On the singular elements of a semisimple Lie algebra and the generalized Amitsur-Levitski Theorem

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1. Introduction

I connect an old result of mine on a Lie algebra generalization of the Amitsur-Levitski Theorem with recent results of Kostant-Wallach on the variety of singular elements in a reductive Lie algebra.

Let R be an associative ring and for any $k \in \mathbb{Z}$ and $x_i, \ldots, x_k$, in R one defines an alternating sum of products

$$[[x_1, \ldots x_k]] = \sum_{\sigma \in \mathrm{Sym\,k}} sg(\sigma)\, x_{\sigma(1)} \cdots x_{\sigma(k)}. \tag{1}$$

One says that R satisfies the standard identity of degree k if $[[x_1, \ldots x_k]] = 0$ for any choice of the $x_i \in R$. Of course R is commutative if and only if it satisfies the standard identity of degree 2.

Now for any $n \in \mathbb{Z}$ and field F, let $M(n, F)$ be the algebra of $n \times n$ matrices over F.

The following is the famous Amitsur-Levitski theorem.

Theorem 1. *$M(n, F)$ satisfies the standard identity of degree 2n.*

Remark 1. By restricting to matrix units for a proof it suffices to take $F = \mathbb{C}$.

Without any knowledge that it was a known theorem, we came upon Theorem 1 a long time ago, from the point of view of Lie algebra cohomology [1]. In fact, the result follows from the fact that if $\mathfrak{g} = M(n, \mathbb{C})$, then the restriction to $\mathfrak{g}$ of the primitive cohomology class of degree $2n+1$ of $M(n+1, \mathbb{C})$ to $\mathfrak{g}$ vanishes.

Of course $\mathfrak{g}_1 \subset \mathfrak{g}$ where $\mathfrak{g}_1 = \mathrm{Lie}\,\mathrm{SO}(n, \mathbb{C})$. Assume n is even. One proves that the restriction to $\mathfrak{g}_1$ of the primitive class of degree $2n-1$ (highest primitive class) of $\mathfrak{g}$ vanishes on $\mathfrak{g}_1$. This leads to a new standard identity, namely

Theorem 2.

$$[[x_1, \ldots, x_{2n-2}]] = 0 \qquad (2)$$

for any choice of $x_i \in \mathfrak{g}_1$*. That is any choice of skew-symmetric matrices.*

Remark 2. Theorem 2 is immediately evident when $n = 2$.

Theorems 1 and 2 suggest that standard identities can be viewed as a subject in Lie theory. Theorem 3 below offers support for this idea.

Let $\mathfrak{r}$ be a complex reductive Lie algebra and let

$$\pi : \mathfrak{r} \to \mathrm{End}\,\mathrm{V} \tag{3}$$

be a finite-dimensional complex completely reducible representation.

If $w \in \mathfrak{r}$ is nilpotent, then $\pi(w)^k = 0$ for some $k \in \mathbb{Z}$. Let $\varepsilon(\pi)$ be the minimal integer k such that $\pi(w)^k = 0$ for all nilpotent $w \in \mathfrak{r}$. In case π is irreducible, one can easily give a formula for $\varepsilon(\pi)$ in terms of the highest weight. If $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) is given as above and π (resp. π_1)) is the defining representation, then $\varepsilon(\pi) = n$ and $\varepsilon(\pi_1) = n - 1$.

Consequently, the following theorem generalizes Theorems 1 and 2.

Theorem 3. *Let $\mathfrak{r}$ be a complex reductive Lie algebra and let π be as above. Then for any $x_i \in \mathfrak{r}$, $i = 1, \ldots, 2\varepsilon(\pi)$, one has*

$$[[\hat{x}_1, \ldots, \hat{x}_{2\,\varepsilon(\pi)}]] = 0 \tag{4}$$

where $\hat{x}_i = \pi(x_i)$. See [4].

2. Henceforth $\mathfrak{g}$, until mentioned otherwise, will be an arbitrary reductive complex finite-dimensional Lie algebra. Let $T(\mathfrak{g})$ be the tensor algebra over $\mathfrak{g}$ and let $S(\mathfrak{g}) \subset T(\mathfrak{g})$(resp. $A(\mathfrak{g}) \subset T(\mathfrak{g})$) be the subspace of symmetric (resp. alternating) tensors in $T(\mathfrak{g})$.

The natural grading on $T(\mathfrak{g})$ restricts to a grading on $S(\mathfrak{g})$ and $A(\mathfrak{g})$. In particular, where multiplication is the tensor product one notes the following.

Proposition 1. *$A^j(\mathfrak{g})$ is the span of $[[x_1, \ldots, x_j]]$ over all choices of x_i, $i = 1, \ldots, j$, in $\mathfrak{g}$.*

Now let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then $U(\mathfrak{g})$ is the quotient algebra of $T(\mathfrak{g})$ so that there is an algebra epimorphism

$$\tau : T(\mathfrak{g}) \to U(\mathfrak{g}).$$

Let $Z = \mathrm{Cent}\,\mathrm{U}(\mathfrak{g})$ and let $E \subset U(g)$ be the graded subspace spanned by all powers $e^j, j = 1, \ldots,$ where $e \in \mathfrak{g}$ is nilpotent. In [2] we proved (where the tensor product identifies with multiplication)

$$U(\mathfrak{g}) = Z \otimes E. \qquad (5)$$

In [4] we proved

Theorem 4. *For any* $k \in (\mathbb{Z})$ *one has*

$$\tau(A^{2k}(\mathfrak{g})) \subset E^k. \tag{6}$$

Theorem 3 is then an immediate consequence of Theorem 4.

Indeed, using the notation of Theorem 3, let $\pi_U : U(\mathfrak{g}) \to \mathrm{End}\, V$ be the algebra extension of π to $U(\mathfrak{g})$, and one then has

Theorem 5. *If* $E^k \subset \mathrm{Ker}\,\pi_{\mathrm{U}}$*, then*

$$[[\hat{x}_1, \ldots, \hat{x}_{2k}]] = 0 \qquad (7)$$

for any $x_i, \ldots, x_{2k}$ *in* $\mathfrak{g}$*.*

3. The Poincaré-Birkhoff-Witt theorem says that the restriction $\tau : S(\mathfrak{g}) \to U(\mathfrak{g})$ is a linear isomorphism. Consequently, given any $t \in T(\mathfrak{g})$ there exists a unique element $\bar{t}$ in $S(\mathfrak{g})$ such that

$$\tau(t) = \tau(\bar{t}). \tag{8}$$

Let $A^{even}(\mathfrak{g})$ be the span of alternating tensors of even degree. Restricting to $A^{even}(\mathfrak{g})$ one has a $\mathfrak{g}$-module map

$$\Gamma_T : A^{even}(\mathfrak{g}) \to S(\mathfrak{g})$$

defined so that if $\mathfrak{a} \in A^{even}(\mathfrak{g})$, then

$$\tau(a) = \tau(\Gamma_T(a)). \tag{9}$$

Now the (commutative) symmetric algebra $P(\mathfrak{g})$ over $\mathfrak{g}$ and exterior algebra $\wedge\mathfrak{g}$ are quotient algebras of $T(\mathfrak{g})$. The restriction of the quotient map clearly induces $\mathfrak{g}$-module isomorphisms

$$\tau_S : S(\mathfrak{g}) \to P(\mathfrak{g}) \tag{10}$$

$$\tau_A : A^{even}(\mathfrak{g}) \to \wedge^{even}\mathfrak{g}$$

where $\wedge^{even}\mathfrak{g}$ is the commutative subalgebra of $\wedge\mathfrak{g}$ spanned by elements of even degree.
We may complete the commutative diagram defining

$$\Gamma : \wedge^{even}\mathfrak{g} \to P(\mathfrak{g}) \tag{11}$$

so that on $A^{even}(\mathfrak{g})$, one has

$$\tau_S \circ \Gamma_T = \Gamma \circ \tau_A. \tag{12}$$

By (6) one notes that for $k \in \mathbb{Z}$ one has

$$\Gamma : \wedge^{2k}\mathfrak{g} \to P^k(\mathfrak{g}). \tag{13}$$

The Killing form extends to a nonsingular symmetric bilinear form on $P(\mathfrak{g})$ and $\wedge\mathfrak{g}$. This enables us to identify $P(\mathfrak{g})$ with the algebra of polynomial functions on $\mathfrak{g}$ and to identify $\wedge\mathfrak{g}$ with its dual space $\wedge\mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual space to $\mathfrak{g}$.

Let $R^k(\mathfrak{g})$ be the image of (13) so that $R^k(\mathfrak{g})$ is a $\mathfrak{g}$-module of homogeneous polynomial functions of degree k on $\mathfrak{g}$. The significance of $R^k(\mathfrak{g})$ has to do with the dimensions of $\mathrm{Ad}\,\mathfrak{g}$ adjoint (= coadjoint) orbits. Any such orbit is symplectic and hence is even dimensional.

For $j \in \mathbb{Z}$ let $\mathfrak{g}^{(2j)} = \{x \in \mathfrak{g} \mid \dim[\mathfrak{g}, x] = 2j\}$.

We recall that a $2j$ $\mathfrak{g}$-sheet is an irreducible component of $\mathfrak{g}^{(2j)}$.
Let $Var\, R^k(\mathfrak{g}) = \{x \in \mathfrak{g} \mid p(x) = 0, \forall p \in R^k(\mathfrak{g})\}$.

We prove the following.

Theorem 6. *One has*

$$Var\ R^k(\mathfrak{g}) = \cup_{2j<2k}\, \mathfrak{g}^{(2j)} \tag{14}$$

or that $\mathrm{Var}\, R^k(\mathfrak{g})$ *is the set of all* $2j$ $\mathfrak{g}$*-sheets for* $j < k$*.*

Let γ be the transpose of Γ. Thus

$$\gamma : P(\mathfrak{g}) \rightarrow \wedge^{even}\mathfrak{g} \tag{15}$$

and one has for $p \in P(\mathfrak{g})$ and $u \in \wedge\mathfrak{g}$,

$$(\gamma(p), u) = (p, \Gamma(u)). \tag{16}$$

One also notes that

$$\gamma : P^k(\mathfrak{g}) \to \wedge^{2k}\mathfrak{g}. \tag{17}$$

The proof of Theorem 6 depends upon establishing some nice algebraic properties of γ. Since we have, via the Killing form, identified $\mathfrak{g}$ with its dual, $\wedge\mathfrak{g}$ is the underlying space for a standard cochain complex $(\wedge\mathfrak{g}, d)$ where d is the coboundary operator of degree $+1$. In particular. if $x \in \mathfrak{g}$, then $dx \in \wedge^2\mathfrak{g}$. Identifying $\mathfrak{g}$ here with $P^1(\mathfrak{g})$ one has a map

$$P^1(\mathfrak{g}) \to \wedge^2\mathfrak{g}. \tag{18}$$

Theorem 7. *The map (15) is the homomorphism of commutative algebras extending (18). In particular for any $x \in \mathfrak{g}$,*

$$\gamma(x^k) = (dx)^k. \tag{19}$$

The connection with Theorem 6 follows from

Proposition 2. *Let* $x \in \mathfrak{g}$. *Then* $x \in \mathfrak{g}^{(2k)}$ *if and only if* k *is maximal such that* $(dx)^k \neq 0$, *in which case there is a scalar* $c \in \mathbb{C}^\times$ *such that*

$$(dx)^k = c\ w_1 \wedge \cdots \wedge w_{2k} \tag{20}$$

where w_i, $i = 1, \ldots, 2k$, *is a basis of* $[x, \mathfrak{g}]$.

3. On the variety of singular elements – joint work with Nolan Wallach

Let $\mathfrak{h}$ be a Cartan sublgebra of $\mathfrak{g}$ and let $\ell = \dim \mathfrak{h}$, so $\ell = \operatorname{rank} \mathfrak{g}$. Let Δ be the set of roots of $(\mathfrak{h}, \mathfrak{g})$ and let $\Delta_+ \subset \Delta$ be a choice of positive roots. Let $r = \operatorname{card} \Delta_+$ so that $n = \ell + 2r$ where we fix $n = \dim \mathfrak{g}$. We assume a well ordering is defined on Δ_+.

For any $\varphi \in \Delta$, let e_φ be a corresponding root vector. The choices will be normalized only insofar as $(e_\varphi, e_{-\varphi}) = 1$ for all $\varphi \in \Delta$. From Proposition 2 one has the well-known fact that $\mathfrak{g}^{(2k)} = 0$ for $k > r$ and $\mathfrak{g}^{(2r)}$ is the set of all regular elements in $\mathfrak{g}$. One also notes then that (16) implies $\operatorname{Var} R^k(\mathfrak{g})$ reduces to 0 if $k > r$, whereas Theorem 6 implies

$$\operatorname{Var}\, \mathrm{R}^{\mathrm{r}}(\mathfrak{g}) \text{ is the set of all singular elements in } \mathfrak{g}. \qquad (21)$$

The paper [5] is mainly devoted to a study of a special construction of $R^r(\mathfrak{g})$ and a determination of its remarkable $\mathfrak{g}$-module structure.

Let $J = P(\mathfrak{g})^{\mathfrak{g}}$ so that J is the ring of $\mathrm{Ad}\,\mathfrak{g}$ polynomial invariants. It is a classic theorem of C. Chevalley that J is a polynomial ring in ℓ-homogeneous generators p_i so that we can write

$$J = \mathbb{C}[\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell].$$

Let $d_i = \deg p_i$. Then if we put $m_i = d_i - 1$, the m_i are referred to as the exponents of $\mathfrak{g}$, and one knows that

$$\sum_{i=1}^{\ell} m_i = r. \tag{22}$$

Let $\mathrm{Diff}\, P(\mathfrak{g})$ be the algebra of differential operators on $P(\mathfrak{g})$ with constant coefficients. One then has an algebra isomorphism

$$P(\mathfrak{g}) \to \mathrm{Diff}\, P(\mathfrak{g}), \quad q \mapsto \partial_q$$

where for $p, q, f \in P(\mathfrak{g})$ one has

$$(\partial_q p, f) = (p, qf) \tag{23}$$

and ∂_x, for $x \in \mathfrak{g}$, is the partial derivative defined by x.

Let $J_+ \subset J$ be the J-ideal of all $p \in J$ with zero constant term and let

$$H = \{q \in P(\mathfrak{g}) \mid \partial_p q = 0 \; \forall p \in J_+\}.$$

H is a graded $\mathfrak{g}$-module whose elements are called harmonic polynomials. Then one knows ([2]) that where tensor product is realized by polynomial multiplication,

$$P(\mathfrak{g}) = J \otimes H. \tag{24}$$

It is immediate from (23) that H is the orthocomplement of the ideal $J_+P(\mathfrak{g})$ in $P(\mathfrak{g})$. However since γ is an algebra homomorphism. one has

$$J_+P(\mathfrak{g}) \subset \operatorname{Ker}\gamma \tag{25}$$

since one easily has that $J_+ \subset \operatorname{Ker}\gamma$.
Indeed this is clear since

$$\begin{aligned}\gamma(J_+) &\subset d(\wedge\,\mathfrak{g}) \cap (\wedge\,\mathfrak{g})^{\mathfrak{g}} \\ &= 0.\end{aligned}$$

But then (16) implies

Theorem 8. *For any $k \in \mathbb{Z}$ one has*

$$R^k(\mathfrak{g}) \subset H.$$

Henceforth assume $\mathfrak{g}$ is simple so that the adjoint representation is irreducible. Let y_j, $j = 1, \ldots, n$, be a basis of $\mathfrak{g}$. One defines a $\ell \times n$ matrix $Q = Q_{ij}$, $i = 1, \ldots, \ell$, $j = 1, \ldots, n$ by putting

$$Q_{ij} = \partial_{y_j} p_i. \tag{26}$$

Let S_i, $i = 1, \ldots, \ell$, be the span of the entries of Q in the i^{th} row. The following is immediate.

Proposition 3. $S_i \subset P_{m_i}(\mathfrak{g})$. *Furthermore S_i is stable under the action of $\mathfrak{g}$ and as a $\mathfrak{g}$-module S_i transforms according to the adjoint representation.*

If V is a $\mathfrak{g}$-module let V_{ad} be the set of all of vectors in V which transform according to the adjoint representation. The equality (24) readily implies $P(\mathfrak{g})_{ad} = J \otimes H_{ad}$.
I proved the following result some time ago.

Theorem 9. *The multiplicity of the adjoint representation in H_{ad} is ℓ. Furthermore the invariants p_i can be chosen so that $S_i \subset H_{ad}$ for all i and the $S_i, i = 1, \ldots, \ell$, are indeed the ℓ occurrences of the adjoint representation in H_{ad}.*

Clearly there are $\binom{n}{\ell}$ $\ell \times \ell$ minors in the matrix Q. The determinant of any of these minors is an element of $P^r(\mathfrak{g})$ by (22). In [5] we exhibit $R^r(\mathfrak{g})$ by proving

Theorem 10. *The determinant of any $\ell \times \ell$ minor of Q is an element of $R^r(\mathfrak{g})$ and indeed $R^r(\mathfrak{g})$ is the span of the determinants of all these minors.*

4. The $\mathfrak{g}$-module structure of $R^r(\mathfrak{g})$

The adjoint action of $\mathfrak{g}$ on $\wedge\mathfrak{g}$ extends to $U(\mathfrak{g})$ so that $\wedge\mathfrak{g}$ is a $U(\mathfrak{g})$-module. If $\mathfrak{s} \subset \mathfrak{g}$ is any subpace and $k = \dim \mathfrak{s}$, let $[\mathfrak{s}] = \wedge^k \mathfrak{s}$ so that $[\mathfrak{s}]$ is a 1-dimensional subspace of $\wedge^k \mathfrak{g}$.

Let $M_k \subset \wedge^k \mathfrak{g}$ be the span of all $[\mathfrak{s}]$ where $\mathfrak{s}$ is any k-dimensional commutative Lie subalgebra of $\mathfrak{g}$. If no such subalgebra exists, put $M_k = 0$.

It is clear that M_k is a $\mathfrak{g}$-submodule of $\wedge^k \mathfrak{g}$. Let $\mathrm{Cas} \in Z$ be the Casimir element corresponding to the Killing form. The following theorem was proved in [3].

Theorem 11. *For any $k \in \mathbb{Z}$ let m_k be the maximal eigenvalue of Cas on $\wedge^k \mathfrak{g}$. Then $m_k \leq k$. Moreover $m_k = k$ if and only if $M_k \neq 0$ in which case M_k is the eigenspace for the maximal eigenvalue k.*

Let Φ be a subset of Δ. Let $k = \mathrm{card}\,\Phi$ and write, in increasing order,

$$\Phi = \{\varphi_1, \ldots, \varphi_k\} \tag{27}$$

Let

$$e_\Phi = e_{\varphi_1} \wedge \cdots \wedge e_{\varphi_k}$$

so that $e_\Phi \in \wedge^k \mathfrak{g}$ is a $(\mathfrak{h})$ weight vector with weight

$$\langle \Phi \rangle = \sum_{i=1}^{k} \varphi_i.$$

Let $\mathfrak{n}$ be the Lie algebra spanned by e_φ for $\varphi \in \Delta_+$ and let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ defined by putting $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$.

Now a subset $\Phi \subset \Delta_+$ will be called an ideal in Δ_+ if the span, $\mathfrak{n}_\Phi$, of e_φ, for $\varphi \in \Phi$, is an ideal of $\mathfrak{b}$.

In such a case $\mathbb{C}e_\Phi$ is stable under the action of $\mathfrak{b}$ and hence if $V_\Phi = U(\mathfrak{g}) \cdot e_\Phi$ then, where $k = \mathrm{card}\,\Phi$,

$$V_\Phi \subset \wedge^k \mathfrak{g}$$

is an irreducible $\mathfrak{g}$-module of highest weight $\langle \Phi \rangle$ having $\mathbb{C}e_\Phi$ as the highest weight space.

We will say that Φ is abelian if $\mathfrak{n}_\Phi$ is an abelian ideal of $\mathfrak{b}$.
Let

$$\mathcal{A}(k) = \{\Phi \mid \Phi \text{ is an abelian ideal of cardinality } k \text{ in } \Delta_+\}.$$

The following theorem was established in [3].

Theorem 12. *If* Φ, Ψ *are distinct ideals in* Δ_+, *then* V_Φ *and* V_Ψ *are inequivalent (i.e.,* $\langle \Phi \rangle \neq \langle \Psi \rangle$*). Furthermore if* $M_k \neq 0$, *then*

$$M_k = \oplus_{\Phi \in \mathcal{A}(k)} V_\Phi \tag{28}$$

so that, in particular, M_k *is a multiplicity* 1 $\mathfrak{g}$*-module.*

We now focus on the case where $k = \ell$.

Clearly $M_\ell \neq 0$ since $\mathfrak{g}^x$ is an abelian subalgebra of dimension ℓ for any regular $x \in \mathfrak{g}$. Let $\mathcal{I}(\ell)$ be the set of all ideals of cardinality ℓ.

The following is one of the main results in [5].

Theorem 13. *One has $\mathcal{I}(\ell) = \mathcal{A}(\ell)$ so that*

$$M_\ell = \oplus_{\Phi \in \mathcal{I}(\ell)} V_\Phi. \tag{29}$$

Moreover as $\mathfrak{g}$-modules, one has the equivalence

$$R^r(\mathfrak{g}) \cong M_\ell, \tag{30}$$

so that $R^r(\mathfrak{g})$ is a multiplicity 1 $\mathfrak{g}$-module with $\operatorname{card} \mathcal{I}(\ell)$ irreducible components and Cas takes the value ℓ on each and every one of the $\mathcal{I}(\ell)$ distinct components.

Example. If $\mathfrak{g}$ is of type A_ℓ, then then the elements of $\mathcal{I}(\ell)$ can identified with Young diagrams of size ℓ. In this case therefore the number of irreducible components in $R^r(\mathfrak{g})$ is $P(\ell)$ where P here is the classical partition function.

5. Appendix: On an explicit construction for $R^k(\mathfrak{g})$ for any k

We wish to explicitly describe this module. Let $\mathrm{Sym}(2k, 2)$ be the subgroup of the symmetric group $\mathrm{Sym}(2k)$, defined by

$$\mathrm{Sym}(2k,2) = \{\sigma \in \mathrm{Sym}(2k) \mid \sigma \text{ permutes the set of unordered pairs } \{(1,2),(3,4),\dots,(2k-1),2k).\}\}$$

That is, if $\sigma \in \mathrm{Sym}(2k,2)$ and $1 \le i \le k$, there exists $1 \le j \le k$ such that as unordered sets

$$(\sigma(2i-1), \sigma(2i)) = ((2j-1, 2j)).$$

It is clear that $\mathrm{Sym}(2k,2)$ is a subgroup of order $2^k \cdot k!$. Let $\Pi(k)$ be a cross-section of the set of left cosets of $\mathrm{Sym}(2k,2)$ in $\mathrm{Sym}(2k)$ so that one has a disjoint union

$$\mathrm{Sym}(2\mathrm{k}) = \cup\, \nu\, \mathrm{Sym}(2\mathrm{k}, 2) \tag{31}$$

indexed by $\nu \in \Pi(k)$.

Remark 3. One notes that the cardinality of $\Pi(k)$ is $(2k-1)(2k-3)\cdots 1$ and the correspondence

$$\nu \mapsto ((\nu(1), \nu(2)), (\nu(3), \nu(4)), \ldots, \nu((2k-1), \nu(2k)))$$

sets up a bijection of $\Pi(k)$ with the set of all partitions of $(1, 2, \ldots, 2k)$ into a union of subsets, each of which have two elements.

We also observe that $\Pi(k)$ may be chosen – and will be chosen – such that $sg\,\nu = 1$ for all $\nu \in \Pi(k)$. This is clear since the *sg* character is not trivial on $\mathrm{Sym}(2k, 2)$ for $k \geq 1$.

Theorem 14 *For any $k \in \mathbb{Z}$ there exists a nonzero scalar c_k such that for any x_i $i = 1, \ldots, 2k$, in $\mathfrak{g}$*

$$\Gamma(x_1 \wedge \cdots \wedge x_{2k}) = c_k \sum_{\nu \in \Pi(k)} [x_{\nu(1)}, x_{\nu(2)}] \cdots [x_{\nu(2k-1)}, x_{\nu(2k)}] \quad (32)$$

Furthermore, Furthermore the homogeneous polynomial of degree k on the right of (32) is harmonic and $R^k(\mathfrak{g})$ is the span of all such polynomials for an arbitrary choice of the x_i.

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