Verifying Kottwitz' conjecture by computer, II

Meinolf Geck

Universität Stuttgart

Salt Lake City, July 2013

B. Casselman, “Verifying Kottwitz’ conjecture by computer”;

R. Kottwitz, “Involutions in Weyl groups”.

(Both in *Representation Theory* **4**, 2000.)

B. Casselman, “Verifying Kottwitz’ conjecture by computer”;

R. Kottwitz, “Involutions in Weyl groups”.

(Both in *Representation Theory* **4**, 2000.)

W finite Weyl group, S simple reflections, $\mathcal{I} = \{w \in W \mid w^2 = 1\}$.
V $\mathbb{Q}$-vector space with basis $\{a_w \mid w \in \mathcal{I}\}$.

B. Casselman, "Verifying Kottwitz' conjecture by computer";

R. Kottwitz, "Involutions in Weyl groups".

(Both in *Representation Theory* **4**, 2000.)

W finite Weyl group, S simple reflections, $\mathcal{I} = \{w \in W \mid w^2 = 1\}$.

V $\mathbb{Q}$-vector space with basis $\{a_w \mid w \in \mathcal{I}\}$.

Kottwitz $\rightsquigarrow$ linear action of W on V:

$$s.a_w = \begin{cases} -a_w & \text{if } sw = ws \text{ and } l(sw) < l(w), \\ a_{sws} & \text{otherwise.} \end{cases}$$

(Formulation of Lusztig–Vogan, 2011.)

B. CASSELMAN, "Verifying Kottwitz' conjecture by computer";

R. KOTTWITZ, "Involutions in Weyl groups".

(Both in *Representation Theory* **4**, 2000.)

W finite Weyl group, S simple reflections, $\mathcal{I} = \{w \in W \mid w^2 = 1\}$.

V $\mathbb{Q}$-vector space with basis $\{a_w \mid w \in \mathcal{I}\}$.

Kottwitz $\rightsquigarrow$ linear action of W on V:

$$s.a_w = \begin{cases} -a_w & \text{if } sw = ws \text{ and } l(sw) < l(w), \\ a_{sws} & \text{otherwise.} \end{cases}$$

(Formulation of Lusztig–Vogan, 2011.)

$\mathrm{Cl}(\mathcal{I})$ conjugacy classes in $\mathcal{I}$. For $C \in \mathrm{Cl}(\mathcal{I})$ let $V_C = \langle a_w \mid w \in C \rangle$.

B. Casselman, “Verifying Kottwitz’ conjecture by computer”;

R. Kottwitz, “Involutions in Weyl groups”.

(Both in *Representation Theory* **4**, 2000.)

W finite Weyl group, S simple reflections, $\mathcal{I} = \{w \in W \mid w^2 = 1\}$.

V $\mathbb{Q}$-vector space with basis $\{a_w \mid w \in \mathcal{I}\}$.

Kottwitz $\rightsquigarrow$ linear action of W on V:

$$s.a_w = \begin{cases} -a_w & \text{if } sw = ws \text{ and } l(sw) < l(w), \\ a_{sws} & \text{otherwise.} \end{cases}$$

(Formulation of Lusztig–Vogan, 2011.)

$\mathrm{Cl}(\mathcal{I})$ conjugacy classes in $\mathcal{I}$. For $C \in \mathrm{Cl}(\mathcal{I})$ let $V_C = \langle a_w \mid w \in C \rangle$.

Then $V = \bigoplus_{C \in \mathrm{Cl}(\mathcal{I})} V_C$ and decomposition into irreducibles is known (Kottwitz: W classical; Casselman: W exceptional).

Since David Vogan briefly mentioned some remarkable properties of the decomposition of V into irreducibles, here are some details.

Since David Vogan briefly mentioned some remarkable properties of the decomposition of V into irreducibles, here are some details.

Let G be a connected reductive algebraic group with Weyl group W, defined and split over $\mathbb{F}_q$.

Since David Vogan briefly mentioned some remarkable properties of the decomposition of V into irreducibles, here are some details.

Let G be a connected reductive algebraic group with Weyl group W, defined and split over $\mathbb{F}_q$. For $w \in W$, let R_w be the virtual character of $G(\mathbb{F}_q)$ defined by Deligne–Lusztig.

Since David Vogan briefly mentioned some remarkable properties of the decomposition of V into irreducibles, here are some details.

Let G be a connected reductive algebraic group with Weyl group W, defined and split over $\mathbb{F}_q$. For $w \in W$, let R_w be the virtual character of $G(\mathbb{F}_q)$ defined by Deligne–Lusztig. For $E \in \mathrm{Irr}(W)$ let

$$R_E := |W|^{-1} \sum_{w \in W} \mathrm{Trace}(w, E)\, R_w.$$

Since David Vogan briefly mentioned some remarkable properties of the decomposition of V into irreducibles, here are some details.

Let G be a connected reductive algebraic group with Weyl group W, defined and split over $\mathbb{F}_q$. For $w \in W$, let R_w be the virtual character of $G(\mathbb{F}_q)$ defined by Deligne–Lusztig. For $E \in \mathrm{Irr}(W)$ let

$$R_E := |W|^{-1} \sum_{w \in W} \mathrm{Trace}(w, E)\, R_w.$$

Kottwitz (+ Casselman), Lusztig–Vogan (reformulation).

$$\langle V, E \rangle_W =$$

Since David Vogan briefly mentioned some remarkable properties of the decomposition of V into irreducibles, here are some details.

Let G be a connected reductive algebraic group with Weyl group W, defined and split over $\mathbb{F}_q$. For $w \in W$, let R_w be the virtual character of $G(\mathbb{F}_q)$ defined by Deligne–Lusztig. For $E \in \mathrm{Irr}(W)$ let

$$R_E := |W|^{-1}\sum_{w\in W} \mathrm{Trace}(w, E)\, R_w.$$

Kottwitz (+ Casselman), Lusztig–Vogan (reformulation).

$$\langle V, E\rangle_W = \sum_{\rho\in\mathrm{Irr}(G(\mathbb{F}_q))} \underbrace{\nu(\rho)}_{\text{Frobenius–Schur indicator}} \cdot \underbrace{\langle \rho, R_E\rangle_{G(\mathbb{F}_q)}}_{\text{Lusztig's orange book}}$$

Since David Vogan briefly mentioned some remarkable properties of the decomposition of V into irreducibles, here are some details.

Let G be a connected reductive algebraic group with Weyl group W, defined and split over $\mathbb{F}_q$. For $w \in W$, let R_w be the virtual character of $G(\mathbb{F}_q)$ defined by Deligne–Lusztig. For $E \in \mathrm{Irr}(W)$ let

$$R_E := |W|^{-1} \sum_{w \in W} \mathrm{Trace}(w, E)\, R_w.$$

Kottwitz (+ Casselman), Lusztig–Vogan (reformulation).

$$\langle V, E \rangle_W = \sum_{\rho \in \mathrm{Irr}(G(\mathbb{F}_q))} \underbrace{\nu(\rho)}_{\text{Frobenius–Schur indicator}} \cdot \underbrace{\langle \rho, R_E \rangle_{G(\mathbb{F}_q)}}_{\text{Lusztig's orange book}}$$

G.–Malle, Represent. Theory **17** (2013):

Extension to "twisted" involution module and non-split groups.

Kottwitz' Conjecture.

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma] \rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma] \rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma]\rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

1. Kottwitz: $V_C \cong \bigoplus_{\lambda \vdash n \,:\, w_{\lambda^*} \in C} E^\lambda$

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma] \rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

1. Kottwitz: $V_C \cong \bigoplus_{\lambda \vdash n \,:\, w_{\lambda^*} \in C} E^\lambda \quad \Rightarrow \quad V \cong \bigoplus_{\lambda \vdash n} E^\lambda.$

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma] \rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

1. Kottwitz: $V_C \cong \bigoplus_{\lambda \vdash n \,:\, w_{\lambda^*} \in C} E^\lambda \quad \Rightarrow \quad V \cong \bigoplus_{\lambda \vdash n} E^\lambda.$
2. Left cells given by Robinson–Schensted correspondence:

   $\{\Gamma_{\mathbb{T}} \mid \mathbb{T} \text{ standard } \lambda\text{-tableau for some } \lambda \vdash n\}.$

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma]\rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

1. Kottwitz: $V_C \cong \bigoplus_{\lambda \vdash n \,:\, w_{\lambda^*} \in C} E^\lambda \quad \Rightarrow \quad V \cong \bigoplus_{\lambda \vdash n} E^\lambda.$
2. Left cells given by Robinson–Schensted correspondence:

   $\{\Gamma_{\mathbb{T}} \mid \mathbb{T} \text{ standard } \lambda\text{-tableau for some } \lambda \vdash n\}.$
3. (1), (2) $\Rightarrow$ Both sides are 0 or 1.

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma] \rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

1. Kottwitz: $V_C \cong \bigoplus_{\lambda \vdash n \,:\, w_{\lambda^*} \in C} E^\lambda \quad \Rightarrow \quad V \cong \bigoplus_{\lambda \vdash n} E^\lambda.$
2. Left cells given by Robinson–Schensted correspondence:
   $\{\Gamma_{\mathbb{T}} \mid \mathbb{T} \text{ standard } \lambda\text{-tableau for some } \lambda \vdash n\}.$
3. (1), (2) $\Rightarrow$ Both sides are 0 or 1. These match up since:

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma]\rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

1. Kottwitz: $V_C \cong \bigoplus_{\lambda \vdash n\,:\, w_{\lambda^*} \in C} E^\lambda \quad \Rightarrow \quad V \cong \bigoplus_{\lambda \vdash n} E^\lambda.$
2. Left cells given by Robinson–Schensted correspondence:
   $\{\Gamma_{\mathbb{T}} \mid \mathbb{T} \text{ standard } \lambda\text{-tableau for some } \lambda \vdash n\}.$
3. (1), (2) $\Rightarrow$ Both sides are 0 or 1. These match up since:
   - $\mathbb{T}, \mathbb{T}'$ have shape $\lambda \quad \Rightarrow \quad [\Gamma_{\mathbb{T}}] \cong [\Gamma_{\mathbb{T}'}] \cong E^\lambda.$

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma] \rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

1. Kottwitz: $V_C \cong \bigoplus_{\lambda \vdash n \,:\, w_{\lambda^*} \in C} E^\lambda \quad \Rightarrow \quad V \cong \bigoplus_{\lambda \vdash n} E^\lambda.$
2. Left cells given by Robinson–Schensted correspondence:
   $\{\Gamma_{\mathbb{T}} \mid \mathbb{T} \text{ standard } \lambda\text{-tableau for some } \lambda \vdash n\}.$
3. (1), (2) $\Rightarrow$ Both sides are 0 or 1. These match up since:
   - $\mathbb{T}, \mathbb{T}'$ have shape $\lambda \quad \Rightarrow \quad [\Gamma_{\mathbb{T}}] \cong [\Gamma_{\mathbb{T}'}] \cong E^\lambda$.
   - All involutions in a given 2-sided cell are contained in one conjugacy class $C \in \mathrm{Cl}(\mathcal{I})$ (Schützenberger, 1976).

Kottwitz' Conjecture.

Let $C \in \mathrm{Cl}(\mathcal{I})$ and $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$\langle V_C, [\Gamma] \rangle_W = |C \cap \Gamma|, \quad \text{where} \quad [\Gamma] = W\text{-module carried by } \Gamma.$$

Example: $W = \mathfrak{S}_n$.

1. Kottwitz: $V_C \cong \bigoplus_{\lambda \vdash n \,:\, w_{\lambda^*} \in C} E^\lambda \quad \Rightarrow \quad V \cong \bigoplus_{\lambda \vdash n} E^\lambda$.
2. Left cells given by Robinson–Schensted correspondence:
   $\{\Gamma_{\mathbb{T}} \mid \mathbb{T} \text{ standard } \lambda\text{-tableau for some } \lambda \vdash n\}$.
3. (1), (2) $\Rightarrow$ Both sides are 0 or 1. These match up since:
   - $\mathbb{T}, \mathbb{T}'$ have shape $\lambda \quad \Rightarrow \quad [\Gamma_{\mathbb{T}}] \cong [\Gamma_{\mathbb{T}'}] \cong E^\lambda$.
   - All involutions in a given 2-sided cell are contained in one conjugacy class $C \in \mathrm{Cl}(\mathcal{I})$ (Schützenberger, 1976).
   - If $\mathbb{T}$ has shape λ and $C \cap \Gamma_{\mathbb{T}} \neq \varnothing$, then $w_{\lambda^*} \in C$.

Casselman: Checked conjecture for F_4 and E_6; similar methods: E_7.
(Explicit computation of all left cells possible in these cases.)

Casselman: Checked conjecture for F_4 and E_6; similar methods: E_7.
(Explicit computation of all left cells possible in these cases.)
Bonnafé and G. (`hal-00698613`, `arXiv:1206.0443`): B_n, D_n.

Casselman: Checked conjecture for F_4 and E_6; similar methods: E_7.

(Explicit computation of all left cells possible in these cases.)

Bonnafé and G. (`hal-00698613`, `arXiv:1206.0443`): B_n, D_n.

Remaining case: type E_8.

Casselman: Checked conjecture for F_4 and E_6; similar methods: E_7.
(Explicit computation of all left cells possible in these cases.)
Bonnafé and G. (hal-00698613, arXiv:1206.0443): B_n, D_n.

Remaining case: type E_8.

- $|\mathcal{I}| = 199\,952$ involutions, in 10 conjugacy classes;
- V_C for each $C \in \mathrm{Cl}(\mathcal{I})$ known from Casselman's computations.

Casselman: Checked conjecture for F_4 and E_6; similar methods: E_7.
(Explicit computation of all left cells possible in these cases.)
Bonnafé and G. (`hal-00698613`, `arXiv:1206.0443`): B_n, D_n.

Remaining case: type E_8.

- $|\mathcal{I}| = 199\,952$ involutions, in 10 conjugacy classes;
- V_C for each $C \in \mathrm{Cl}(\mathcal{I})$ known from Casselman's computations.
- 101 796 left cells, 46 two-sided cells.
- The vectors $\left(\langle[\Gamma], E\rangle_W\right)_{E\in \mathrm{Irr}(W)}$ are known (Lusztig 1986).

Casselman: Checked conjecture for F_4 and E_6; similar methods: E_7.
(Explicit computation of all left cells possible in these cases.)
Bonnafé and G. (`hal-00698613`, `arXiv:1206.0443`): B_n, D_n.

Remaining case: type E_8.

- $|\mathcal{I}| = 199\,952$ involutions, in 10 conjugacy classes;
- V_C for each $C \in \mathrm{Cl}(\mathcal{I})$ known from Casselman's computations.
- 101 796 left cells, 46 two-sided cells.
- The vectors $\big(\langle [\Gamma], E\rangle_W\big)_{E\in \mathrm{Irr}(W)}$ are known (Lusztig 1986).

Problem: We need to know partition of $\mathcal{I}$ into left cells, but E_8 is too big to compute systematically Kazhdan–Lusztig polynomials $P_{y,w}$.

Casselman: Checked conjecture for F_4 and E_6; similar methods: E_7.
(Explicit computation of all left cells possible in these cases.)
Bonnafé and G. (`hal-00698613`, `arXiv:1206.0443`): B_n, D_n.

Remaining case: type E_8.

- $|\mathcal{I}| = 199\,952$ involutions, in 10 conjugacy classes;
- V_C for each $C \in \mathrm{Cl}(\mathcal{I})$ known from Casselman's computations.
- 101 796 left cells, 46 two-sided cells.
- The vectors $\big(\langle[\Gamma], E\rangle_W\big)_{E\in\mathrm{Irr}(W)}$ are known (Lusztig 1986).

Problem: We need to know partition of $\mathcal{I}$ into left cells, but E_8 is too big to compute systematically Kazhdan–Lusztig polynomials $P_{y,w}$.

Main tool (both for handling B_n, D_n and E_8): Lusztig's theory of "Leading coefficients of character values of Hecke algebras" (1987).

H generic Iwahori–Hecke algebra of W over $\mathbb{Q}(v)$.

Basis $\{T_w \mid w \in W\}$; $T_s^2 = T_1 + (v - v^{-1})T_s$ for $s \in S$.

$\mathbf{H}$ generic Iwahori–Hecke algebra of W over $\mathbb{Q}(v)$.

Basis $\{T_w \mid w \in W\}$; $\qquad T_s^2 = T_1 + (v - v^{-1})T_s \quad$ for $s \in S$.

- Tits' Deformation Theorem: $v \to 1$ induces bijection

$$\boxed{\operatorname{Irr}(W) \longleftrightarrow \operatorname{Irr}(\mathbf{H}), \qquad E \leftrightarrow E_v}$$

$\mathbf{H}$ generic Iwahori–Hecke algebra of W over $\mathbb{Q}(v)$.

Basis $\{T_w \mid w \in W\}$; $\qquad T_s^2 = T_1 + (v - v^{-1})T_s \quad$ for $s \in S$.

- Tits' Deformation Theorem: $v \to 1$ induces bijection

$$\boxed{\operatorname{Irr}(W) \longleftrightarrow \operatorname{Irr}(\mathbf{H}), \qquad E \leftrightarrow E_v}$$

such that $\operatorname{Trace}(w, E) = \operatorname{Trace}(T_w, E_v)|_{v \to 1}$ for all $w \in W$.

H generic Iwahori–Hecke algebra of W over $\mathbb{Q}(v)$.

Basis $\{T_w \mid w \in W\}$; $\quad T_s^2 = T_1 + (v - v^{-1})T_s \quad$ for $s \in S$.

- Tits' Deformation Theorem: $v \to 1$ induces bijection

  $$\mathrm{Irr}(W) \longleftrightarrow \mathrm{Irr}(\mathbf{H}), \qquad E \leftrightarrow E_v$$

  such that $\mathrm{Trace}(w, E) = \mathrm{Trace}(T_w, E_v)|_{v \to 1}$ for all $w \in W$.

- $E \in \mathrm{Irr}(W) \;\rightsquigarrow\; D_E \in \mathbb{Q}[u]$ "generic degree" (where $u = v^2$).

  ($D_E(q)$ = dimension of principal series representation of $G(\mathbb{F}_q)$ corresponding to E.)

$\mathbf{H}$ generic Iwahori–Hecke algebra of W over $\mathbb{Q}(v)$.

Basis $\{T_w \mid w \in W\}$; $\qquad T_s^2 = T_1 + (v - v^{-1})T_s \quad$ for $s \in S$.

- Tits' Deformation Theorem: $v \to 1$ induces bijection

  $$\mathrm{Irr}(W) \longleftrightarrow \mathrm{Irr}(\mathbf{H}), \qquad E \leftrightarrow E_v$$

  such that $\mathrm{Trace}(w, E) = \mathrm{Trace}(T_w, E_v)|_{v\to 1}$ for all $w \in W$.

- $E \in \mathrm{Irr}(W) \rightsquigarrow D_E \in \mathbb{Q}[u]$ "generic degree" (where $u = v^2$).

  ($D_E(q)$ = dimension of principal series representation of $G(\mathbb{F}_q)$ corresponding to E.)

  Lusztig's $\mathbf{a}$-invariant:

  $$D_E = f_E^{-1}\, u^{\mathbf{a}_E} + \text{ higher powers of } u$$

  where $\mathbf{a}_E \geqslant 0$ and $f_E > 0$ are integers.

H generic Iwahori–Hecke algebra of W over $\mathbb{Q}(v)$.

Basis $\{T_w \mid w \in W\}$; $\quad T_s^2 = T_1 + (v - v^{-1})T_s \quad$ for $s \in S$.

- Tits' Deformation Theorem: $v \to 1$ induces bijection

  $$\mathrm{Irr}(W) \longleftrightarrow \mathrm{Irr}(\mathbf{H}), \qquad E \leftrightarrow E_v$$

  such that $\mathrm{Trace}(w, E) = \mathrm{Trace}(T_w, E_v)|_{v \to 1}$ for all $w \in W$.

- $E \in \mathrm{Irr}(W) \rightsquigarrow D_E \in \mathbb{Q}[u]$ "generic degree" (where $u = v^2$).

  ($D_E(q) =$ dimension of principal series representation of $G(\mathbb{F}_q)$ corresponding to E.)

  Lusztig's **a**-invariant:

  $$D_E = f_E^{-1}\, u^{\mathbf{a}_E} + \text{ higher powers of } u$$

  where $\mathbf{a}_E \geqslant 0$ and $f_E > 0$ are integers.

- D_E and, hence, $\mathbf{a}_E$ and f_E are explicitly known in all cases.

Now: $v^{\mathbf{a}_E}\,\mathrm{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

Now: $v^{\mathbf{a}_E}\,\mathrm{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

$$c_{w,E} := \text{constant term of } (-1)^{l(w)}\, v^{\mathbf{a}_E}\, \mathrm{Trace}(T_w, E_v)$$

The numbers $c_{w,E} \in \mathbb{Z}$ are Lusztig's "leading coefficients".

Now: $v^{\mathbf{a}_E} \mathrm{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

$$c_{w,E} := \text{constant term of } (-1)^{l(w)}\, v^{\mathbf{a}_E}\, \mathrm{Trace}(T_w, E_v)$$

The numbers $c_{w,E} \in \mathbb{Z}$ are Lusztig's "leading coefficients".

- **They behave as if there were character values of an algebra:**

Now: $v^{\mathbf{a}_E}\mathrm{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

$$c_{w,E} := \text{constant term of } (-1)^{l(w)}\, v^{\mathbf{a}_E}\, \mathrm{Trace}(T_w, E_v)$$

The numbers $c_{w,E} \in \mathbb{Z}$ are Lusztig's "leading coefficients".

- **They behave as if there were character values of an algebra:**

$$\sum_{w\in W} c_{w,E}\, c_{w,E'} = \begin{cases} f_E \dim E & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

Now: $v^{\mathbf{a}_E}\mathrm{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

$$c_{w,E} := \text{constant term of } (-1)^{l(w)}\, v^{\mathbf{a}_E}\, \mathrm{Trace}(T_w, E_v)$$

The numbers $c_{w,E} \in \mathbb{Z}$ are Lusztig's "leading coefficients".

- **They behave as if there were character values of an algebra:**

$$\sum_{w\in W} c_{w,E}\, c_{w,E'} = \begin{cases} f_E \dim E & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

- **There are not many** $w \in W$ **such that** $c_{w,E} \neq 0$:

Now: $v^{\mathbf{a}_E}\mathrm{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

$$c_{w,E} := \text{constant term of } (-1)^{l(w)}\, v^{\mathbf{a}_E}\, \mathrm{Trace}(T_w, E_v)$$

The numbers $c_{w,E} \in \mathbb{Z}$ are Lusztig's "leading coefficients".

- **They behave as if there were character values of an algebra:**

$$\sum_{w\in W} c_{w,E}\, c_{w,E'} = \begin{cases} f_E \dim E & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

- **There are not many $w \in W$ such that $c_{w,E} \neq 0$:**
    - $c_{w,E} \neq 0$ for some $E \quad \Leftrightarrow \quad w, w^{-1}$ are in the same left cell.

Now: $v^{\mathbf{a}_E}\operatorname{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

$$c_{w,E} := \text{constant term of } (-1)^{l(w)}\, v^{\mathbf{a}_E} \operatorname{Trace}(T_w, E_v)$$

The numbers $c_{w,E} \in \mathbb{Z}$ are Lusztig's "leading coefficients".

- **They behave as if there were character values of an algebra:**

$$\sum_{w\in W} c_{w,E}c_{w,E'} = \begin{cases} f_E \dim E & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

- **There are not many** $w \in W$ **such that** $c_{w,E} \neq 0$:
  - $c_{w,E} \neq 0$ for some $E \quad \Leftrightarrow \quad w, w^{-1}$ are in the same left cell.
  - In particular: $w \in \mathcal{I}$ involution $\quad \Rightarrow \quad c_{w,E} \neq 0$ for some E.

Now: $v^{\mathbf{a}_E}\,\mathrm{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

$$c_{w,E} := \text{constant term of } (-1)^{l(w)}\, v^{\mathbf{a}_E}\, \mathrm{Trace}(T_w, E_v)$$

The numbers $c_{w,E} \in \mathbb{Z}$ are Lusztig's "leading coefficients".

- **They behave as if there were character values of an algebra:**

$$\sum_{w \in W} c_{w,E}\, c_{w,E'} = \begin{cases} f_E \dim E & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

- **There are not many $w \in W$ such that $c_{w,E} \neq 0$:**
  - $c_{w,E} \neq 0$ for some E $\Leftrightarrow$ w, w^{-1} are in the same left cell.
  - In particular: $w \in \mathcal{I}$ involution $\Rightarrow$ $c_{w,E} \neq 0$ for some E.

- **Refined orthogonality relations:**

Now: $v^{\mathbf{a}_E}\mathrm{Trace}(T_w, E_v) \in \mathbb{Z}[v]$ for all $w \in W$.

$$c_{w,E} := \text{constant term of } (-1)^{l(w)}\, v^{\mathbf{a}_E}\, \mathrm{Trace}(T_w, E_v)$$

The numbers $c_{w,E} \in \mathbb{Z}$ are Lusztig's "leading coefficients".

- **They behave as if there were character values of an algebra:**

$$\sum_{w\in W} c_{w,E}c_{w,E'} = \begin{cases} f_E \dim E & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

- **There are not many** $w \in W$ **such that** $c_{w,E} \neq 0$:
    - $c_{w,E} \neq 0$ for some $E \quad \Leftrightarrow \quad w, w^{-1}$ are in the same left cell.
    - In particular: $w \in \mathcal{I}$ involution $\quad \Rightarrow \quad c_{w,E} \neq 0$ for some E.

- **Refined orthogonality relations:** Let Γ be a left cell.

$$\sum_{w\in \Gamma} c_{w,E}c_{w,E'} = \begin{cases} f_E\, \langle [\Gamma], E\rangle_W & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

Define graph with vertices $\mathcal{I}$.

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)"

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)" Can assume $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some E.

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)" Can assume $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some E. Let Γ, Γ' be the left cells such that $w \in \Gamma$, $w' \in \Gamma'$.

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)" Can assume $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some E.
Let Γ, Γ' be the left cells such that $w \in \Gamma$, $w' \in \Gamma'$.
Refined orthogonality relations:

$$\sum_{y \in \Gamma} (c_{y,E})^2 = f_E \, \langle [\Gamma], E \rangle_W$$

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)" Can assume $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some E.
Let Γ, Γ' be the left cells such that $w \in \Gamma$, $w' \in \Gamma'$.
Refined orthogonality relations:

$$\sum_{y\in\Gamma}(c_{y,E})^2 = f_E\,\langle[\Gamma], E\rangle_W \qquad \Rightarrow \qquad \langle[\Gamma], E\rangle_W \neq 0.$$

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)" Can assume $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some E.
Let Γ, Γ' be the left cells such that $w \in \Gamma$, $w' \in \Gamma'$.
Refined orthogonality relations:

$$\sum_{y\in\Gamma}(c_{y,E})^2 = f_E \langle[\Gamma], E\rangle_W \qquad \Rightarrow \qquad \langle[\Gamma], E\rangle_W \neq 0.$$

$$\sum_{y\in\Gamma'}(c_{y,E})^2 = f_E \langle[\Gamma'], E\rangle_W \qquad \Rightarrow \qquad \langle[\Gamma'], E\rangle_W \neq 0.$$

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)" Can assume $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some E.
Let Γ, Γ' be the left cells such that $w \in \Gamma$, $w' \in \Gamma'$.
Refined orthogonality relations:

$$\sum_{y \in \Gamma} (c_{y,E})^2 = f_E \, \langle [\Gamma], E \rangle_W \qquad \Rightarrow \qquad \langle [\Gamma], E \rangle_W \neq 0.$$

$$\sum_{y \in \Gamma'} (c_{y,E})^2 = f_E \, \langle [\Gamma'], E \rangle_W \qquad \Rightarrow \qquad \langle [\Gamma'], E \rangle_W \neq 0.$$

So $\langle [\Gamma], [\Gamma'] \rangle_W \neq 0$,

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)" Can assume $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some E.
Let Γ, Γ' be the left cells such that $w \in \Gamma$, $w' \in \Gamma'$.
Refined orthogonality relations:

$$\sum_{y\in\Gamma}(c_{y,E})^2 = f_E\,\langle[\Gamma], E\rangle_W \qquad \Rightarrow \qquad \langle[\Gamma], E\rangle_W \neq 0.$$
$$\sum_{y\in\Gamma'}(c_{y,E})^2 = f_E\,\langle[\Gamma'], E\rangle_W \qquad \Rightarrow \qquad \langle[\Gamma'], E\rangle_W \neq 0.$$

So $\langle[\Gamma], [\Gamma']\rangle_W \neq 0$, hence Γ, Γ' are contained in the same 2-sided cell.

Define graph with vertices $\mathcal{I}$. Two vertices $w \neq w'$ in $\mathcal{I}$ are connected if $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some $E \in \mathrm{Irr}(W)$.

Lemma. Let $w, w' \in \mathcal{I}$. The following are equivalent.

1. w, w' are in the same connected component of this graph.
2. w, w' are in the same 2-sided Kazhdan–Lusztig cell.

"(1) $\Rightarrow$ (2)" Can assume $c_{w,E} \neq 0$ and $c_{w',E} \neq 0$ for some E.
Let Γ, Γ' be the left cells such that $w \in \Gamma$, $w' \in \Gamma'$.
Refined orthogonality relations:

$$\sum_{y\in\Gamma}(c_{y,E})^2 = f_E \langle[\Gamma], E\rangle_W \qquad \Rightarrow \qquad \langle[\Gamma], E\rangle_W \neq 0.$$

$$\sum_{y\in\Gamma'}(c_{y,E})^2 = f_E \langle[\Gamma'], E\rangle_W \qquad \Rightarrow \qquad \langle[\Gamma'], E\rangle_W \neq 0.$$

So $\langle[\Gamma], [\Gamma']\rangle_W \neq 0$, hence Γ, Γ' are contained in the same 2-sided cell.
"(2) $\Rightarrow$ (1)" A bit more complicated but similar.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\Gamma \to \mathbb{Z}, \qquad w \mapsto l(w) - 2\deg P_{1,w},$

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\quad \Gamma \to \mathbb{Z}, \qquad w \mapsto l(w) - 2\deg P_{1,w},$

  reaches its minimum at exactly one element of Γ, denoted d_Γ.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\Gamma \to \mathbb{Z}, \qquad w \mapsto l(w) - 2\deg P_{1,w},$ reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\Gamma \to \mathbb{Z}, \qquad w \mapsto l(w) - 2\deg P_{1,w},$
  reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.
- The decomposition of $[\Gamma]$ into irreducibles is determined by:
  $$\langle [\Gamma], E\rangle_W = c_{d_\Gamma,E} \quad \text{for all } E \in \mathrm{Irr}(W).$$

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\Gamma \to \mathbb{Z}$, $\quad w \mapsto l(w) - 2\deg P_{1,w}$,
  reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.
- The decomposition of $[\Gamma]$ into irreducibles is determined by:
  $$\langle [\Gamma], E\rangle_W = c_{d_\Gamma,E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

We set $\mathcal{D} := \{d_\Gamma \mid \Gamma \text{ left cell of } W\}$ "distinguished involutions".

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\quad \Gamma \to \mathbb{Z}, \qquad w \mapsto l(w) - 2\deg P_{1,w},$
  reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.
- The decomposition of $[\Gamma]$ into irreducibles is determined by:
  $$\langle [\Gamma], E\rangle_W = c_{d_\Gamma, E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

We set $\mathcal{D} := \{d_\Gamma \mid \Gamma \text{ left cell of } W\}$ "distinguished involutions".

Consequence: Each $w \in \mathcal{I}$ is directly connected to some $d \in \mathcal{D}$.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\Gamma \to \mathbb{Z}, \qquad w \mapsto l(w) - 2\deg P_{1,w},$
  reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.
- The decomposition of $[\Gamma]$ into irreducibles is determined by:
  $$\langle [\Gamma], E\rangle_W = c_{d_\Gamma,E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

We set $\mathcal{D} := \{d_\Gamma \mid \Gamma \text{ left cell of } W\}$ "distinguished involutions".

Consequence: Each $w \in \mathcal{I}$ is directly connected to some $d \in \mathcal{D}$.

Let Γ be the left cell such that $w \in \Gamma$.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\Gamma \to \mathbb{Z}$, $w \mapsto l(w) - 2\deg P_{1,w}$, reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.
- The decomposition of $[\Gamma]$ into irreducibles is determined by:

$$\langle [\Gamma], E\rangle_W = c_{d_\Gamma,E} \quad \text{for all } E \in \mathrm{Irr}(W).$$

We set $\mathcal{D} := \{d_\Gamma \mid \Gamma \text{ left cell of } W\}$ "distinguished involutions".

Consequence: Each $w \in \mathcal{I}$ is directly connected to some $d \in \mathcal{D}$.

Let Γ be the left cell such that $w \in \Gamma$.

Let $E \in \mathrm{Irr}(W)$ be such that $c_{w,E} \neq 0$. As before: $\langle [\Gamma], E\rangle_W \neq 0$.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\Gamma \to \mathbb{Z}$, $w \mapsto l(w) - 2\deg P_{1,w}$, reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.
- The decomposition of $[\Gamma]$ into irreducibles is determined by:

$$\langle [\Gamma], E\rangle_W = c_{d_\Gamma,E} \quad \text{for all } E \in \mathrm{Irr}(W).$$

We set $\mathcal{D} := \{d_\Gamma \mid \Gamma \text{ left cell of } W\}$ "distinguished involutions".

Consequence: Each $w \in \mathcal{I}$ is directly connected to some $d \in \mathcal{D}$.

Let Γ be the left cell such that $w \in \Gamma$.

Let $E \in \mathrm{Irr}(W)$ be such that $c_{w,E} \neq 0$. As before: $\langle [\Gamma], E\rangle_W \neq 0$.

Then Theorem shows $c_{d_\Gamma,E} \neq 0$.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\Gamma \to \mathbb{Z}$, $w \mapsto l(w) - 2\deg P_{1,w}$, reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.
- The decomposition of $[\Gamma]$ into irreducibles is determined by:
$$\langle [\Gamma], E\rangle_W = c_{d_\Gamma,E} \quad \text{for all } E \in \mathrm{Irr}(W).$$

We set $\mathcal{D} := \{d_\Gamma \mid \Gamma \text{ left cell of } W\}$ "distinguished involutions".

Consequence: Each $w \in \mathcal{I}$ is directly connected to some $d \in \mathcal{D}$.

Let Γ be the left cell such that $w \in \Gamma$.

Let $E \in \mathrm{Irr}(W)$ be such that $c_{w,E} \neq 0$. As before: $\langle [\Gamma], E\rangle_W \neq 0$.

Then Theorem shows $c_{d_\Gamma,E} \neq 0$. So w, d_Γ are connected in the graph.

Theorem (Lusztig 1986). Let Γ be a left cell in W.

- The function $\quad \Gamma \to \mathbb{Z}, \qquad w \mapsto l(w) - 2\deg P_{1,w},$
  reaches its minimum at exactly one element of Γ, denoted d_Γ.
- The element d_Γ is an involution.
- The decomposition of $[\Gamma]$ into irreducibles is determined by:
  $$\langle [\Gamma], E\rangle_W = c_{d_\Gamma, E} \quad \text{for all } E \in \mathrm{Irr}(W).$$

We set $\mathcal{D} := \{d_\Gamma \mid \Gamma \text{ left cell of } W\}$ "distinguished involutions".

Consequence: Each $w \in \mathcal{I}$ is directly connected to some $d \in \mathcal{D}$.

Let Γ be the left cell such that $w \in \Gamma$.

Let $E \in \mathrm{Irr}(W)$ be such that $c_{w,E} \neq 0$. As before: $\langle [\Gamma], E\rangle_W \neq 0$.

Then Theorem shows $c_{d_\Gamma, E} \neq 0$. So w, d_Γ are connected in the graph.

(Implication "(2) $\Rightarrow$ (1)" of Lemma follows easily from this.)

Summary. The leading coefficients $c_{w,E}$ determine:

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.
- The set of distinguished involutions:

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.
- The set of distinguished involutions:

$$\mathcal{D} = \Big\{ w \in W \;\Big|\; \sum_{E \in \mathrm{Irr}(W)} f_E^{-1} \, c_{w,E} \neq 0 \Big\}.$$

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.
- The set of distinguished involutions:

$$\mathcal{D} = \left\{ w \in W \;\middle|\; \sum_{E \in \mathrm{Irr}(W)} f_E^{-1} c_{w,E} \neq 0 \right\}.$$

- Given a left cell Γ, the decomposition of $[\Gamma]$ into irreducibles:

$$\langle [\Gamma], E \rangle_E = c_{d_\Gamma, E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.
- The set of distinguished involutions:

$$\mathcal{D} = \Big\{ w \in W \;\Big|\; \sum_{E \in \mathrm{Irr}(W)} f_E^{-1}\, c_{w,E} \neq 0 \Big\}.$$

- Given a left cell Γ, the decomposition of $[\Gamma]$ into irreducibles:

$$\langle [\Gamma], E \rangle_E = c_{d_\Gamma, E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

Theorem (Bonnafé–G., 2012). Assume W is of classical type.

Let $C \in \mathrm{Cl}(\mathcal{I})$.

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.
- The set of distinguished involutions:
$$\mathcal{D} = \Big\{ w \in W \;\Big|\; \sum_{E \in \mathrm{Irr}(W)} f_E^{-1}\, c_{w,E} \neq 0 \Big\}.$$
- Given a left cell Γ, the decomposition of $[\Gamma]$ into irreducibles:
$$\langle [\Gamma], E \rangle_E = c_{d_\Gamma, E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

Theorem (Bonnafé–G., 2012). Assume W is of classical type.

Let $C \in \mathrm{Cl}(\mathcal{I})$. Let $\mathcal{F}$ be a 2-sided cell and $E_0 \in \mathrm{Irr}(W)$ be the unique "special" representation corresponding to $\mathcal{F}$. Then

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.
- The set of distinguished involutions:

$$\mathcal{D} = \left\{ w \in W \;\middle|\; \sum_{E \in \mathrm{Irr}(W)} f_E^{-1} c_{w,E} \neq 0 \right\}.$$

- Given a left cell Γ, the decomposition of $[\Gamma]$ into irreducibles:

$$\langle [\Gamma], E \rangle_E = c_{d_\Gamma, E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

Theorem (Bonnafé–G., 2012). Assume W is of classical type.

Let $C \in \mathrm{Cl}(\mathcal{I})$. Let $\mathcal{F}$ be a 2-sided cell and $E_0 \in \mathrm{Irr}(W)$ be the unique "special" representation corresponding to $\mathcal{F}$. Then

$$|C \cap \Gamma| \dim E_0 = |C \cap \mathcal{F}| \qquad \text{for every left cell } \Gamma \subseteq \mathcal{F},$$

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.
- The set of distinguished involutions:

$$\mathcal{D} = \Big\{ w \in W \;\Big|\; \sum_{E \in \mathrm{Irr}(W)} f_E^{-1}\, c_{w,E} \neq 0 \Big\}.$$

- Given a left cell Γ, the decomposition of $[\Gamma]$ into irreducibles:

$$\langle [\Gamma], E \rangle_E = c_{d_\Gamma, E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

Theorem (Bonnafé–G., 2012). Assume W is of classical type.

Let $C \in \mathrm{Cl}(\mathcal{I})$. Let $\mathcal{F}$ be a 2-sided cell and $E_0 \in \mathrm{Irr}(W)$ be the unique "special" representation corresponding to $\mathcal{F}$. Then

$$|C \cap \Gamma| \dim E_0 = |C \cap \mathcal{F}| \qquad \text{for every left cell } \Gamma \subseteq \mathcal{F},$$

and this is the key to proving Kottwitz' conjecture for classical type.

Summary. The leading coefficients $c_{w,E}$ determine:

- The partition of the set of involutions $\mathcal{I}$ into 2-sided cells.
- The set of distinguished involutions:

$$\mathcal{D} = \Big\{ w \in W \;\Big|\; \sum_{E \in \mathrm{Irr}(W)} f_E^{-1} c_{w,E} \neq 0 \Big\}.$$

- Given a left cell Γ, the decomposition of $[\Gamma]$ into irreducibles:

$$\langle [\Gamma], E \rangle_E = c_{d_\Gamma, E} \qquad \text{for all } E \in \mathrm{Irr}(W).$$

Theorem (Bonnafé–G., 2012). Assume W is of classical type.

Let $C \in \mathrm{Cl}(\mathcal{I})$. Let $\mathcal{F}$ be a 2-sided cell and $E_0 \in \mathrm{Irr}(W)$ be the unique "special" representation corresponding to $\mathcal{F}$. Then

$$|C \cap \Gamma| \dim E_0 = |C \cap \mathcal{F}| \qquad \text{for every left cell } \Gamma \subseteq \mathcal{F},$$

and this is the key to proving Kottwitz' conjecture for classical type. (Global identity $|\mathcal{I} \cap \Gamma| \dim E_0 = |\mathcal{I} \cap \mathcal{F}|$ due to Lusztig, 1985.)

Theorem (Abbie Halls, 2012). Let W be of type E_8.

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w'$$

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ \\ \end{cases}$$

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ \end{cases}$$

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ w, w' \text{ have same generalized } \tau\text{-invariant.} \end{cases}$$

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ w, w' \text{ have same generalized } \tau\text{-invariant.} \end{cases}$$

(2) Kottwitz's conjecture holds for W.

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ w, w' \text{ have same generalized } \tau\text{-invariant.} \end{cases}$$

(2) Kottwitz's conjecture holds for W.

- Need to know all leading coefficients $c_{w,E}$ for $w \in \mathcal{I}$.

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ w, w' \text{ have same generalized } \tau\text{-invariant.} \end{cases}$$

(2) Kottwitz's conjecture holds for W.

- Need to know all leading coefficients $c_{w,E}$ for $w \in \mathcal{I}$.
- $\left|\{w \in W \mid c_{w,E} \neq 0 \text{ for some } E\}\right|$

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ w, w' \text{ have same generalized } \tau\text{-invariant.} \end{cases}$$

(2) Kottwitz's conjecture holds for W.

- Need to know all leading coefficients $c_{w,E}$ for $w \in \mathcal{I}$.
- $|\{w \in W \mid c_{w,E} \neq 0 \text{ for some } E\}| = \sum_\Gamma \langle [\Gamma], [\Gamma] \rangle_W = 208\,422$.

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ w, w' \text{ have same generalized } \tau\text{-invariant.} \end{cases}$$

(2) Kottwitz's conjecture holds for W.

- Need to know all leading coefficients $c_{w,E}$ for $w \in \mathcal{I}$.
- $|\{w \in W \mid c_{w,E} \neq 0 \text{ for some } E\}| = \sum_\Gamma \langle [\Gamma], [\Gamma] \rangle_W = 208\,422$.
- $|\operatorname{Irr}(W)| = 112$. So aiming at 208422×112 matrix of integers.

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ w, w' \text{ have same generalized } \tau\text{-invariant.} \end{cases}$$

(2) Kottwitz's conjecture holds for W.

- Need to know all leading coefficients $c_{w,E}$ for $w \in \mathcal{I}$.
- $|\{w \in W \mid c_{w,E} \neq 0 \text{ for some } E\}| = \sum_\Gamma \langle [\Gamma], [\Gamma] \rangle_W = 208\,422$.
- $|\operatorname{Irr}(W)| = 112$. So aiming at 208422×112 matrix of integers.
- Lusztig (1987): $|c_{w,E}| \leqslant 8$ for all w, E. (Determined above matrix but without identifying elements of W labelling the rows.)

Theorem (Abbie Halls, 2012). Let W be of type E_8.

(1) Let $w, w' \in \mathcal{I}$ be involutions. Then:

$$w \sim_L w' \quad \Leftrightarrow \quad \begin{cases} w, w' \text{ in same component of graph,} \\ w_J \sim_L w'_J \text{ for every proper } J \subset S, \\ w, w' \text{ have same generalized } \tau\text{-invariant.} \end{cases}$$

(2) Kottwitz's conjecture holds for W.

- Need to know all leading coefficients $c_{w,E}$ for $w \in \mathcal{I}$.
- $|\{w \in W \mid c_{w,E} \neq 0 \text{ for some } E\}| = \sum_\Gamma \langle [\Gamma], [\Gamma] \rangle_W = 208\,422$.
- $|\operatorname{Irr}(W)| = 112$. So aiming at 208422×112 matrix of integers.
- Lusztig (1987): $|c_{w,E}| \leqslant 8$ for all w, E. (Determined above matrix but without identifying elements of W labelling the rows.)
- Computation of this 208422×112 matrix $(c_{w,E})$ takes about 3 weeks and 24 GB of main memory.

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C \in \text{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).

For $y, w \in W$ write $w \to y$ for transitive closure of:

"$y = sws$ for some $s \in S$ where $l(y) \leqslant l(w)$."

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).

For $y, w \in W$ write $w \to y$ for transitive closure of:

$$\text{“ } y = sws \text{ for some } s \in S \text{ where } l(y) \leqslant l(w).\text{ ”}$$

Theorem (G.–Pfeiffer, 1993). Let $C \in \mathrm{Cl}(W)$.

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).

For $y, w \in W$ write $w \to y$ for transitive closure of:

$$\text{“ } y = sws \text{ for some } s \in S \text{ where } l(y) \leqslant l(w). \text{ ”}$$

Theorem (G.–Pfeiffer, 1993). Let $C \in \mathrm{Cl}(W)$.

1. Let $w, w' \in C_{\min}$. Then $T_w, T_{w'} \in \mathbf{H}$ are conjugate in $\mathbf{H}$.

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).

For $y, w \in W$ write $w \to y$ for transitive closure of:

“ $y = sws$ for some $s \in S$ where $l(y) \leqslant l(w)$. ”

Theorem (G.–Pfeiffer, 1993). Let $C \in \mathrm{Cl}(W)$.

1. Let $w, w' \in C_{\min}$. Then $T_w, T_{w'} \in \mathbf{H}$ are conjugate in $\mathbf{H}$.
2. Let $w \in C$. Then there exists some $y \in C_{\min}$ such that $w \to y$.

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).

For $y, w \in W$ write $w \to y$ for transitive closure of:

$$\text{“ } y = sws \text{ for some } s \in S \text{ where } l(y) \leqslant l(w). \text{ ”}$$

Theorem (G.–Pfeiffer, 1993). Let $C \in \mathrm{Cl}(W)$.

1. Let $w, w' \in C_{\min}$. Then $T_w, T_{w'} \in \mathbf{H}$ are conjugate in $\mathbf{H}$.
2. Let $w \in C$. Then there exists some $y \in C_{\min}$ such that $w \to y$.

(1) $\rightsquigarrow$ Fix $w_C \in C_{\min}$ for all $C \in \mathrm{Cl}(W)$.

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.
Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).
For $y, w \in W$ write $w \to y$ for transitive closure of:

$$\text{“ } y = sws \text{ for some } s \in S \text{ where } l(y) \leqslant l(w). \text{ ”}$$

Theorem (G.–Pfeiffer, 1993). Let $C \in \mathrm{Cl}(W)$.

1. Let $w, w' \in C_{\min}$. Then $T_w, T_{w'} \in \mathbf{H}$ are conjugate in $\mathbf{H}$.
2. Let $w \in C$. Then there exists some $y \in C_{\min}$ such that $w \to y$.

(1) $\rightsquigarrow$ Fix $w_C \in C_{\min}$ for all $C \in \mathrm{Cl}(W)$. Character table:

$$X(\mathbf{H}) := \big(\mathrm{Trace}(T_{w_C}, E_v)\big)_{E \in \mathrm{Irr}(W),\, C \in \mathrm{Cl}(W)}.$$

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).

For $y, w \in W$ write $w \to y$ for transitive closure of:

$$\text{“ } y = sws \text{ for some } s \in S \text{ where } l(y) \leqslant l(w).\text{ ”}$$

Theorem (G.–Pfeiffer, 1993). Let $C \in \mathrm{Cl}(W)$.

1. Let $w, w' \in C_{\min}$. Then $T_w, T_{w'} \in \mathbf{H}$ are conjugate in $\mathbf{H}$.
2. Let $w \in C$. Then there exists some $y \in C_{\min}$ such that $w \to y$.

(1) $\rightsquigarrow$ Fix $w_C \in C_{\min}$ for all $C \in \mathrm{Cl}(W)$. Character table:

$$X(\mathbf{H}) := \Big(\mathrm{Trace}(T_{w_C}, E_v)\Big)_{E \in \mathrm{Irr}(W),\, C \in \mathrm{Cl}(W)}.$$

(2) $\rightsquigarrow$ For any $w \in W$,

Let $C \in \mathrm{Cl}(W)$ and set $d_C := \min\{l(w) \mid w \in C\}$.

Let $C_{\min} := \{w \in C \mid l(w) = d_C\}$ (elements of minimal length in C).

For $y, w \in W$ write $w \to y$ for transitive closure of:

$$\text{“ } y = sws \text{ for some } s \in S \text{ where } l(y) \leqslant l(w). \text{ ”}$$

Theorem (G.–Pfeiffer, 1993). Let $C \in \mathrm{Cl}(W)$.

1. Let $w, w' \in C_{\min}$. Then $T_w, T_{w'} \in \mathbf{H}$ are conjugate in $\mathbf{H}$.
2. Let $w \in C$. Then there exists some $y \in C_{\min}$ such that $w \to y$.

(1) $\rightsquigarrow$ Fix $w_C \in C_{\min}$ for all $C \in \mathrm{Cl}(W)$. Character table:

$$X(\mathbf{H}) := \big(\mathrm{Trace}(T_{w_C}, E_v)\big)_{E \in \mathrm{Irr}(W),\, C \in \mathrm{Cl}(W)}.$$

(2) $\rightsquigarrow$ For any $w \in W$, there are unique $f_{w,C} \in \mathbb{Z}[v, v^{-1}]$ such that

$$T_w \equiv \sum_{C \in \mathrm{Cl}(W)} f_{w,C}\, T_{w_C} \mod [\mathbf{H}, \mathbf{H}].$$

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

	T_1	T_{s_1}	$T_{s_1 s_2}$
$E_v^{(3)}$	1	v	v^2
$E_v^{(21)}$	2	$v - v^{-1}$	-1
$E_v^{(111)}$	1	$-v^{-1}$	v^{-2}

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

	T_1	T_{s_1}	$T_{s_1 s_2}$
$E_v^{(3)}$	1	v	v^2
$E_v^{(21)}$	2	$v - v^{-1}$	-1
$E_v^{(111)}$	1	$-v^{-1}$	v^{-2}

All these tables in: `http://www.math.rwth-aachen.de/~CHEVIE/`

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

	T_1	T_{s_1}	$T_{s_1 s_2}$
$E_v^{(3)}$	1	v	v^2
$E_v^{(21)}$	2	$v-v^{-1}$	-1
$E_v^{(111)}$	1	$-v^{-1}$	v^{-2}

All these tables in: `http://www.math.rwth-aachen.de/~CHEVIE/`

Let $w = s_1 s_2 s_1$.

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

	T_1	T_{s_1}	$T_{s_1 s_2}$
$E_v^{(3)}$	1	v	v^2
$E_v^{(21)}$	2	$v - v^{-1}$	-1
$E_v^{(111)}$	1	$-v^{-1}$	v^{-2}

All these tables in: `http://www.math.rwth-aachen.de/~CHEVIE/`

Let $w = s_1 s_2 s_1$. Then $w \to s_2$ via conjugation with s_1; furthermore, s_2 conjugate to s_1.

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

	T_1	T_{s_1}	$T_{s_1 s_2}$
$E_v^{(3)}$	1	v	v^2
$E_v^{(21)}$	2	$v-v^{-1}$	-1
$E_v^{(111)}$	1	$-v^{-1}$	v^{-2}

All these tables in: `http://www.math.rwth-aachen.de/~CHEVIE/`

Let $w = s_1 s_2 s_1$. Then $w \to s_2$ via conjugation with s_1; furthermore, s_2 conjugate to s_1. So

$$T_{s_1 s_2 s_1} \equiv T_{s_1} + (v - v^{-1}) T_{s_1 s_2} \quad \text{mod } [\mathbf{H}, \mathbf{H}].$$

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

	T_1	T_{s_1}	$T_{s_1 s_2}$
$E_v^{(3)}$	1	v	v^2
$E_v^{(21)}$	2	$v - v^{-1}$	-1
$E_v^{(111)}$	1	$-v^{-1}$	v^{-2}

All these tables in: `http://www.math.rwth-aachen.de/~CHEVIE/`

Let $w = s_1 s_2 s_1$. Then $w \to s_2$ via conjugation with s_1; furthermore, s_2 conjugate to s_1. So

$$T_{s_1 s_2 s_1} \equiv T_{s_1} + (v - v^{-1}) T_{s_1 s_2} \quad \mod [\mathbf{H}, \mathbf{H}].$$

Hence:

$\text{Trace}(T_{s_1 s_2 s_1}, E_v^{(21)}) =$

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

	T_1	T_{s_1}	$T_{s_1 s_2}$
$E_v^{(3)}$	1	v	v^2
$E_v^{(21)}$	2	$v-v^{-1}$	-1
$E_v^{(111)}$	1	$-v^{-1}$	v^{-2}

All these tables in: `http://www.math.rwth-aachen.de/~CHEVIE/`

Let $w = s_1 s_2 s_1$. Then $w \to s_2$ via conjugation with s_1; furthermore, s_2 conjugate to s_1. So

$$T_{s_1 s_2 s_1} \equiv T_{s_1} + (v - v^{-1}) T_{s_1 s_2} \mod [\mathbf{H}, \mathbf{H}].$$

Hence:

$$\mathrm{Trace}(T_{s_1 s_2 s_1}, E_v^{(21)}) = v - v^{-1} + (v - v^{-1})(-1) = 0$$

Example. $W(A_2) = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$. Character tables:

	$\mathbf{a}_E$	1	s_1	$s_1 s_2$
$E^{(3)}$	0	1	1	1
$E^{(21)}$	1	2	0	-1
$E^{(111)}$	3	1	-1	1

	T_1	T_{s_1}	$T_{s_1 s_2}$
$E_v^{(3)}$	1	v	v^2
$E_v^{(21)}$	2	$v-v^{-1}$	-1
$E_v^{(111)}$	1	$-v^{-1}$	v^{-2}

All these tables in: `http://www.math.rwth-aachen.de/~CHEVIE/`

Let $w = s_1 s_2 s_1$. Then $w \to s_2$ via conjugation with s_1; furthermore, s_2 conjugate to s_1. So

$$T_{s_1 s_2 s_1} \equiv T_{s_1} + (v - v^{-1}) T_{s_1 s_2} \quad \mod [\mathbf{H}, \mathbf{H}].$$

Hence:

$$\text{Trace}(T_{s_1 s_2 s_1}, E_v^{(21)}) = v - v^{-1} + (v - v^{-1})(-1) = 0 \rightsquigarrow c_{s_1 s_2 s_1, E^{(21)}} = 0.$$

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\operatorname{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then

   $f_{w,C} = f_{y,C}$

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then
$$f_{w,C} = f_{y,C} = f_{sys,C} + (v - v^{-1}) f_{sy,C} \qquad \text{for all } C \in \mathrm{Cl}(W).$$

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then

$$f_{w,C} = f_{y,C} = f_{sys,C} + (v - v^{-1})f_{sy,C} \qquad \text{for all } C \in \mathrm{Cl}(W).$$

Note: $l(sy) = l(w) - 1$ and $l(sys) = l(w) - 2$. Apply induction.

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then
$$f_{w,C} = f_{y,C} = f_{sys,C} + (v - v^{-1}) f_{sy,C} \qquad \text{for all } C \in \mathrm{Cl}(W).$$
Note: $l(sy) = l(w) - 1$ and $l(sys) = l(w) - 2$. Apply induction.
2. $w \in C'_{\min}$ for some $C' \in \mathrm{Cl}(W)$.

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then
$$f_{w,C} = f_{y,C} = f_{sys,C} + (v - v^{-1})f_{sy,C} \qquad \text{for all } C \in \mathrm{Cl}(W).$$
Note: $l(sy) = l(w) - 1$ and $l(sys) = l(w) - 2$. Apply induction.
2. $w \in C'_{\min}$ for some $C' \in \mathrm{Cl}(W)$. Identify C'; then $f_{w,C} = \delta_{C,C'}$.

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then

$$f_{w,C} = f_{y,C} = f_{sys,C} + (v - v^{-1})f_{sy,C} \qquad \text{for all } C \in \mathrm{Cl}(W).$$

   Note: $l(sy) = l(w) - 1$ and $l(sys) = l(w) - 2$. Apply induction.

2. $w \in C'_{\min}$ for some $C' \in \mathrm{Cl}(W)$. Identify C'; then $f_{w,C} = \delta_{C,C'}$.

If $l(w) = m$, "only" need to know all $f_{y,C}$ where $l(y) = m-1, m-2$.

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then
$$f_{w,C} = f_{y,C} = f_{sys,C} + (v - v^{-1})f_{sy,C} \qquad \text{for all } C \in \mathrm{Cl}(W).$$
Note: $l(sy) = l(w) - 1$ and $l(sys) = l(w) - 2$. Apply induction.
2. $w \in C'_{\min}$ for some $C' \in \mathrm{Cl}(W)$. Identify C'; then $f_{w,C} = \delta_{C,C'}$.

If $l(w) = m$, "only" need to know all $f_{y,C}$ where $l(y) = m-1, m-2$.

Worst case in E_8: 18 210 722 elements of length 60.

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then

   $$f_{w,C} = f_{y,C} = f_{sys,C} + (v - v^{-1}) f_{sy,C} \qquad \text{for all } C \in \mathrm{Cl}(W).$$

   Note: $l(sy) = l(w) - 1$ and $l(sys) = l(w) - 2$. Apply induction.

2. $w \in C'_{\min}$ for some $C' \in \mathrm{Cl}(W)$. Identify C'; then $f_{w,C} = \delta_{C,C'}$.

If $l(w) = m$, "only" need to know all $f_{y,C}$ where $l(y) = m-1, m-2$.

Worst case in E_8: 18 210 722 elements of length 60.

A priori $\approx 112 \times 2 \times 18\,000\,000 \approx 4 \times 10^9$ polynomials required.

Recursive computation of the polynomials $f_{w,C}$ (induction on $l(w)$).

Given $w \in W$, compute the "cyclic shift orbit"

$$\mathrm{Cyc}(w) = \{y \in W \mid w \to y \text{ and } l(w) = l(y)\}.$$

1. For some $y \in \mathrm{Cyc}(w)$ and $s \in S$ we have $l(sys) < l(y)$. Then

$$f_{w,C} = f_{y,C} = f_{sys,C} + (v - v^{-1}) f_{sy,C} \qquad \text{for all } C \in \mathrm{Cl}(W).$$

   Note: $l(sy) = l(w) - 1$ and $l(sys) = l(w) - 2$. Apply induction.

2. $w \in C'_{\min}$ for some $C' \in \mathrm{Cl}(W)$. Identify C'; then $f_{w,C} = \delta_{C,C'}$.

If $l(w) = m$, "only" need to know all $f_{y,C}$ where $l(y) = m-1, m-2$.

Worst case in E_8: 18 210 722 elements of length 60.

A priori $\approx 112 \times 2 \times 18\,000\,000 \approx 4 \times 10^9$ polynomials required.

Use: $f_{w,C} = f_{w',C}$ if $w' \in \mathrm{Cyc}(w)$. Gain factor ≈ 100.

```
Python 2.7.4 (default, Apr 19 2013, 18:28:01)
[GCC 4.7.3] on linux2
###################################################################
##  A PYTHON VERSION OF CHEVIE-GAP FOR (FINITE) COXETER GROUPS   ##
##      (by Meinolf Geck,  version 1r6p18,  20 Dec 2012)         ##
##                                                               ##
##  To get started type  "help(coxeter)" or  "allfunctions()";   ##
##  see also http://dx.doi.org/10.1112/S1461157012001064.        ##
##  For notes about this version type  "versioninfo(1.6)".       ##
##  Check www.mathematik.uni-stuttgart.de/~geckmf for updates.   ##
##                                                               ##
##  Import into "sage" (4.7 or higher, www.sagemath.org) works.  ##
##                                                               ##
##       The proposed name for this module is "PyCox".           ##
##                    All comments welcome!                      ##
###################################################################
```