

# The Langlands Classification

## Background for the Atlas workshop2013

Jeffrey Adams

June 28, 2013

### 1 Representations of $SL(2, \mathbb{R})$

Elsewhere, and in the background reading, we discuss how to go from representations of a group on a Hilbert space, to algebraic representations, or  $(\mathfrak{g}, K)$ -modules. Once in the setting of  $(\mathfrak{g}, K)$ -modules it is possible to do some explicit calculations. The case of  $SL(2, \mathbb{R})$  is very informative, and we will treat this in some detail in the lectures. Here is some background.

We take  $G = SL(2, \mathbb{R})$ , with Lie algebra  $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$ , the two-by-two real matrices of trace 0.

Set  $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$ . Let  $t(e^{i\theta}) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ , so  $t$  is an isomorphism from the circle to the subgroup  $K = \{g \in G \mid g^t g = I\} = SO(2) \subset SL(2, \mathbb{R})$ . This is a maximal compact subgroup of  $G$ . We identify the irreducible representation  $\widehat{K}$  of  $K$  with  $\mathbb{Z}$ . We assume familiarity with representations of compact groups.

Define the following basis of  $\mathfrak{g}$ :

$$(1.1) \quad E = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad F = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

**Exercise 1.2** Check that  $H, E, F$  satisfy the familiar identities  $[H, E] = 2E$ ,  $[H, F] = -2F$ ,  $[E, F] = H$ . Also  $iH$  is a basis of  $\mathfrak{k}_0 = \text{Lie}(K) \subset \mathfrak{g}_0$ .

**Definition 1.3** For  $\nu \in \mathbb{C}, \epsilon \in \mathbb{Z}/2\mathbb{Z}$  define a vector space  $I(\nu, \epsilon)$ , with representations of  $\mathfrak{g}$  and  $K$  as follows. It has a basis  $\{v_j \mid j \in \epsilon + 2\mathbb{Z}\}$ . The action of  $K$  is:

$$(1.4)(a) \quad \pi(t(e^{i\theta}))(v_k) = e^{ik\theta} v_k$$

The action of  $\mathfrak{g}$  is given by

$$\begin{aligned} (1.4)(b) \quad & \pi(H)v_j = jv_j \\ (1.4)(c) \quad & \pi(E)v_j = 1/2(\nu + (j + 1))v_{j+2} \\ (1.4)(d) \quad & \pi(F)v_j = 1/2(\nu - (j - 1))v_{j-2} \end{aligned}$$

**Exercise 1.5** Verify that  $I(\nu, \epsilon)$  is a  $(\mathfrak{g}, K)$ -module. This means:

- (1) This is a representation of  $\mathfrak{g}$  and  $K$ ;
- (2) Every vector is  $K$ -finite:  $\dim\langle \pi(k)v \mid k \in K \rangle < \infty$  for all  $v \in V$ ;
- (3) The representation of  $\mathfrak{g}$ , restricted to  $\mathfrak{k}_0$ , is the differential of the representation of  $K$ .

(We have omitted a fourth condition which is not necessary since  $K$  is connected.)

We say a  $(\mathfrak{g}, K)$ -module is irreducible if it does not contain a proper subspace  $W$ , invariant under the action of  $\mathfrak{g}$  and  $K$ .

**Lemma 1.6**  $I(\nu, \epsilon)$  is reducible if and only if

$$(1.7) \quad \nu \in \epsilon + 2\mathbb{Z} + 1.$$

**Lemma 1.8** Suppose  $\nu = n \in \epsilon + 2\mathbb{Z} + 1$  with  $n \geq 0$ . Then  $I(\nu, \epsilon)$  has two infinite dimensional subrepresentations

$$\begin{aligned} I_+(\nu, \epsilon) &= \langle v_{n+1}, v_{n+3}, \dots \rangle \\ I_-(\nu, \epsilon) &= \langle v_{-n-1}, v_{-n-3}, \dots \rangle \end{aligned}$$

Furthermore  $V/(I_+ \oplus I_-)$  is an irreducible finite dimensional representation  $I_0(\nu, \epsilon)$  of dimension  $n$  with basis  $v_{-n+1}, v_{-n+3}, \dots, v_{n-1}$  (the image of these vectors in the quotient).

In other words there is an exact sequence of  $(\mathfrak{g}, K)$ -modules:

$$0 \rightarrow I_+(\nu, \epsilon) \oplus I_-(\nu, \epsilon) \rightarrow I(\nu, \epsilon) \rightarrow I_0(\nu, \epsilon) \rightarrow 0$$

Furthermore:

$$\begin{aligned} I_{\pm}(\nu, \epsilon) &\simeq I_{\pm}(-\nu, \epsilon) \\ I_0(\nu, \epsilon) &\simeq I_0(-\nu, \epsilon) \end{aligned}$$

An important special case is  $\epsilon = 1, \nu = 0$ , in which case  $V = I_+ \oplus I_-$  and  $I_0$  vanishes.

**Lemma 1.9** Suppose  $\nu = n \in \epsilon + 2\mathbb{Z} + 1$ , and  $n \leq 0$ . Then  $I(\nu, \epsilon)$  has a finite dimensional subrepresentation  $I_0(\nu, \epsilon) = \langle v_{n+1}, v_{n+3}, \dots, v_{-n-1} \rangle$  of dimension

$n$  (vanishing if  $n = 0$ ), and  $V/I_0$  is the direct sum of two irreducible, infinite dimensional representations:

$$\begin{aligned} I_+(\nu, \epsilon) &= \langle v_{-n+1}, v_{-n+3}, \dots \rangle \\ I_-(\nu, \epsilon) &= \langle v_{n-1}, v_{n-3}, \dots \rangle \end{aligned}$$

In other words there is an exact sequence

$$0 \rightarrow I_0(\nu, \epsilon) \rightarrow I(\nu, \epsilon) \rightarrow I_+(\nu, \epsilon) \oplus I_-(\nu, \epsilon) \rightarrow 0$$

**Exercise 1.10** Prove both of these Lemmas. The main point is that  $I(\nu, \epsilon)$  is irreducible if and only if  $E v_j \neq 0$  and  $F v_j \neq 0$  for all  $j$ . Furthermore  $E v_n = 0$  if and only if  $F v_{-n} = 0$ , and the behavior of the submodules/quotients depends on whether  $\nu$  is positive or negative.

Here are a few useful formulas.

$$(1.11) \quad \begin{aligned} \pi(E)v_{j-2} &= 1/2(\nu + (j-1))v_j \\ \pi(F)v_{j+2} &= 1/2(\nu - (j+1))v_j \\ \pi(E)\pi(F)v_j &= \frac{1}{4}(\nu^2 - (j-1)^2)v_j \\ \pi(F)\pi(E)v_j &= \frac{1}{4}(\nu^2 - (j+1)^2)v_j \end{aligned}$$

Let  $C$  be the Casimir element:

$$(1.12) \quad \begin{aligned} C &= H^2 + 2EF + 2FE + 1 \\ &= H^2 + 2H + 4FE + 1 \\ &= H^2 - 2H + 4EF + 1 \end{aligned}$$

This is an element of the universal enveloping algebra of  $\mathfrak{g}$ , and it acts on a representation using any one of these formulas, for example  $\pi(C)(v) = (\pi(H)^2 + 2\pi(E)\pi(F) + 2\pi(F)\pi(E) + 1)v$ .

**Exercise 1.13** Show that  $\pi(C)\pi(X) = \pi(X)\pi(C)$  for all  $X \in \mathfrak{g}$ , and that given  $\nu, \epsilon$ ,

$$\pi(C)(v) = \nu^2 v \quad \text{for all } v \in I(\nu, \epsilon).$$

In fact  $C$  is in the center of the universal enveloping algebra.

**Theorem 1.14** Consider the irreducible  $(\mathfrak{g}, K)$ -modules:

- (a)  $I(\nu, \epsilon)$  ( $\nu \notin \epsilon + 2\mathbb{Z} + 1$ );
- (b)  $I_0(n, \epsilon)$  ( $n = 1, 2, 3, \dots$ )
- (c)  $I_{\pm}(n, \epsilon)$  ( $n = 0, 1, 2, 3, \dots$ )

In (a)  $I(\nu, \epsilon) \simeq I(-\nu, \epsilon)$ ; there are no other isomorphisms between these modules. Every irreducible  $(\mathfrak{g}, K)$  module is isomorphic to one of these.

Most of this is straightforward, except for the fact that every irreducible representation is isomorphic to one of these. See [11, Proposition 1.2.14].

Define an action of  $K$  on the algebraic dual space  $\text{Hom}(V, \mathbb{C})$  as usual:  $\pi(k)(f)(v) = f(\pi(k^{-1})v)$ . Let  $V^*$  be the  $K$ -finite vectors. Then  $\mathfrak{g}$  acts on  $V^*$  as usual:  $\pi(X)(f)(v) = -f(\pi(X)v)$ .

**Exercise 1.15** Show that  $V^*$  is a  $(\mathfrak{g}, K)$ -module. What is  $I(\nu, \epsilon)^*$ ? Include the cases when this module is reducible. What about  $I_0(\nu, \epsilon)^*$  and  $I_{\pm}(\nu, \epsilon)^*$ ?

## 2 Hermitian forms

Given a  $(\mathfrak{g}, K)$ -module  $(\pi, V)$  for  $SL(2, \mathbb{R})$ , suppose  $(, )$  is a Hermitian form on the complex vector space  $V$ . The natural notion of invariance under the action of the Lie algebra is

$$(2.1)(a) \quad (\pi(X)v, w) + (v, \pi(X)w) = 0 \quad (X \in \mathfrak{g}_0).$$

It is essential that  $X$  is in the *real* Lie algebra. To give a valid formula on  $\mathfrak{g}$  observe that  $\pi(iX) = i\pi(X)$ , and (a) implies

$$(2.1)(b) \quad (\pi(iX)v, w) + (v, \pi(-iX)w) = i(\pi(X)v, w) + \overline{-i}(v, \pi(X)w) = 0$$

In other words if  $\sigma$  is complex conjugation of  $\mathfrak{g}$  with respect to  $\mathfrak{g}_0$ , then (a) is equivalent to

$$(2.1)(c) \quad (\pi(X)v, w) + (v, \pi(\sigma(X))w) = 0 \quad (X \in \mathfrak{g}).$$

**Definition 2.2** An invariant Hermitian form on a  $(\mathfrak{g}, K)$ -module  $(\pi, V)$  is a Hermitian form satisfying

$$\begin{aligned} (\pi(k)v, \pi(k)w) &= (v, w) \quad (k \in K) \\ (\pi(X)v, w) + (v, \pi(\sigma(X))w) &= 0 \quad (X \in \mathfrak{g}). \end{aligned}$$

Note that  $K = K(\mathbb{R}) = SO(2)$ . It is natural to replace  $K$  with  $K(\mathbb{C}) = \mathbb{C}^*$ , in which case the first formula would have an extra  $\sigma$  on the right. We don't do this here.

**Exercise 2.3** Show that the invariance condition is equivalent to

$$(2.4) \quad \begin{aligned} (\pi(E)v, w) &= -(v, \pi(F)w) \\ (\pi(F)v, w) &= -(v, \pi(E)w) \\ (\pi(H)v, w) &= (v, \pi(H)w) \end{aligned}$$

Consider the module  $I(\nu, \epsilon)$  with  $\nu \in \mathbb{R}$ , and basis  $\{v_j\}$ .

**Exercise 2.5** Suppose  $(\cdot, \cdot)$  is an invariant Hermitian form on  $I(\nu, \epsilon)$ . Then  $(v_j, v_k) = 0$  for  $j \neq k$ .

**Exercise 2.6** Suppose  $I(\nu, \epsilon)$  is irreducible, and  $I(\nu, \epsilon)$  has a invariant form. Show that (2.4) and (1.4)(b-d) imply for all  $j$

$$(2.7) \quad (v_{j+2}, v_{j+2}) = \frac{(-\bar{\nu} + (j+1))}{(\nu + (j+1))} (v_j, v_j).$$

Using the fact that  $(v, v) \in \mathbb{R}$  for all  $v$  implies  $I(\nu, \epsilon)$  supports an invariant Hermitian form if and only if  $\nu \in i\mathbb{R} \cup \mathbb{R}$ .

- (1) If  $\nu \in i\mathbb{R}$  this form can be taken to be identically 1.
- (2) If  $\nu \in \mathbb{R}$  and  $\epsilon = 0$  this form can be uniquely normalized so that  $(v_0, v_0) = 1$ .
- (3) If  $\nu \in \mathbb{R}^*$  and  $\epsilon = 1$  then  $(v_{-1}, v_{-1}) = -(v_1, v_1)$ .

The irreducibility assumption is only to avoid the denominator being 0. With a little more care this can be dropped.

The representations with  $\nu \in i\mathbb{R}$  are the tempered, unitary principal series. The important case is  $\nu \in \mathbb{R}$  (the case of real infinitesimal character).

The last case in the exercise illustrates a crucial problem: there is no canonical way to normalize the form on the lowest K-types  $v_{\pm 1}$  in this case.

The remedy is to use a modification of the invariant Hermitian form. Define

$$\sigma_c(X) = -\overline{\epsilon X}$$

Then  $\mathfrak{g}^{\sigma_c} = \mathfrak{su}(2)$ , the Lie algebra of the compact group  $SU(2)$ , the subscript stands for compact. Notice that  $\sigma_c(\text{Lie}(K)) = \text{Lie}(K)$ .

**Definition 2.8** A  $c$ -invariant Hermitian form on a  $(\mathfrak{g}, K)$ -module  $(\pi, V)$  is a Hermitian form  $(\cdot, \cdot)_c$  satisfying

$$\begin{aligned} (\pi(k)v, \pi(k)w)_c &= (v, w)_c \quad (k \in K) \\ (\pi(X)v, w)_c + (v, \pi(\sigma_c(X))w)_c &= 0 \quad (X \in \mathfrak{g}). \end{aligned}$$

**Exercise 2.9** Suppose  $I(\nu, \epsilon)$  is irreducible, and  $I(\nu, \epsilon)$  has a  $c$ -invariant form. Show that (2.4) and (1.4)(b-d) imply for all  $j$

$$(2.10) \quad (v_{j+2}, v_{j+2})_c = \frac{(\bar{\nu} - (j+1))}{(\nu + (j+1))} (v_j, v_j)_c$$

Conclude that if  $\nu \in \mathbb{R}$  then  $I(\nu, \epsilon)$  supports a unique  $c$ -invariant Hermitian form normalized so that  $(v_0, v_0)_c = 1$  ( $\epsilon = 0$ ) or  $(v_1, v_1)_c = (v_{-1}, v_{-1})_c = 1$  ( $\epsilon = 1$ ).

**Exercise 2.11** Show that the  $n$ -dimensional irreducible representation has a positive definite  $c$ -invariant Hermitian form. It supports an invariant Hermitian form, which is not positive definite unless  $n = 1$ .

**Exercise 2.12** Suppose  $\nu > 0$ ,  $\nu \notin \mathbb{Z}$ .

- (1) Show that  $I(\nu, 1)$  has an invariant Hermitian form, which is not positive definite.
- (2) Show that  $I(\nu, 0)$  has an invariant Hermitian form, which is positive definite if and only if  $\nu < 1$ .

The representations (2) are the complementary series for  $SL(2, \mathbb{R})$ .

### 3 Tori

An important role is played by real algebraic tori and their representations. We discuss representations of real algebraic tori. For background on algebraic groups see [10], [4] or [6]. and [3, Section 3] for the representations of tori.

Start with a complex torus  $H = H(\mathbb{C}) \simeq \mathbb{C}^{*n}$ . Let  $X^*$  be the holomorphic characters of  $H$ . If  $n = 1$  these are the maps  $z \rightarrow z^n$  ( $n \in \mathbb{Z}$ ). Let  $X_*$  be the algebraic co-characters, i.e. holomorphic group homomorphisms  $\mathbb{C}^* \rightarrow H$ . Then  $X^*, X_*$  are isomorphic to  $\mathbb{Z}^n$ , and there is a perfect pairing  $\langle, \rangle : X^* \times X_* \rightarrow \mathbb{Z}$ . If  $\alpha(\gamma^\vee(z)) = z^k$  ( $z \in \mathbb{C}^*, k \in \mathbb{Z}$ ) then  $\langle \alpha, \gamma^\vee \rangle = k$ .

**Lemma 3.1** Suppose  $\theta$  is a holomorphic involution of  $H = H(\mathbb{C})$ . Then there is an isomorphism  $H \simeq \mathbb{C}^{*n}$  so that

$$\theta(z_1, \dots, z_a, w_1, \dots, w_b, u_1, v_1, \dots, u_c, v_c) = (z_1, \dots, z_a, w_1^{-1}, \dots, w_b^{-1}, v_1, u_1, \dots, v_c, u_c).$$

**Exercise 3.2** Try proving this result. It is *not* elementary. It is equivalent to proving that any involution in  $GL(n, \mathbb{Z})$  is conjugate to a matrix with diagonal

entries 1,  $-1$ , or  $2 \times 2$  blocks  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

**Lemma 3.3** Suppose  $\sigma$  is an anti-holomorphic involution of  $H$ . Then there is an isomorphism  $H \simeq \mathbb{C}^{*n}$  so that

$$\sigma(z_1, \dots, z_a, w_1, \dots, w_b, u_1, v_1, \dots, u_c, v_c) = (\bar{z}_1^{-1}, \dots, \bar{z}_a^{-1}, \bar{w}_1, \dots, \bar{w}_b, \bar{v}_1^{-1}, \bar{u}_1^{-1}, \dots, \bar{v}_c^{-1}, \bar{u}_c^{-1}).$$

Write  $H(\mathbb{R}) = H^\sigma$ . This is a real Lie group. With these coordinates we have:

$$\begin{aligned} H(\mathbb{R}) &= S^{1a} \times \mathbb{R}^{*b} \times \mathbb{C}^{*c} \quad (\text{the real torus}) \\ T(\mathbb{R}) &= H(\mathbb{R})^\theta = S^{1a} \times (\mathbb{Z}/2\mathbb{Z})^b \times S^{1c} \quad (\text{the maximal compact subgroup}) \\ T(\mathbb{C}) &= H^\theta = \mathbb{C}^{*a} \times (\mathbb{Z}/2\mathbb{Z})^b \times \mathbb{C}^{*c} \quad (\text{the complexified maximal compact subgroup}) \end{aligned}$$

**Lemma 3.4** *There is a natural isomorphism  $\widehat{T(\mathbb{R})} \simeq X^*/(1-\theta)X^*$ .*

**Exercise 3.5** Prove the Lemma. First of all, the characters of  $T(\mathbb{R})$  are the same thing as the algebraic characters of  $T(\mathbb{C}) = H^\theta$ . You may assume that any algebraic character of  $H^\theta$  is the restriction of a character of  $H$ .

A character of  $H(\mathbb{R})$  may be identified with a  $(\mathfrak{h}, H^\theta)$ -module.

**Proposition 3.6** *The  $(\mathfrak{h}, H^\theta)$  modules are parametrized by pairs  $(\nu, \kappa)$  satisfying:*

- (1)  $\nu \in X^* \otimes \mathbb{C} \simeq \mathfrak{h}^*$ ;
- (2)  $\kappa \in X^*/(1-\theta)X^*$ ;
- (3)  $(1+\theta)\nu = (1+\theta)\kappa$ .

*The corresponding character  $\nu$  of  $H(\mathbb{R})$  has differential  $\nu$ , and the restriction of  $\nu$  to  $H(\mathbb{R})^\theta$  is  $\kappa$ .*

**Exercise 3.7** Prove the Proposition. Note that  $H(\mathbb{R}) = T(\mathbb{R}) \exp(\mathfrak{h}_0)$ .

**Example 3.8** Suppose  $\theta = 1$ , so  $H(\mathbb{R}) \simeq S^{1n}$  is compact. The characters are parametrized by  $\kappa \in X^*$ .

If  $\theta = -1$ ,  $H(\mathbb{R}) \simeq \mathbb{R}^{*n}$ , and characters are parametrized by pairs  $(\nu, \kappa)$  with  $\nu \in X^* \otimes \mathbb{C} \simeq \mathbb{C}^n$  and  $\kappa \in X^*/2X^* (\simeq \mathbb{Z}/2\mathbb{Z})^n$ .

**Exercise 3.9** Work out the case of  $H(\mathbb{R}) \simeq \mathbb{C}^*$ . In this case  $H(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ ,  $\sigma(z, w) = (\bar{w}^{-1}, \bar{z}^{-1})$ , and  $\theta(z, w) = (w, z)$ .

### 3.1 Digression: Characters of compact groups

Let  $G = SU(2)$ , with  $T \simeq S^1$  the diagonal Cartan subgroup. The character of the  $n$ -dimensional irreducible representation ( $n = 1, 2, \dots$ ) is

$$e^{i(-n+1)\theta} + e^{i(-n+3)\theta} + \dots + e^{i(n-3)\theta} + e^{i(n-1)\theta} = \frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}$$

This is the Weyl character formula in this case.

Now consider the similar case of  $G = SO(3)$ . Again a Cartan subgroup  $T$  is

the circle, for example take  $\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$  The irreducible representations of  $G$  are odd dimensional; the character of the  $n$ -dimensional irreducible representation for ( $n = 1, 3, 5, \dots$ ) is:

$$(3.1.1)(a) \quad e^{i(\frac{-n+1}{2})\theta} + e^{i(\frac{-n+3}{2})\theta} + \dots + e^{i(\frac{n-3}{2})\theta} + e^{i(\frac{n-1}{2})\theta} = \frac{e^{in\theta/2} - e^{-in\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}$$

The numerator and denominator of the quotient are not well defined functions on  $T$ , because of the  $2s$  in the denominator. The quotient is well defined; after multiplying by  $\frac{e^{i\theta/2}}{e^{i\theta/2}}$  it can be written

$$(3.1.1)(b) \quad \frac{e^{i(\frac{n+1}{2})\theta} - e^{i(\frac{-n+1}{2})\theta}}{1 - e^{-i\theta}}$$

Since  $n$  is odd both numerator and denominator are well defined. However (a) is clearly a more symmetric, and therefore preferable, expression.

For a general connected compact group the Weyl denominator is

$$\prod_{\alpha>0} (e^{-\alpha/2} - e^{-\alpha/2}) = \prod_{\alpha>0} (1 - e^{-\alpha}) e^\rho$$

Here  $\rho = \frac{1}{2} \sum_{\alpha>0} \alpha$  as usual. The first expression isn't really well defined; it is shorthand for the second. The second expression is well defined if  $\rho$  exponentiates to a character of  $T$ . This holds for  $SL(2)$ , but not  $PSL(2)$ .

The conclusion is it is very useful to introduce a two-fold cover of the torus on which  $\rho$  is well defined. These play an important role in the Langlands classification. If  $\rho$  exponentiates this cover can be ignored, and it is reasonable to focus on this case first time around.

### 3.2 Covers of tori

Now fix an element  $\gamma \in \frac{1}{2}X^*$ . Let

$$(3.2.1) \quad H_\gamma = \{(h, z) \in H \times \mathbb{C}^* \mid 2\gamma(h) = z^2\}.$$

This is a two-fold cover of  $H$  via the map  $(h, z) \rightarrow h$ ; write  $\zeta$  for the nontrivial element in the kernel of this map. We call this the  $\gamma$ -cover of  $H$ . Note that  $(h, z) \rightarrow z$  is a character of  $H_\gamma$ , and is a canonical square root of  $2\gamma$ , denoted  $\gamma$ .

**Exercise 3.2.2** Prove this cover splits, i.e.  $H_\gamma \simeq H \times \mathbb{Z}/2\mathbb{Z}$  if and only if  $\gamma \in X^*$ .

Now assume  $H$  is defined over  $\mathbb{R}$ , with Cartan involution  $\theta$ . The  $\gamma$  cover of  $H(\mathbb{R})$  is defined to be the inverse image of  $H(\mathbb{R})$  in  $H_\gamma$ . A character of  $H(\mathbb{R})_\gamma$  is said to be genuine if it is nontrivial on  $\zeta$ .

**Lemma 3.2.3** *The genuine characters of  $H(\mathbb{R})_\gamma$  are canonically parametrized by the set of pairs  $(\nu, \kappa)$  with  $\nu \in \mathfrak{h}^*$ ,  $\kappa \in \gamma + X^*/(1 - \theta)X^*$ , and satisfying  $(1 + \theta)\nu = (1 + \theta)\kappa$ .*

**Exercise 3.2.4** Prove the Lemma.

**Example 3.2.5** Suppose  $H$  is a Cartan subgroup of a reductive group  $G$ . Choose a set of positive roots  $\Delta^+$  and let  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ , and consider the cover

$H_\rho$  of  $H$ . This is independent of the choice of  $\Delta^+$  up to *canonical* isomorphism (prove this!).

Suppose  $G = SL(2, \mathbb{R})$ . For any real Cartan subgroup  $H(\mathbb{R})_\rho \simeq H(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$ .

Let  $G = PSL(2, \mathbb{R}) \simeq SO(2, 1)$  (this group is disconnected). If  $H(\mathbb{R}) \simeq S^1$ , then  $H(\mathbb{R})_\rho \simeq S^1$ , with projection map  $z \rightarrow z^2$ .

If  $H(\mathbb{R}) \simeq \mathbb{R}^*$  then  $H(\mathbb{R})_\rho \simeq \mathbb{R}^* \cup i\mathbb{R}^*$ . Note this has an element of order 4. Although  $H_\rho$  is a real algebraic group,  $H(\mathbb{R})_\rho$  is not its real points;  $H_\rho(\mathbb{R}) \simeq \mathbb{R}^*$  is a subgroup of index 2 in  $H(\mathbb{R})_\rho$ .

**Exercise 3.2.6** Think through the final example .

## 4 Cartan subgroups

If  $G$  is compact or complex, its Cartan subgroups are unique up to conjugacy. In a real Lie group there are a finite number of conjugacy classes of Cartan subgroups, and these play an important role in representation theory.

If  $G = SL(2, \mathbb{R})$  there are two Cartan subgroups, up to conjugacy. The diagonal  $\{\text{diag}(x, \frac{1}{x})\} \simeq \mathbb{R}^*$ , and the circle  $SO(2) \simeq S^1$ .

**Exercise 4.1** Show that every semisimple element of  $GL(2, \mathbb{R})$  is conjugate to either  $\text{diag}(x, y)$  or  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ . Conclude that  $GL(2, \mathbb{R})$  has two Cartan subgroups (up to conjugacy), one  $\mathbb{R}^{*2}$  and  $\mathbb{C}^*$ .

**Exercise 4.2** Find representatives of all conjugacy classes of Cartan subgroups in  $GL(n, \mathbb{R})$  and  $SL(n, \mathbb{R})$ .

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