Two triangularity results and invariants of $(\mathfrak{g}, Sp(p) \times Sp(q))$ modules

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- G compact Lie group.

Irred. rep of G are "determined" by their characters.

Roughly: Understanding the characters of irred. rep. means that the Representation Theory of G is understood.

Weaker Invariant: dimension of the rep.

- G simple Lie group.

Global character of irred. rep. (infinite dimensional) has been define by Harish-Chandra (difficult to compute).

A number of other invariants contain relevant info about irred. rep.

Examples: Associated variety, Associated cycle, Annihilator, Characteristic cycles (can we compute some of them?)

- G is a complex simple algebraic group with involution θ.
- $K = G^\theta$. The real form defined by θ has a maximal compact subgroup with complexification K. Write $G_\mathbb{R}, K_\mathbb{R}$ for the real points.
- $\mathcal{N}$ denotes the nilpotent cone.
- We denote by $X = \{\mathfrak{b} \subset \mathfrak{g} \text{ Borel subalgebra}\}$ the flag variety. (All borel subalg. are G-conjugate. Hence if $B = N_G(\mathfrak{b})$ then $X \simeq G/B$). $(\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n})$
- The cotangent bundle is $T^*X = \{(\mathfrak{b}, \xi) : \xi \in (\mathfrak{g}/\mathfrak{b})^*\}$.
- The moment map for the Hamiltonian action of G on $T^*(X)$ is

$$\mu : T^*X = \{(\mathfrak{b}, \xi) : \xi \in (\mathfrak{g}/\mathfrak{b})^*\} \to \mathcal{N}$$
$$(\mathfrak{b}, \xi) \to \xi.$$

$\mathcal{M}_\rho(\mathfrak{g}, B)$

- Let $\mathcal{M}_\rho(\mathfrak{g}, B)$ be the abelian category of f. generated B-finite $(\mathfrak{g}, B)$-modules with inf. char. ρ. [The Grothendieck group $\mathcal{K}(\mathcal{M}_\rho(\mathfrak{g}, B)) = \oplus \mathbb{Z} L_w$.]

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- $\{B\text{-orbits on } X\} = \{X(w)\}_{w \in W}$ (Shubert cells)

The characteristic variety is a union of several conormal bundles to B-orbits in X:

$$CV(M) = \cup \overline{T^*_{X(y)} X} = \text{supp}(gr(\mathcal{M}_M)).$$

The characteristic cycle keeps track of multiplicities along the components:

$$CC(M) = CC(\mathcal{M}_M) = \sum n_y [\overline{T^*_{X(y)} X}].$$

Recall the moment map $\mu : T^*(X) \to \mathcal{N}$. There is a special nilpotent G-orbit in $\mathcal{N}$ so that

$$\mu(CV(L_w)) = \Upsilon_1 \cup \ldots \cup \Upsilon_r = AV(L_w)$$

where $\{\Upsilon_i\}$ are irred. components of $\overline{\mathcal{O}} \cap \mathfrak{n}$. (Orbital Varieties)

Joseph attaches to each Υ_i a polynomial $p_{\Upsilon_i} \in \mathcal{P}(\mathfrak{h}^*)$:

- $\{p_\Upsilon : \Upsilon$ irred. component of $\overline{\mathcal{O}} \cap \mathfrak{n}\}$.

[p_Υ measure the growth rate of the H-weights on $\mathcal{S}(\overline{n})/I(\Upsilon)$ where $I(\Upsilon)$ is ideal of definition of Υ.]

- span$\{p_\Upsilon : \Upsilon$ irred. component of $\overline{\mathcal{O}} \cap \mathfrak{n}\} = Sp(\mathcal{O})$ Irred. W-module.

A second basis of $Sp(\mathcal{O})$

Define:

$$\mathrm{Prim}_\rho(\mathcal{O}) = \{I \text{ :primitive 2-sided ideals in } \mathcal{U}(\mathfrak{g}) \\ \text{infinitesimal character } \rho \in \mathfrak{h}^* \\ \text{the variety of zeros of } gr(I) = \overline{\mathcal{O}}\}.$$

Joseph attached to each $I \in \mathrm{Prim}_\rho(\mathcal{O})$ a polynomial $q_I \in \mathcal{P}(\mathfrak{h}^*)$, the so called Goldie rank polynomial. $[\lambda \to \ \mathrm{rank}(\mathcal{U}(\mathfrak{g})/Ann(L(\lambda))]$

- $\mathrm{span}_\mathbb{C}\{q_I : I \in \mathrm{Prim}_\rho(\mathcal{O})\} \simeq Sp(\mathcal{O})$, as W modules.

We are interested in: (a) the change of bases matrix
(b) the info about CC of $(\mathfrak{g}, B)$ and $(\mathfrak{g}, K)$ modules encoded in the matrix.

Monty McGovern defined a bijection:

$$\mathrm{Prim}_\rho(\mathcal{O}) \leftrightarrow \{\Upsilon \text{irred components of } \overline{\mathcal{O}} \cap \mathfrak{n}\}$$

[combinatorial in nature: both $\mathrm{Prim}_\rho(\mathcal{O})$ and $\{\Upsilon\}$ are parametrized by SDT of special shape. Roughly : Υ corresponds to I iff their SDT agree.]

The notion of τ^r-invariant and τ^r_∞ is well defined at the level of tabeaux.

Monty introduces an order on $\{\Upsilon\}$

$$\Upsilon_i < \Upsilon_j \leftrightarrow \tau^r_\infty(SDT_i) \subset \tau^r_\infty(SDT_j).$$

- He shows that the matrix relating $\{q_I\}$ to $\{p_\Upsilon\}$, in this order, is upper triangular.

- McGovern's triangularity result says: If

$$CC(L_w) = T^*_{X(w)} + \sum m_{y,w} \; T^*_{X(y)}(X)$$

$$\text{and } m_{y,w} \neq 0, \text{then } \tau^r_\infty(w) \subset \tau^r_\infty(y)$$

Key Theorem (Joseph)

Write $q_{\mathrm{Ann}(L_{w^{-1}})} = \sum \; m_i \; p_{\Upsilon_i}$. Then

$$m_i \neq 0 \Leftrightarrow \Upsilon_i \text{ is open in } AV(L_w).$$

McGovern's result

- imposes strong restriction on the "shape" of $AV(L_w)$,
- generalizes a triangularity result in type A_n by Joseph,
- proves a conjecture by Tanisaki.

- Let $\mathcal{M}_\rho(\mathfrak{g}, K)$ the category of Harish-Chandra modules with inf. ch. ρ.

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Peter Trapa says:

1. There is a bijection between $\mathrm{Prim}_\rho(\mathcal{O})$ and $\{\Upsilon$irred components of $\overline{\mathcal{O}} \cap \mathfrak{n}\}$ "geometric" in nature. Moreover, geometric orders can be defined on $\{\Upsilon\}$ so that the matrix relating Goldie rank polynomials and Joseph's characteristic polynomials is upper triangular.

2. Joseph's theorem is closely related to *CC* of Harish-Chandra $(\mathfrak{g}, K)$-modules.

3. The leading term cycle (" a piece of CC") encodes all needed info to compute Ann. and AV.

4. Question: Is it true that *LTC* of H-C module for $G_{\mathbb{R}} = Sp(p, q)$ are irreducible?

- Beilinson-Berstein equivalence of categories

$$\begin{aligned} \mathcal{M}_\rho(\mathfrak{g}, K) &\simeq \mathcal{M}_{\mathsf{coh}}(\mathcal{D}_X, K) \\ M \to \mathcal{M}_M &= \mathcal{D}_X \otimes_{\mathcal{U}(\mathfrak{g})} M \end{aligned}$$

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$$\{\text{Irred. H-C modules with inf. cha. } \rho\} \leftrightarrow \{(\mathcal{Q}, \chi) K\text{-orbits on } X, \text{ local char.}\}$$

- Similar to what we have done in $\mathcal{M}(\mathfrak{g}, B)$

The characteristic cycle is the support of $gr(\mathcal{M}_M)$ keeping track of multiplicities

$$CC(M) = CC(\mathcal{M}_M) = \sum n_{\mathcal{Q}}[\overline{T^*_{\mathcal{Q}}X}], \qquad CV(M) = \cup \overline{T^*_{\mathcal{Q}}X} = \text{supp}(gr(\mathcal{M}_M))$$

- $\mu(\overline{T^*_{\mathcal{Q}}(X)}) = \overline{\mathcal{O}_K}$, nilpotent K-orbit.

The leading characteristic cycle is

$$\sum_{\mathcal{Q}:\mu(T^*_{\mathcal{Q}}(X)) \text{ of max dim}} n_{\mathcal{Q}} \; \overline{T^*_{\mathcal{Q}}(X)}$$

.

To simplify exposition assume $G_{\mathbb{R}} = Sp(p,q)$ or $SO^*(2n)$.

1. Fix $\mathcal{O}$ a nilpotent G-orbit and a real form $\mathcal{O}_K = K \cdot f \subset \overline{\mathcal{O}}$.
2. $\mu^{-1}(f) = \cup C_i$. Let $A_G(f)$ the group of components of $Z_G(f)$.
3. (Spaltenstein) There is a bijection

$$\{\Upsilon : \text{ irred. comp. } \overline{\mathcal{O}} \cap \mathfrak{n}\} \leftrightarrow \{A_G(f)\text{-orbits in } \mathrm{Irr}(\mu^{-1}(f))\}$$

4. On the other hand, conormal bundles partition Springer fibers, i.e $T^*_{\mathcal{Q}}(X) \cap \mu^{-1}(f)$ is dense in a unique component $C_{\mathcal{Q}}$. Hence,

$$\blacksquare \Upsilon_i \equiv A_G(f) \cdot C_i \leftrightarrow \{\mathcal{Q}_{i,t} \in K \backslash X : \mu(\overline{T^*_{\mathcal{Q}_{i,t}}(X)}) = \overline{\mathcal{O}_K} \text{ has } C_{i,t} \in A_G(f) \cdot C_i\}$$

Orders on $\{\mathcal{Q} \in K \backslash X\}$ compatible with orbit closure inclusion induce orders on $\{\Upsilon\}$.

Continue with $G_{\mathbb{R}} = Sp(p,q)$ or $SO^*(2n)$ and $\mathcal{O}_K = K \cdot f \subset \overline{\mathcal{O}}$.

- Trapa defines a bijection between $\mathrm{Prim}_\rho(\mathcal{O})$ and $\{\Upsilon \text{irred components of } \overline{\mathcal{O}} \cap \mathfrak{n}\}$ (a little technical for me to explain here.) In any of the orders just described on $\{\Upsilon\}$ the matrix relating Goldie rank polynomials and Joseph's characteristic polynomials is upper triangular.

- Trapa's triangularity result says: If

$$LTC(M) = T^*_{\mathrm{supp(M)}}(X) + \sum m_{\mathcal{Q}_i}\ T^*_{\mathcal{Q}_i}(X),$$

and $m_i \neq 0,$ then $A_G(f) \cdot C_i < A_G(f) \cdot C_{\mathrm{supp}(M)}$

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  [The Jacquet-Casselman functor $\mathbb{J} : \mathcal{M}(\mathfrak{g}, K)_\rho \to \mathcal{M}(\mathfrak{g}, B)_\rho$ should play a central role here. For us, $\mathbb{J}$ is in the background, through McGovern's work.]

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- A practical link, $G_{\mathbb{R}} = Sp(p, q)$ or $SO^*(2n)$

Theorem (Trapa) A Geometric interpretation of Joseph's Thm

Let M be an irred. H-C module with $AV(M) = \overline{K \cdot f}$ and $\mathrm{Ann}(M) = \mathrm{Ann}\ (L_{w^{-1}})$. Write

$$\star LTC(M) = \overline{T^*_{\mathrm{supp(M)}}(X)} + \sum m_{\mathcal{Q}_i}\ \overline{T^*_{\mathcal{Q}_i}(X)},\ \ \star AV(L_w) = \cup \Upsilon_k$$

Then, $m_{\mathcal{Q}_i} \neq 0$ " iff" the orbital variety Υ_i bijective to $A_G(f) \cdot C_{\mathcal{Q}_i}$ is open in $AV(L_w)$.:

- If we want to understand CC we can not forget of a powerful tool: (Tanisaki)

$$CC : \mathcal{K}(\mathcal{M}(\mathfrak{g}, K)_{\rho}) \to H_{\text{top}}(T_K^*(X), \mathbb{Q})$$

is W-equivariant.

- $\mathcal{M}(\mathfrak{g}, K)_{\rho}$ partitions into "equivalence classes", the HC cells $\{\mathcal{C}_{HC,i}\}$. (designed to capture information on tensoring a HC modules with a finite dim. rep.)
  Key property:
  (McGovern) Assume $(\mathfrak{g}, Sp(p) \times Sp(q))$. Let $\mathcal{O}_K$ be a nilpotent K-orbit. Then,

$$\mathcal{C}_{HC} = \{X \in \mathcal{M}(\mathfrak{g}, K)_{\rho} : AV(X) = \overline{\mathcal{O}_K}\}.$$

- For each $\mathcal{C}_{HC}$, there is a W-cell representation $V_{\mathcal{C}_{HC}}$ (minimal sub-quotient of the coherent continuation rep $\mathcal{K}(\mathcal{M}(\mathfrak{g}, K)_{\rho})$ that is spanned by $\mathcal{C}_{HC}$).

When $G_{\mathbb{R}} = U(p,q)$.

1. Each irreducible $(\mathfrak{g}, K)$ with inf. chr. ρ has irreducible associated variety, $AV(X) = \overline{\mathcal{O}_K} = \overline{K \cdot f}$.
2. $V_{\mathcal{C}_{HC}} \simeq H_{\text{top}}(\mu^{-1}(f))$ is an irreducible W-representation.
3. The component groups $A_G(f)$ of the centralizer of f are trivial. For $X, Y \in \mathcal{C}_{HC}$, then $Ann(X) \neq Ann(Y)$.
4. $\mathcal{C}_{HC}$ contains an $A_{\mathfrak{q}}$ for which the associated cycle can be computed.
5. As $V_{\mathcal{C}_{HC}} = \mathbb{Q}[W] \cdot A_{\mathfrak{q}}$, (indeed any $M \in \mathcal{C}_{HC}$ has such property.) The point is that we can compute multiplicity polynomials using coherent continuation rep.
6. Many irreducible $M \in \mathcal{C}_{HC}$, have $LTC(M) = 1 \cdot \overline{T^*_{\mathfrak{Q}}(X)}$ ($\overline{\mathfrak{Q}}$ is the support of M).

We would like to emphasize that none of the above facts hold in general.

Proposition

Assume $G_{\mathbb{R}} = Sp(p,q)$ or $SO^*(2n)$.
Let M_1, M_2 irreducible $(\mathfrak{g}, K)$-modules with $AV(M_i) = \overline{K \cdot f}$.
Write $\mathcal{Q}_{M_i}$ the support of M_i, and let $C_{\mathcal{Q}_{M_i}}$ be the irred. component of $(\mu^{-1}(f))$ corresponding to $\mathcal{Q}_{M_i}$. Then,

$$\mathrm{Ann}(M_1) = \mathrm{Ann}(M_2) \Longleftrightarrow C_{\mathcal{Q}_{M_1}} \in A_G(f) \cdot C_{\mathcal{Q}_{M_2}}.$$

Remark

1. In view of the prop. $T : \mathrm{Prim}_\rho(\mathcal{O}) \leftrightarrow \{\Upsilon \text{irred components of } \overline{\mathcal{O}} \cap \mathfrak{n}\}$ can be described as follows.: given $I \in \mathrm{Prim}_\rho(\mathcal{O})$ take any $M(\mathcal{Q})$ with $AV(M(\mathcal{Q})) = \overline{K \cdot f}$ and $Ann(M(\mathcal{Q})) = I$, then $T(I) = \Upsilon \leftrightarrow A_G(f) \cdot C_{\mathcal{Q}}$.
2. The prop. will allow us to compare Trapa's bijection and McGovern bijection leading to restrictions on shape of LTC.
3. The prop. (combined with other tools) give some info on the structure of $V_{\mathcal{C}_{HC}}$.

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Theorem

$G_{\mathbb{R}} = Sp(p,q)$ or $SO^*(2n)$.

1. $Q[W]M \simeq V_{\mathcal{C}_{HC}}$ if and only if $\mathrm{card}\{M' \in \mathcal{C}_{HC} : \mathrm{Ann}(M') = \mathrm{Ann}(M)\}$ is maximal for $\mathcal{C}_{HC}$.
2. If $M' \in \mathbb{C}_{HC}$, then

$$\mathbb{Q}[W]M' \simeq H_{\mathrm{top}}(\mu^{-1}(f))^{A_G(f, C_{\mathcal{Q}_{M'}})}$$

where, $A_G(f, C_{\mathcal{Q}_{M'}}) = \{z \in A_G(f) : z \cdot C_{\mathcal{Q}_{M'}} = C_{\mathcal{Q}_{M'}}\}$.

Remark: The proof of Theorem relies in the Proposition and an result of McGovern, i.e for $\mathfrak{sp}(p,q)$ $V_{\mathcal{C}_{HC}} \simeq H_{\mathrm{top}}(\mu^{-1}(f), \mathbb{Q})$ as W-modules.

Application: Theorem (4) (B.-Zierau) Assume $\mathfrak{g} = \mathfrak{sp}(p,q)$. There is an effective algorithm to compute multiplicity polynomials for each irreducible HC module $X \in \mathcal{C}_{HC}$ provided $\mathcal{C}_{HC}$ contains a rep. in the discrete series.

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- $G_{\mathbb{R}} = Sp(1,1)$
- $\mathcal{C}_{HC}$ is the H-C cell with $AV(Ann) = [2,2]$ and $AV = \overline{K \cdot f}$ a real form of $[2,2]$.
- $\mathcal{C}_{HC} = \{\pi_1 = ds_1, \pi_2 = ds_2, \pi_3 = A_{\mathfrak{q}}(\lambda)\}$. where $\text{Ann}(ds_1) = \text{Ann}(ds_2) \neq \text{Ann}(A_{\mathfrak{q}}(\lambda)$.
- $V_{\mathcal{C}_{HC}} \equiv \mathbb{Q}[W] \cdot ds_i$ but $Q[W] \cdot A_{\mathfrak{q}}(\lambda) \subsetneq V_{\mathcal{C}_{HC}}$.
- $CC(\pi_i) = T^*_{\mathcal{Q}_i}(X)$ with $\overline{\mathcal{Q}_i} = \text{supp}(\pi_i)$. Write C_i for the component of the Springer fiber corresponding to $T^*_{\mathcal{Q}_i}(X) \cap \mu^{-1}(f)$.
- A basis for $H_{top}(\mu^{-1}(f))^{A_G(f,C_2)}$ is $\{[C_0 + C_1], [C_2]\}$ and

$$
\begin{aligned}
s_{\alpha_1} C_2 &= -C_2 & \qquad s_{\alpha_1}[C_0 + C_1] &= [C_0 + C_1] + 2C_2 \\
s_{\alpha_2} C_2 &= C_2 + [C_0 + C_1] & \qquad s_{\alpha_2}[C_0 + C_1] &= -[C_0 + C_1].
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Assume $G_{\mathbb{R}} = Sp(p,q)$ or $SO^*(2n)$.
Fix $\mathcal{C}_{HC}$ with $AV(\mathcal{C}_{HC}) = \overline{K \cdot f}$. Write $M(\mathcal{Q})$ for $M \in \mathcal{C}_{HC}$ with $\mathrm{supp}(M(\mathcal{Q})) = \overline{\mathcal{Q}}$ and write

$$LTC(M(\mathcal{Q})) = T^*_{\mathcal{Q}}(X) + \sum m_{\mathcal{Q}_j}\ T^*_{\mathcal{Q}_j}(X), \text{ then}$$

$$\tau^r_\infty(M(\mathcal{Q})) \subset \tau^r_\infty(M(\mathcal{Q}_j))$$

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Remark:

- We show that $T : \mathrm{Prim}_\rho(\mathcal{O}) \leftrightarrow \{\Upsilon \text{irred components of } \overline{\mathcal{O}} \cap \mathfrak{n}\}$ sends $I \to \Upsilon(A_G(f)C_{\mathcal{Q}}) = \Upsilon_{SDT(I)}$.
- There are other restrictions that come from Trapa's order and also $\mathcal{Q}_j \subset \overline{\mathcal{Q}}$.
- Restrictions + ATLAS (to compute coherent continuation, for example) yield many examples of HC modules with irred. LTC.

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Key to the answer is:

Theorem (Trapa) A Geometric interpretation of Joseph's Thm

Assume M is irred., $AV(M) = \overline{K \cdot f}$ and $\mathrm{Ann}(M) = \mathrm{Ann}\ (L_{w^{-1}})$. Write $\star LTC(M) = \overline{T^*_{\mathrm{supp(M)}}(X)} + \sum m_{\mathcal{Q}_i}\ \overline{T^*_{\mathcal{Q}_i}(X)}$, $\star AV(L_w) = \cup \Upsilon_k$
Then, $m_{\mathcal{Q}_i} \neq 0$ " iff" the orbital variety Υ_i bijective to $A_G(f) \cdot C_{\mathcal{Q}_i}$ is open in $AV(L_w)$.:

Answer to the question: NO.
The strategy is to find a Highest Weight module L_w so that

1. $AV(\mathrm{Ann}(L_w)) = \overline{\mathcal{O}}$ is the associated variety of the annihilator of a $(\mathfrak{sp}(2n), \mathrm{Sp}(p) \times \mathrm{Sp}(q))$-module and
2. $AV(L_w)$ is reducible.

Once this is achieved, the theorem guarantees the existence of a Harish-Chandra module with reducible leading term cycle.

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Key to the answer is:

Theorem (Trapa) A Geometric interpretation of Joseph's Thm

Assume M is irred., $AV(M) = \overline{K \cdot f}$ and $\mathrm{Ann}(M) = \mathrm{Ann}\ (L_{w^{-1}})$. Write $\star LTC(M) = \overline{T^*_{\mathrm{supp(M)}}(X)} + \sum m_{\mathbb{Q}_i}\ \overline{T^*_{\mathbb{Q}_i}(X)}$, $\star AV(L_w) = \cup \Upsilon_k$
Then, $m_{\mathbb{Q}_i} \neq 0$ " iff" the orbital variety Υ_i bijective to $A_G(f) \cdot C_{\mathbb{Q}_i}$ is open in $AV(L_w)$.:

Answer to the question: NO.
The strategy is to find a Highest Weight module L_w so that

1. $AV(\mathrm{Ann}(L_w)) = \overline{\mathcal{O}}$ is the associated variety of the annihilator of a $(\mathfrak{sp}(2n), \mathrm{Sp}(p) \times \mathrm{Sp}(q))$-module and
2. $AV(L_w)$ is reducible.

Once this is achieved, the theorem guarantees the existence of a Harish-Chandra module with reducible leading term cycle.

How do we find such L_w?

We change real form.

We find a $\mathrm{Sp}(2n,\mathbb{R})$ Highest Weight module M so that

1. $AV(\mathrm{Ann}(M)) = \overline{\mathcal{O}}$ and $\mathcal{O}$ is also the associated variety of the annihilator of some $\mathrm{Sp}(p,q)$-module.
2. $LTC(M)$ is reducible.

The Highest Weight L_w we are looking for, has $\mathrm{Ann}(L_{w^{-1}}) = \mathrm{Ann}(M)$.

A concrete example:

- We consider $\mathcal{O} \simeq [2,2,2,2]$.
- There is a $(\mathfrak{sp}(8,\mathbb{C}), GL(4))$-module $R_{\mathfrak{q}}(Y)$ with (Y a $\mathfrak{sp}(4)$-module with red. CC and irr. LTC) $LTC(R_{\mathfrak{q}}(Y)) = CC(R_{\mathfrak{q}}(Y)) = T^*_{\mathcal{Q}'_1}(X) + T^*_{\mathcal{Q}'_2}(X)$.
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