of deg $-\nu - 1$

Need "signature" of invt Herm form on inf-diml space.

Basis $\{w_m \mid m \in 2\mathbb{Z}\}$, $w_m \leftrightsquigarrow e^{im\theta}$, $H \cdot w_m = m w_m$,

$$X \cdot w_m = \frac{1}{2}(\nu + m + 1) w_{m+2}, \quad Y \cdot w_m = \frac{1}{2}(\nu - (m-1)) w_{m-2}.$$

Requirements for invariant Hermitian form $\langle , \rangle_\nu$:

$$\langle H \cdot w, w' \rangle_\nu = \langle w, H \cdot w' \rangle_\nu, \qquad \langle X \cdot w, w' \rangle_\nu + \langle w, Y \cdot w' \rangle_\nu = 0.$$

Apply first requirement to $w = w_m$, $w' = w_{m'}$; get

$$m \langle w_m, w_{m'} \rangle_\nu = m' \langle w_m, w_{m'} \rangle_\nu,$$

and therefore $\langle w_m, w_{m'} \rangle_\nu = 0$ for $m \neq m'$.

So only need $\langle w_m, w_m \rangle_\nu \quad (m \in 2\mathbb{Z})$. Second reqt says

$$((m+1) + \nu) \langle w_{m+2}, w_{m+2} \rangle_\nu = ((m+1) - \overline{\nu}) \langle w_m, w_m \rangle_\nu.$$

Easy solution: ν imaginary, all $\langle w_m, w_m \rangle_\nu$ equal

THM: For $\nu \in i\mathbb{R}$, $L^2(S^1/\{\pm 1\}) \rightsquigarrow I(\nu, 0)$ unitary rep of G.

Invariant forms on $I(\nu)$ by hand, continued

Recall $I(\nu)$ = even functions on $\mathbb{R}^2$, homog deg $-\nu - 1$; seeking invt Herm form $\langle , \rangle_\nu$, specified by values on basis

$$w_m(r,\theta) = r^{-\nu-1} e^{im\theta} \qquad (m \in 2\mathbb{Z}).$$

$$((m+1)+\nu)\langle w_{m+2}, w_{m+2}\rangle_\nu = ((m+1)-\overline{\nu})\langle w_m, w_m\rangle_\nu.$$

Non-imag ν: nonzero (real) solns exist iff $\nu \in \mathbb{R}$:

$$((m+1)+\nu)\langle w_{m+2}, w_{m+2}\rangle_\nu = ((m+1)-\nu)\langle w_m, w_m\rangle_\nu \qquad (\nu \in \mathbb{R}).$$

Natural to normalize $\langle w_0, w_0\rangle_\nu = 1$, calculate

$$\langle w_{\pm 2}, w_{\pm 2}\rangle_\nu = \frac{(1-\nu)}{(1+\nu)}, \quad \langle w_{\pm 4}, w_{\pm 4}\rangle_\nu = \frac{(1-\nu)(3-\nu)}{(1+\nu)(3+\nu)}$$

$$\vdots$$

$$\langle w_{\pm 2m}, w_{\pm 2m}\rangle_\nu = \frac{(1-\nu)(3-\nu)\cdots(2m-1-\nu)}{(1+\nu)(3+\nu)\cdots(2m-1+\nu)}$$

If $\nu \in (2m-1, 2m+1)$, sign alternates on $w_0, w_2, \ldots w_{2m}$.

pos def for $0 \le \nu < 1$; for $\nu > 1$, sign diff on w_0, w_2.

$\langle , \rangle_\nu$ "meromorphic" in (real) ν

One K-type-at-a-time calc too complicated to generalize.

Deforming signatures for $SL(2,\mathbb{R})$

Here's representation-theoretic picture of deforming $\langle , \rangle_\nu$.

$\nu = 0$, $I(0)$ "$\subset$" $L^2(\mathbb{H})$: unitary, signature positive.

$0 < \nu < 1$, $I(\nu)$ irr: signature remains positive.

$\nu = 1$: form finite pos on quotient $J(1) \leftrightsquigarrow SO(2)$ rep 0.

$\nu = 1$: form has simple zero, pos residue on $\ker(I(1) \to J(1))$.

$1 < \nu < 3$, across zero at $\nu = 1$, signature changes.

$\nu = 3$: form finite $- + -$ on quotient $J(3)$.

$\nu = 3$: form has simple zero, neg residue on $\ker(I(3) \to J(3))$.

$3 < \nu < 5$, across zero at $\nu = 3$, signature changes. ETC.

Conclude: $J(\nu)$ unitary, $\nu \in [0,1]$; nonunitary, $\nu \in (1,\infty)$.

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$	$SO(2)$ reps
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 0$
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$0 < \nu < 1$
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 1$
$\cdots$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$\cdots$	$1 < \nu < 3$
$\cdots$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$\cdots$	$\nu = 3$
$\cdots$	$+$	$+$	$-$	$+$	$-$	$+$	$+$	$\cdots$	$3 < \nu < 5$

From $SL(2,\mathbb{R})$ to reductive G

Calculated signatures of invt Herm forms on spherical reps of $SL(2,\mathbb{R})$.

Seek to do "same" for real reductive group. Need...

List of irr reps $=$ ctble union (cplx vec space)/(fin grp).

reps for purely imag points "$\subset$" $L^2(G)$: unitary!

Natural (orth) decomp of any irr (Herm) rep into fin-diml subspaces $\rightsquigarrow$ define signature subspace-by-subspace.

Signature at $\nu + i\tau$ by analytic cont $t\nu + i\tau$, $0 \le t \le 1$.

Precisely: start w unitary (pos def) signature at $t = 0$; add contribs of sign changes from zeros/poles of odd order in $0 \le t \le 1 \rightsquigarrow$ signature at $t = 1$.

How to think about the unitary dual.

Know a lot about complex repns of Γ algebraically.

Want to study unitarity of repns algebraically.

Helpful to step back, ask what we know about the **set** of representations of Γ.

Short answer: it's a complex **algebraic variety**.

Then ask Felix Klein question: what natural **automorphisms** exist on set of representations?

Short answer: from **auts of** Γ and from **lin alg**.

Try to relate unitary structure to these natural things.

Short answer: they're related to $\mathbb{R}$-rational structure on complex variety of repns.

What's a set of irr reps look like?

Γ fin gen group, gens $S = \{\sigma\}$, relations $R = \{\rho\}$.

Relation is a noncomm word $\rho = \sigma_1^{m_1} \cdots \sigma_n^{m_n} (\sigma_i \in S,\ m_i \in \mathbb{Z})$.

N-dim rep $\pi \leftrightsquigarrow N \times N$ matrices $\{\pi(\sigma) \mid \sigma \in S\}$ subject to alg rels $\pi(\rho) = I$ for $\rho \in R$: $\pi(\sigma_1)^{m_1}\pi(\sigma_2)^{m_2} \cdots \pi(\sigma_n)^{m_n} = I$.

Conclude: $\{N\text{-dim reps of } \Gamma\}$ = aff alg var in $GL(N,\mathbb{C})^S$.

Reduc reps are closed $\bigcup_{0 \subsetneq W \subsetneq \mathbb{C}^N} \{\pi \mid \pi(\sigma)W = W \ \ (\sigma \in S)\}$, so irr N-dimls reps are open-in-affine alg variety.

Reps up to equiv: divide by $GL(N,\mathbb{C})$ conj; still more or less alg variety. (Possibly not *separated*, etc.)

Thm. Set $\widehat{\Gamma}_{\text{fin}}$ of equiv classes of fin-diml reps of fin-gen Γ is (approx) disjt union of complex alg vars.

Similar ideas apply to $(\mathfrak{g}, K)$-modules: reps containing fixed rep of K with mult N are N-diml modules for a fin-gen cplx algebra.

Thm. Set $\widehat{G(\mathbb{R})}$ of equiv classes of irr $(\mathfrak{g}, K)$-mods is (approx) disjt union of complex alg vars.

Langlands identifies alg vars as $\mathfrak{a}^*/W^\delta$.

Group automorphisms acting on reps

Γ fin gen group, $\tau \in \mathrm{Aut}(\Gamma)$, (π, V) rep of $\Gamma \rightsquigarrow (\pi^\tau, V)$ new rep on same space, $\pi^\tau(\gamma) =_{\mathrm{def}} \pi(\tau(\gamma))$.

Gives (right) action of $\mathrm{Aut}(G)$ on $\widehat{\Gamma}$.

Inner auts act trivially: linear isom $\pi(\gamma_0)$ intertwines π and $\pi^{\mathrm{Int}(\gamma_0)}$ since $\pi^{\mathrm{Int}(\gamma_0)}(\gamma)\pi(\gamma_0) = \pi(\gamma_0)\pi(\gamma)$.

(Easy) Thm. $\mathrm{Out}(\Gamma) =_{\mathrm{def}} \mathrm{Aut}(\Gamma)/\,\mathrm{Int}(\Gamma)$ acts by algebraic variety automorphisms on $\widehat{\Gamma}_{\mathrm{fin}}$.

(Easy) Thm. $\mathrm{Out}(G(\mathbb{R}))$ acts by algebraic variety automorphisms on $\widehat{G(\mathbb{R})}$.

Main technical point: each aut of $G(\mathbb{R})$ can be modified by inner aut so as to preserve K; so get action on $(\mathfrak{g}, K)$-modules.

Bilinear forms and dual spaces

V cplx vec space (or $(\mathfrak{g}, K)$-module).

Dual of V $V^* = \{\xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = z\xi(v)\}$

(V alg K-rep $\rightsquigarrow$ require ξ K-finite; V topolog. $\rightsquigarrow$ require ξ cont.)

$V = \mathbb{C}^N$ $N \times 1$ column vectors $\rightsquigarrow$ $V^h = \mathbb{C}^N$, $\xi(v) = {}^t\xi v$.

Bilinear pairings between V and W

$$\mathrm{Bil}(V, W) = \{\langle , \rangle \colon V \times W \to \mathbb{C}, \text{lin in } V, \text{lin in } W\}$$
$$\mathrm{Bil}(V, W) \simeq \mathrm{Hom}(V, W^*), \quad \langle v, w\rangle_T = (Tv)(w).$$

Exchange vars in forms to get linear isom

$$\mathrm{Bil}(V, W) \simeq \mathrm{Bil}(W, V).$$

Corr lin isom on maps is transpose:

$$\mathrm{Hom}(V, W^*) \simeq \mathrm{Hom}(W, V^*), \quad (T^t w)(v) = (Tv)(w).$$
$$(TS)^t = S^t T^t, \qquad (zT)^t = z(T^h).$$

Bil form $\langle , \rangle_T$ on V ($\leftrightsquigarrow$ $T \in \mathrm{Hom}(V, V^h)$) orthogonal if

$$\langle v, v'\rangle_T = \langle v', v\rangle_T \iff T^t = T.$$

Bil form $\langle , \rangle_T$ on V ($\leftrightsquigarrow$ $T \in \mathrm{Hom}(V, V^h)$) symplectic if

$$\langle v, v'\rangle_S = -\langle v', v\rangle_T \iff S^t = -S.$$

Defining contragredient repn

(π, V) $(\mathfrak{g}, K)$-module; had (K-finite) dual space V^* of V.

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^*, V^*)$$

using transpose map of operators.

Because transpose is antiaut REQUIRES twisting by antiaut of $(\mathfrak{g}, K)$.

$X \mapsto -X$ is Lie alg antiaut, and $k \mapsto k^{-1}$ group antiaut

Define contragredient $(\mathfrak{g}, K)$-module π^* on V^*,

$$\pi^*(Z) \cdot \xi =_{\text{def}} [\pi(-Z)]^t \cdot \xi \qquad (Z \in \mathfrak{g},\ \xi \in V^*),$$

$$\pi^*(k) \cdot \xi =_{\text{def}} [\pi(k^{-1})]^t \cdot \xi \qquad (k \in K,\ \xi \in V^*).$$

Thm. If Γ is a fin gen group, passage to contragredient is an involutive automorphism of the algebraic variety $\widehat{\Gamma}$.

Thm. If $G(\mathbb{R})$ real reductive, passage to contragredient is an involutive automorphism of the algebraic variety $\widehat{G(\mathbb{R})}$.

Invariant bilinear forms

$V = (\mathfrak{g}, K)$-module, τ involutive aut of $(\mathfrak{g}, K)$.

An invt bilinear form on V is bilinear pairing $\langle , \rangle$ such that

$$\langle Z \cdot v, w\rangle = \langle v, -Z \cdot w\rangle, \quad \langle k \cdot v, w\rangle = \langle v, k^{-1} \cdot w\rangle$$

$$(Z \in \mathfrak{g}; k \in K; v, w \in V).$$

Proposition

Invt bilinear form on V $\leftrightsquigarrow$ *$(\mathfrak{g}, K)$-map $T \colon V \to V^*$:*

$$\langle v, w\rangle_T = (Tv)(w).$$

Form is orthogonal $\iff T^* = T$.

Form is symplectic $\iff T^* = -T$.

Assume from now on V is irreducible.

$V \simeq V^* \iff \exists$ *invt bilinear form on V*

Invt bil form on V unique up to real scalar mult.; non-deg whenever nonzero.

Invt bil form must be either orthogonal or symplectic.

$T \to T^* \leftrightsquigarrow$ involution of cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^*)$.

Existence of invt bil form $\leftrightsquigarrow$ compute $V \mapsto V^*$ on $\widehat{G(\mathbb{R})}$.

Deciding orth/symp usually somewhat harder.

Hermitian forms and dual spaces

V cplx vec space (or $(\mathfrak{g}, K)$-module).

Herm dual of V $V^h = \{\xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v)\}$

(V alg K-rep $\rightsquigarrow$ require ξ K-finite; V topolog. $\rightsquigarrow$ require ξ cont.)

$V = \mathbb{C}^N$ $N \times 1$ column vectors $\rightsquigarrow$ $V^h = \mathbb{C}^N$, $\xi(v) = {}^t\overline{\xi}v$.

Sesquilinear pairings between V and W

$$\mathrm{Sesq}(V, W) = \{\langle, \rangle \colon V \times W \to \mathbb{C}, \text{lin in } V, \text{conj-lin in } W\}$$

$$\mathrm{Sesq}(V, W) \simeq \mathrm{Hom}(V, W^h), \quad \langle v, w\rangle_T = (Tv)(w).$$

Cplx conj of forms is (conj linear) isom

$$\mathrm{Sesq}(V, W) \simeq \mathrm{Sesq}(W, V).$$

Corr (conj lin) isom on maps is Hermitian transpose:

$$\mathrm{Hom}(V, W^h) \simeq \mathrm{Hom}(W, V^h), \quad (T^h w)(v) = \overline{(Tv)(w)}.$$

$$(TS)^h = S^h T^h, \qquad (zT)^h = \overline{z}(T^h).$$

Sesq form $\langle, \rangle_T$ on V ($\leftrightsquigarrow$ $T \in \mathrm{Hom}(V, V^h)$) Hermitian if

$$\langle v, v'\rangle_T = \overline{\langle v', v\rangle_T} \iff T^h = T.$$

Defining Herm dual repn(s)

(π, V) $(\mathfrak{g}, K)$-module; Recall Herm dual V^h of V.

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators.

Definition REQUIRES twisting by conj lin antiaut of $\mathfrak{g}$, gp antiaut of K.

Since $\mathfrak{g}$ equipped with a real form $\mathfrak{g}_0$, have natural conj-lin aut $\sigma_0(X + iY) = X - iY$ $(X, Y \in \mathfrak{g}_0)$. Also $X \mapsto -X$ is Lie alg antiaut, and $k \mapsto k^{-1}$ gp antiaut.

Define Hermitian dual $(\mathfrak{g}, K)$-module π^h on V^h,

$$\pi^h(Z) \cdot \xi =_{\text{def}} [\pi(-\sigma_0(Z))]^h \cdot \xi \quad (Z \in \mathfrak{g},\ \xi \in V^h),$$

$$\pi^h(k) \cdot \xi =_{\text{def}} [\pi(k^{-1})]^h \cdot \xi \quad (k \in K,\ \xi \in V^h).$$

Need also a variant: suppose τ inv aut of $G(\mathbb{R})$ preserving K. Define τ-herm dual $(\mathfrak{g}, K)$-module $\pi^{h,\tau}$ on V^h,

$$\pi^{h,\tau}(X) \cdot \xi = [\pi(-\tau(\sigma_0(Z))]^h \cdot \xi \quad (Z \in \mathfrak{g},\ \xi \in V^h),$$

$$\pi^{h,\tau}(k) \cdot \xi = [\pi(\tau(k)^{-1})]^h \cdot \xi \quad (k \in K,\ \xi \in V^h).$$

Invariant Hermitian forms

For τ an inv aut of $(G(\mathbb{R}), K)$, defined τ-herm dual

$$\pi^{h,\tau}(X)\cdot\xi = [\pi(-\tau(\sigma_0(Z))]^h\cdot\xi \quad (Z\in\mathfrak{g},\ \xi\in V^h),$$

$$\pi^{h,\tau}(k)\cdot\xi = [\pi(\tau(k)^{-1})]^h\cdot\xi \qquad (k\in K,\ \xi\in V^h).$$

Thm. τ-herm dual is Galois for $\mathbb{R}$-struc on alg var $\widehat{G(\mathbb{R})}$.

Reason: conj transpose is real Galois action on $GL(N,\mathbb{C})$.

A τ-invt sesq form on $(\mathfrak{g},K)$-module V is pairing $\langle,\rangle^\tau$ with

$$\langle Z\cdot v, w\rangle = \langle v, -\tau(\sigma_0(Z))\cdot w\rangle, \quad \langle k\cdot v, w\rangle = \langle v, \tau(k^{-1})\cdot w\rangle$$
$$(Z\in\mathfrak{g};\ k\in K;\ v,w\in V).$$

Prop. τ-invt sesq form on V $\leftrightsquigarrow$ $(\mathfrak{g},K)$-map $T\colon V\to V^{h,\tau}$:

$$\langle v,w\rangle_T = (Tv)(w).$$

Form is Hermitian $\iff$ $T^h = T$.

Assume from now on V is irreducible.

$V\simeq V^{h,\tau} \iff \exists\tau$-invt sesq $\iff \exists\tau$-invt Herm

τ-invt Herm form on V unique up to real scalar mult.

$T\to T^h \leftrightsquigarrow$ real form of cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^{h,\tau})$.

Deciding existence of τ-invt Hermitian form amounts to computing the involution $V\mapsto V^{h,\tau}$ on $\widehat{G}$; easy.