On K-spherical flag varieties

—joint work with

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Representations of Reductive Groups
University of Utah
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1. Motivation & Problems : multipleflag variety
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Relation to derived functor modules (by Yoshiki Oshima)
Classifications of spherical action on flag varieties

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   classical finite type $\Leftarrow$ classification by Magyar-Weyman-Zelevinsky
   Recently also by Matsuki

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   $\mathfrak{X}_{P_\lambda}$: G-spherical $\quad \rightsquigarrow \quad X_\lambda$: $G \times \mathbb{C}^\times$-spherical

   $\therefore \qquad \mathbb{C}[X_\lambda] \simeq \bigoplus_{k \geq 0} V_{k\lambda}^*$: mult-free decomp

[Remark: actually X_λ is G-spherical]

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- Extra feature (KGB-theory):

K-orbits on $\mathfrak{X}_B = G/B$ with local system

$\longleftrightarrow$ K-equiv $\mathscr{D}$-module on $\mathfrak{X}_B$

$\xleftrightarrow{\text{localization}}$ Harish-Chandra $(\mathfrak{g}, K)$-modules

3. $G \curvearrowright X = \mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_B$: triple flag variety

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$\Longrightarrow X_\lambda \times X_\mu$: $G \times \mathbb{C}^\times \times \mathbb{C}^\times$-spherical

$\rightsquigarrow V_{k\lambda}^* \otimes V_{\ell\mu}^* \simeq \bigoplus_\eta V_\eta$: mult-free decomp $(\forall k, \ell \geq 0)$

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Interesting generalization:

Next to spherical (complexity 1) $\cdots$ Ponomareva (2012, arXiv)

$\exists$ Open orbit on mult flag var $\cdots$ Popov (2007)

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$\cdots\cdots$ want to extend it to a symmetric pair

Double flag variety — definition

(G, K) : symmetric pair $/\mathbb{C}$ $K \leftrightarrow \theta$: involution

Ex. $(G, K) = (GL_{p+q}, GL_p \times GL_q)$, (SL_n, O_n), (SL_{2n}, Sp_{2n}), $(Sp_{2n}, GL_n), \ldots$

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Hecke alg module structure: $\mathscr{H}(G, B) \curvearrowright H^*(\mathfrak{X}_P \times \mathcal{Z}_Q) \curvearrowleft \mathscr{H}(K, B_K)$

$\mathfrak{X}_P = G/P$: PFV of G, $\mathcal{Z}_Q = K/Q$: PFV of K

Examples of $\mathfrak{X}_P \times \mathcal{Z}_Q$: double flag var (DFV) of finite type

Type AI : $G/K = \mathrm{SL}_n/\mathrm{SO}_n$ $(n \geq 3)$

P	Q	$\mathfrak{X}_P$	$\mathcal{Z}_Q$	extra condition
maximal	any	$\mathrm{Grass}_m(\mathbb{C}^n)$	$\mathcal{Z}_Q$	
$(\lambda_1, \lambda_2, \lambda_3)$	Siegel	$\mathfrak{X}_P$	$\mathrm{LGrass}(\mathbb{C}^n)$	n is even

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Type AIII : $G/K = \mathrm{GL}_n/\mathrm{GL}_p \times \mathrm{GL}_q$ $(n = p + q)$

P	Q_1	Q_2	$\mathfrak{X}_P$	$\mathcal{Z}_Q$
any	mirabolic	GL_q	$\mathfrak{X}_P$	$\mathbb{P}(\mathbb{C}^p)$
any	GL_p	mirabolic	$\mathfrak{X}_P$	$\mathbb{P}(\mathbb{C}^q)$
maximal	any	any	$\mathrm{Grass}_m(\mathbb{C}^n)$	$\mathcal{Z}_Q$
$(\lambda_1, \lambda_2, \lambda_3)$	maximal	maximal	$\mathfrak{X}_P$	$\mathrm{Grass}_k(\mathbb{C}^p) \times \mathrm{Grass}_\ell(\mathbb{C}^q)$

Relation to multiple flag varieties for G

1. Triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$ with G-action
   $\cdots$ special case of double flag variety $\mathfrak{X}_P \times \mathcal{Z}_Q$ with K-action

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$(\because)$ Take $\mathbb{G} = G \times G$ and $\mathbb{K} = \Delta G$ as usual

$$\mathbb{P} = P_1 \times P_2, \quad \mathbb{Q} = \Delta P_3$$

$$\rightsquigarrow \ \mathbb{G}/\mathbb{P} \times \mathbb{K}/\mathbb{Q} = G/P_1 \times G/P_2 \times G/P_3 \qquad \square$$

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Relation to multiple flag varieties for G

1. Triple flag variety $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_{P_3}$ with G-action
   $\cdots$ special case of double flag variety $\mathfrak{X}_P \times \mathcal{Z}_Q$ with K-action

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   In general $\#K \backslash (\mathfrak{X}_P \times \mathfrak{X}_{P'}) = \infty$

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Key idea to describe K orbits on $\mathfrak{X}_P \times \mathcal{Z}_Q$

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Strategy 1: θ-twisted embedding à la Miličić

$$\begin{cases} \Delta_\theta : \mathfrak{X}_P \hookrightarrow \mathfrak{X}_P \times \mathfrak{X}_{\theta(P)} : \theta\text{-twisted embedding} \\ \iota : \mathcal{Z}_Q \hookrightarrow \mathfrak{X}_{P'} : \text{closed embedding} \end{cases}$$

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$\rightsquigarrow$ parametrization of $(\mathfrak{X}_P \times \mathcal{Z}_Q)/K$ roughly by

$$\underline{((\mathfrak{X}_P \times \mathfrak{X}_{\theta(P)} \times \mathfrak{X}_{P'})/G)} \times \underline{(W_P\backslash W/W_{P'})} \times \underline{(\text{conn comp})}$$

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Parametrization

Reduces to paramet'n of $P \backslash P w P' / Q$ for $w \in {}^J W^{J'}$

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Assume $B \supset T$: θ-stable $\quad B \leftrightarrow \Delta^+ \supset \Pi$: simple roots

$P \leftrightarrow J \subset \Pi$ and $P' \leftrightarrow J' \subset \Pi$

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Reduction map :

$$\begin{array}{ccc} P \backslash PwP'/Q & \xrightarrow{\text{surj}} & P_{L'}(w) \backslash L' / L'_K = V(w) \\ \Cup & & \Cup \\ PwaQ & \longmapsto & P_{L'}(w) \ell_a L'_K \end{array}$$

where $a = \ell_a u_a$ is Levi decomp along $P' = L'U'$

For $w \in {}^{J}W^{J'}, v \in V(w)$, put

$$\begin{cases} \mathscr{U}(w,v) := (U' \cap P(wv)) \backslash U' / (U' \cap K) & : \text{variety of unipotent elts} \\ L'_K(w,v) := L' \cap K \cap P(wv) \subset L'_K \end{cases}$$

Notation: $P(g) := g^{-1}Pg \in \mathfrak{X}_P$

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Theorem (He-N-Ochiai-Y.Oshima)

Recall ${}^J W^{J'} = W_J \backslash W / W_{J'}$ *and* $V(w) = P_{L'}(w) \backslash L' / L'_K$

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We have bijection of orbits (parametrization):

$$K \backslash \mathfrak{X}_P \times \mathcal{Z}_Q \simeq \coprod_{w \in {}^J W^{J'}} \coprod_{v \in V(w)} \mathscr{U}(w,v) / \operatorname{Ad}(L'_K(w,v))$$

$\rightsquigarrow$ criterion of finiteness of orbits

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In particular, $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ *is of finite type*

$$\iff \# \Big((U_0 \cap P(w))\backslash U_0/(U_0 \cap K) \Big) \Big/ \operatorname{Ad} T < \infty \text{ for } \forall w$$

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Interesting connection to (co-)normal bundles:

$$T^*_{\mathcal{O}} \mathfrak{X}_P \simeq K \times_R \mathfrak{u}_P^{-\theta} \ : \ \text{conormal bundle over } \mathcal{O} \ (R := K \cap P)$$

Geometric & Representation Theoretic interpretation

1. We can deduce the former theorem from Panyushev's thm

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TFAE

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2. Multiplicity free derived functor modules:

Theorem (Y.Oshima)

Assume P *is* θ*-stable*

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$\iff$ *derived functor module* $A_{\mathfrak{p}}(\lambda)$ *has mult-free* K*-types for* $\forall \lambda$ *: dom regular integral*

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Classification of K-spherical flag variety

G : simply connected, connected simple group

Possible to classify (G, K, P) for which $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ is of finite type

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Howe, Wallach and Horvath classified them in 80's (unpublished note)

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Possible to classify (G, K, P) for which $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ is of finite type

Theorem (HNOO)

Complete cassification of $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ of finite type

(*including exceptional type*)

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Strategy

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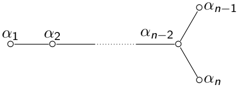
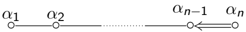
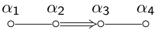
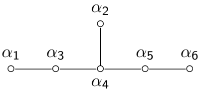
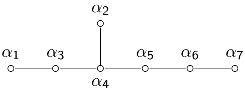
2. Use criterion in Theorem (Existence of open orbit)

$$L_P^\theta := L_P \cap K : \text{reductive} \curvearrowright \mathfrak{u}_P^{-\theta} \text{ is mult-free (or spherical)}$$

$\exists$ classification of mult-free space by Benson-Ratcliff (2004)

$\mathfrak{g}$	$\mathfrak{k}$	$\Pi \setminus J$ $(P = P_J)$
$\mathfrak{sl}_{n+1}$		
$\mathfrak{sl}_{n+1}$	$\mathfrak{so}_{n+1}$	$\{\alpha_i\}(\forall i)$
$\mathfrak{sl}_{2m}$ $2m = n+1$	$\mathfrak{sp}_m$	$\{\alpha_i\}(\forall i)$, $\{\alpha_i, \alpha_{i+1}\}(\forall i)$, $\{\alpha_1, \alpha_i\}(\forall i)$, $\{\alpha_i, \alpha_n\}(\forall i)$, $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$, $\{\alpha_1, \alpha_2, \alpha_n\}$, $\{\alpha_1, \alpha_{n-1}, \alpha_n\}$
$\mathfrak{sl}_{p+q}$ $p+q = n+1$	$\mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{C}$ $1 \le p \le q$	$\{\alpha_i\}(\forall i)$, $\{\alpha_i, \alpha_{i+1}\}(\forall i)$, $\{\alpha_1, \alpha_i\}(\forall i)$, $\{\alpha_i, \alpha_n\}(\forall i)$, $\{\alpha_i, \alpha_j\}(\forall i, j)$ if $p = 2$, any subset of Π if $p = 1$
$\mathfrak{so}_{2n+1}$		
$\mathfrak{so}_{p+q}$ $p+q = 2n+1$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$ $1 \le p \le q$	$\{\alpha_1\}$, $\{\alpha_n\}$, $\{\alpha_i\}(\forall i)$ if $p = 2$, any subset of Π if $p = 1$
$\mathfrak{so}_{2n}$		
$\mathfrak{so}_{p+q}$ $p+q = 2n$ $n \ge 4$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$ $1 \le p \le q$	$\{\alpha_1\}$, $\{\alpha_{n-1}\}$, $\{\alpha_n\}$, $\{\alpha_i\}(\forall i)$ if $p = 2$, $\{\alpha_i, \alpha_{n-1}\}(\forall i)$ if $p = 2$, $\{\alpha_i, \alpha_n\}(\forall i)$ if $p = 2$, any subset of Π if $p = 1$
$\mathfrak{so}_{2n}$ $n \ge 4$	$\mathfrak{sl}_n \oplus \mathbb{C}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_{n-1}\}$, $\{\alpha_n\}$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_{n-1}\}$, $\{\alpha_1, \alpha_n\}$, $\{\alpha_{n-1}, \alpha_n\}$, $\{\alpha_2, \alpha_3\}$ if $n = 4$

$\mathfrak{sp}_n$		
$\mathfrak{sp}_n$	$\mathfrak{sl}_n \oplus \mathbb{C}$	$\{\alpha_1\}$, $\{\alpha_n\}$
$\mathfrak{sp}_{p+q}$ $p+q=n$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$ $1 \le p \le q$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_n\}$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_i\}(\forall i)$ if $p \le 2$, $\{\alpha_i, \alpha_j\}(\forall i, j)$ if $p = 1$
$\mathfrak{f}_4$		
$\mathfrak{f}_4$	$\mathfrak{so}_9$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_4\}$, $\{\alpha_1, \alpha_4\}$
$\mathfrak{e}_6$		
$\mathfrak{e}_6$	$\mathfrak{sp}_4$	$\{\alpha_1\}$, $\{\alpha_6\}$
$\mathfrak{e}_6$	$\mathfrak{sl}_6 \oplus \mathfrak{sl}_2$	$\{\alpha_1\}$, $\{\alpha_6\}$
$\mathfrak{e}_6$	$\mathfrak{so}_{10} \oplus \mathbb{C}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_5\}$, $\{\alpha_6\}$, $\{\alpha_1, \alpha_6\}$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_5\}$, $\{\alpha_6\}$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_2, \alpha_6\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_5, \alpha_6\}$
$\mathfrak{e}_7$		
$\mathfrak{e}_7$	$\mathfrak{sl}_8$	$\{\alpha_7\}$
$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{sl}_2$	$\{\alpha_7\}$
$\mathfrak{e}_7$	$\mathfrak{e}_6 \oplus \mathbb{C}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_7\}$

Application 2: when $P = B$ is Borel

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$$\mathfrak{X}_B \times \mathcal{Z}_Q : \text{finite type} \iff G/Q : \underline{G}\text{-spherical}$$

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WLOG **assume** $P = B = TU_0 \subset G$: θ-stable Borel subgrp
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Recall θ-stable P' s.t. $Q = P' \cap K$
$P' = L'U'$: Levi decomp $\rightsquigarrow$ $Q = L'_K U'_K$: Levi decomp

Theorem

TFAE

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$\because$ (3) $\Longrightarrow$ $L'_K \curvearrowright \mathfrak{u}'^{-\theta}$ is mult-free action

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$$\mathfrak{X}_B \times \mathcal{Z}_Q \ : \ \text{finite type} \implies \mathfrak{X}_{P'} \times \mathcal{Z}_{B_K} \ : \ \text{finite type}$$

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$\therefore$ $\mathfrak{X}_B \times \mathcal{Z}_Q$: finite type $\Longleftrightarrow$ $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_B$: finite type

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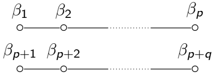
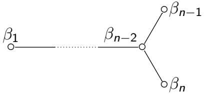
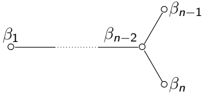
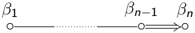
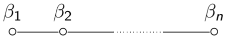
   $\therefore$ $\mathfrak{X}_B \times \mathcal{Z}_Q$: finite type $\iff$ $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2} \times \mathfrak{X}_B$: finite type

   $\exists$ classification of spherical $\mathfrak{X}_{P_1} \times \mathfrak{X}_{P_2}$ by Stembridge (2003)

$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_K \setminus J_K$ $(Q = Q_{J_K})$
$\mathfrak{sl}_{2n}$ $n \geq 2$	$\mathfrak{sp}_n$	 $\{\beta_1\}$, $\{\beta_3\}$ if $n = 3$, any subset of Π_K if $n = 2$
$\mathfrak{sl}_{p+q+2}$ $p + q \geq 1$	$\mathfrak{sl}_{p+1} \oplus \mathfrak{sl}_{q+1} \oplus \mathbb{C}$ $p \leq q$	 $\{\beta_1\}$, $\{\beta_p\}$, $\{\beta_{p+1}\}$, $\{\beta_{p+q}\}$, $\{\beta_i\}(\forall i)$ if $p = 1$, any subset of Π_K if $p = 0$
$\mathfrak{so}_{2n+2}$ $n \geq 3$	$\mathfrak{so}_{2n} \oplus \mathbb{C}$	 $\{\beta_{n-1}\}$, $\{\beta_n\}$
$\mathfrak{so}_{2n+1}$ $n \geq 3$	$\mathfrak{so}_{2n}$	 any subset of Π_K
$\mathfrak{so}_{2n+2}$ $n \geq 3$	$\mathfrak{so}_{2n+1}$	 any subset of Π_K

<table>
<tr><th>$\mathfrak{g}$</th><th>$\mathfrak{k}$</th><th>$\Pi_K \setminus J_K \ (Q = Q_{J_K})$</th></tr>
<tr><td>$\mathfrak{so}_{2n+2}$
$n \geq 3$</td><td>$\mathfrak{sl}_{n+1} \oplus \mathbb{C}$</td><td>

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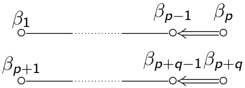
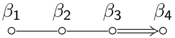
<tr><td>$\mathfrak{sp}_{p+q}$</td><td>$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$
$1 \leq p \leq q$</td><td>

$\{\beta_1\}$, $\{\beta_{p+1}\}$,
$\{\beta_p\}$ if $p \leq 3$, $\{\beta_{p+q}\}$ if $p \leq 2$, $\{\beta_{p+q}\}$ if $q \leq 3$,
$\{\beta_1, \beta_2\}$ if $p = 2$, $\{\beta_{p+1}, \beta_{p+2}\}$ if $q = 2$,
$\{\beta_i\}(\forall i)$ if $p = 1$, $\{\beta_i, \beta_j\}(\forall i, j)$ if $p = 1$</td></tr>
<tr><td>$\mathfrak{f}_4$</td><td>$\mathfrak{so}_9$</td><td>

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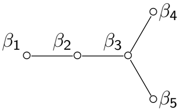
<tr><td>$\mathfrak{e}_6$</td><td>$\mathfrak{so}_{10} \oplus \mathbb{C}$</td><td>

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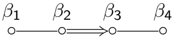
<tr><td>$\mathfrak{e}_6$</td><td>$\mathfrak{f}_4$</td><td>

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</table>

Thank you for your attention!!