The Atlas Algorithm
Workshop, Salt Lake City
July 20-24, 2009

Jeffrey Adams

August 14, 2009

Note This is a preliminary version of these notes. A final version will be available on Monday July 20, the first day of the workshop.

Introduction

These talks are an introduction to an algorithm for computing the admissible dual of a real reductive group. This algorithm forms the basis of the `atlas` software. These talks are being given at the Atlas workshop at the University of Utah, July 20-24, 2009. Frequent reference will be made to the concurrent lectures by Peter Trapa and David Vogan. The notes are available on the workshop web page `www.math.utah.edu/realgroups`.

Here is an outline of the lectures. We begin with an overview of root data and conected complex reductive algebraic groups. In Lecture II we turn to our primary class of groups: real forms of connected complex reductive algebraic groups. We discuss these from two points of view: anti-holomorphic involutions and holomorphic, or Cartan, involutions. In what follows we work exclusiely with the latter.

In Lecture III we discuss the group $\mathrm{Out}(G)$ of outer automorphism of G, and inner classes of real forms, This leads to the *extended group* G^Γ, and *strong real forms* in Lecture IV. With this machinery in place we desribe the space of K orbits on the flag variety G/B.

In lecture V apply the preceding construction to the dual group $G^\vee$. This leads directly to our main result, a computable set $\mathcal{Z}$ which parametrizes the admissible dual of real forms of G.

There will be informal evening sessions where Annegret Paul and Scott Crofts will do some examples, answer questions, and do some computer demonstrations. These notes include a number of exercises which they will be available to discuss.

Some illustrations using the `atlas` software discussed in the Appendix (currently being written).

ndWe assume the reader is familiar with the basic theory of Lie groups, and has some familiarity with root systems and algebraic groups. There are a number of references at the end of these notes. In particular we recommend Springer's *Linear Algebraic Groups* [11] for background on algebraic groups, and his Corvallis article [10] for a succint introduction. *Guide to the Atlas Software: Computational Representation Theory of Real Reductive Groups* [2] has a number of examples and is intended to be reasonably accessible. Details and proofs of the results sketched here are in *Algorithms for representation theory of real reductive groups* [3]. Much more information is available in the papers section of the `atlas` web site `www.liegroups.org`.

Lecture I: Root Data and Complex Groups

1 Root Data

1.1 Coxeter Groups

A *Coxeter group* is a group W defined by generators S, subject only to relations

(1.1) $$(st)^{m(s,t)} = 1 \quad (s, t \in S)$$

where $m(s,s) = 1$ and $m(s,t) = m(t,s) \geq 2$ for all $s \neq t \in S$.

A Coxeter group W is efficiently described by a *Coxeter graph*: this is an undirected graph with each edge labelled by an integer ≥ 3 or ∞. Such a graph defines a Coxeter group, with one generator for each vertex, $m(s,t) = 2$ unless s and t are joined by an edge, in which case $m(s,t)$ is the label on the edge. By convention the label 3 is omitted.

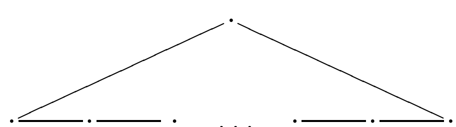
The *Coxeter matrix* of W is the matrix $\{m(s,t) \mid s, t \in S\}$. This is symmetric, with $1's$ on the diagonal.

Example 1.2

(1.3)

With n vertices this is the group generated by $\{s_1, \ldots s_n\}$ subject to $s_i^2 = 1$, $(s_i s_{i+1})^3 = 1$. This is the symmetric group S_{n+1}.

(1.4)



This the Coxeter matrix for the infinite, affine Weyl group of type A_n.

Example of Coxeter groups include: symmetric groups, finite reflection groups, Weyl groups, and affine reflection groups

1.2 Root Data

Just as a large collection of interesting groups (Coxeter groups) are concisely parametrized by a small set of combinatorial data (Coxeter graphs), *complex reductive groups* are parametrized by *root data*.

1.3 Root Data of $GL(n)$

Let $G = GL(n, \mathbb{C})$, the $n \times n$ invertible complex matrices. Let $T \simeq \mathbb{C}^{*n}$ be the usual Cartan subgroup $T = \{\text{diag}(z_1, \dots, z_n \,|\, z_i \neq 0)\}$. Let $X^*(T)$ be the set of *algebraic* characters of T: the group homomorphisms $T \to \mathbb{C}^*$ which can be expressed as a polynomial in $z_1^{\pm 1}, \dots, z_n^{\pm 1}$.

Exercise 1.5 $X^*(T) \simeq \mathbb{Z}^n$: the character $\text{diag}(z_1, \dots, z_n) \to z_i$ goes to the standard basis element e_i.

Consider the action of T on $V = M_n(\mathbb{C})$ by conjugation. View this as a representation of T, and decompose it into one-dimensional representations. Thus V has a basis $\{X_{i,j} \,|\, 1 \leq i, j \leq n\}$ where $X_{i,j}$ has a 1 in the i^{th} row and j^{th} column. These are eigenvectors for the action of T; if $t = \text{diag}(z_1, \dots, z_n)$ then

$$t.X_{i,j} = (z_i/z_j)X_{i,j}. \tag{1.6}$$

In other words let $\alpha_{i,j} \in X^*(T)$ be the character

$$\alpha_{i,j}(\text{diag}(z_1, \dots, z_n)) = z_i/z_j. \tag{1.7}$$

Lemma 1.8 *V decomposes, as a representation of T, into the direct sum of $\{\alpha_{i,j} \,|\, 1 \leq i \neq j \leq n\}$, together with the trivial representation of multiplicity n.*

Under the isomorphism $X^*(T) \simeq \mathbb{Z}^n$, the set of characters $\alpha_{i,j}$ goes to

$$\Delta = \{e_i - e_j \,|\, 1 \leq i \neq j \leq n\}. \tag{1.9}$$

Definition 1.10 *$X_*(T)$ is the set of algebraic group homomorphisms $\mathbb{C}^* \to T$.*

Exercise 1.11 $X_*(T) \simeq \mathbb{Z}^n$; the map $z \to (1, \dots, z, \dots, 1)$ (z in the i^{th} place) goes to the standard basis vector e_i.

Define $\alpha_{i,j}^\vee \in X_*(T)$:

$$\alpha_{i,j}^\vee(z) = \text{diag}(1, \dots, z, \dots, z^{-1}, \dots, 1) \tag{1.12}$$

with z in the i^{th} place and z^{-1} in the j^{th}, and

$$\Delta^\vee = \{\alpha_{i,j}^\vee\} \subset X_*(T). \tag{1.13}$$

Using the isomorphism $X_*(T) \simeq \mathbb{Z}^n$ we have:

$$\Delta^\vee = \{e_i - e_j \,|\, 1 \le i \ne j \le n\}. \tag{1.14}$$

Exercise 1.15 There is a natural pairing $\langle\, , \,\rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$: define $\langle \chi, \phi \rangle = k$ if $\chi(\phi(z)) = z^k$ for all $z \in \mathbb{C}^*$. This is a perfect pairing, and gives an isomorphism $X_*(T) \simeq \mathrm{Hom}(X^*(T), \mathbb{Z})$.

Using the isomorphisms $X^*(T) \simeq \mathbb{Z}^n$ and $X_*(T) \simeq \mathbb{Z}^n$ the pairing $\langle\, , \,\rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ becomes the standard dot product.

For $\alpha = \alpha_{i,j} \in \Delta$ define $s_\alpha \in \mathrm{Hom}(X^*(T), X^*(T))$:

$$s_\alpha(\gamma) = \gamma - \langle \gamma, \alpha^\vee \rangle \alpha \quad (\gamma \in X^(T)). \tag{1.16}$$

Define $s_{\alpha^\vee} \in \mathrm{Hom}(X_*(T), X_*(T))$ similarly.

Exercise 1.17 For all $\alpha, \beta \in \Delta$:
(1) $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$,
(2) $s_\alpha(\Delta) = \Delta$
(2) $s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee$.

The quadruple

$$(X^*(T), \Delta, X_*(T), \Delta^\vee) \tag{1.18}$$

is an example of a *root datum*.

1.4 Root Datum of $SL(n)$

Let $G = SL(n, \mathbb{C}) = \{g \in GL(n, \mathbb{C}) \,|\, \det(g) = 1\}$. There is an exact sequence

$$1 \longrightarrow SL(n, \mathbb{C}) \longrightarrow GL(n, \mathbb{C}) \xrightarrow{det} \mathbb{C}^* \longrightarrow 1. \tag{1.19}$$

Let $T = \{\mathrm{diag}(z_1, \dots, z_n) \,|\, z_1 z_2 \dots z_n = 1\}$. Consider the action on $V = \{X \in M_n(\mathbb{C}) \,|\, \mathrm{trace}(X) = 0\}$.

A simliar calculation to the one for $GL(n)$ gives a root datum for $SL(n, \mathbb{C})$:

$$(X^*(T), \Delta, X_*(T), \Delta^\vee). \tag{1.20}$$

Here $X^*(T), X_*(T)$ are lattices of rank $n-1$, and $\Delta, \Delta^\vee$ are as before.

Exercise 1.21 The lattices for $SL(n,\mathbb{C})$ are:

$$X^*(T) \simeq \mathbb{Z}^n/\{(k,k,\dots,k) \mid k \in \mathbb{Z}\} \tag{1.22)(a}$$

and

$$X_*(T) \simeq \mathbb{Z}^n_0 = \{(a_1,\dots,a_n) \in \mathbb{Z}^n \mid \sum a_i = 0\}. \tag{1.22)(b}$$

Exercise 1.23 Let $R = \mathbb{Z}\langle\Delta\rangle$. Then R is a lattice of rank $n-1$, $R \simeq \mathbb{Z}^n_0$, and $X^*(T)/R \simeq \mathbb{Z}/n\mathbb{Z}$.

1.5 Root Data

Here is the abstract definition of root data.

Fix a pair $X, X^\vee$ of free abelian groups of finite rank, together with a perfect pairing $\langle\,,\,\rangle : X \times X^\vee \to \mathbb{Z}$. (In other words we can find isomorphism $X \simeq \mathbb{Z}^n$, $X^\vee \simeq \mathbb{Z}^n$ such that $\langle\,,\,\rangle$ becomes the dot product).

A *root datum* is a quadruple

$$D = (X, \Delta, X^\vee, \Delta^\vee) \tag{1.24}$$

where $\Delta \subset X$ and $\Delta^\vee \subset X^\vee$ are finite subsets, equipped with a bijection $\alpha \to \alpha^\vee$ such that for all $\alpha \in \Delta$:

$$\langle\alpha,\alpha^\vee\rangle = 2,\ s_\alpha(\Delta) = \Delta,\ s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee. \tag{1.25}$$

Here

$$\begin{aligned} s_\alpha(\gamma) &= \gamma - \langle\gamma,\alpha^\vee\rangle\alpha \quad (\gamma \in X^*(T)) \\ s_{\alpha^\vee}(\gamma^\vee) &= \gamma^\vee - \langle\alpha,\gamma^\vee\rangle\alpha^\vee \quad (\beta^\vee \in X_*(T)). \end{aligned} \tag{1.26}$$

We say $(X_i, \Delta_i, X_i^\vee, \Delta_i^\vee)$ are isomorphic if there is an isomorphism $\phi : X_1 \simeq X_2$ taking Δ_1 to Δ_2, and so that ϕ^t takes $\Delta_1^\vee$ to $\Delta_2^\vee$.

The *rank* of a root datum $D = (X, \Delta, X^\vee, \Delta^\vee)$ is defined to be the rank of X, and and the *semisimple rank* of D is defined to be the rank of the *root lattice* $R = \mathbb{Z}\langle\Delta\rangle$.

Remark 1.27 (Important Point) The roots Δ are a subset of the real vector space $V = X \otimes \mathbb{R}$, and $\Delta^\vee \subset X^\vee \otimes \mathbb{R}$, which is naturally isomorphic to $V^* = \mathrm{Hom}(V, \mathbb{R})$. It is (almost) *never* a good idea to identify V with V^*, and $\alpha^\vee$ with $2\alpha/(\alpha,\alpha)$ (for some non-canonical bilinear form $(\ ,\ )$). The roots and coroots live in different places. Writing $\langle \alpha, \beta^\vee \rangle$ is natural, while writing (α, β) is almost always a bad idea.

Example 1.28 The root datum of $GL(n, \mathbb{C})$ has rank n and semisimple rank $n-1$. Both the rank and semisimple rank of the root datum of $SL(n, \mathbb{C})$ are $n-1$.

Root data are an extraordinarly compact way to encode reductive groups. If we fix isomorphisms of X and $X^\vee$ with $\mathbb{Z}^n$, then Δ and $\Delta^\vee$ each become a set of m integral column vectors.

Exercise 1.29 Define an equivalence relation on ordered pairs of integral matrices with n rows and m columns as follows: $(A, B) \sim (g^t AP, g^{-1} BP)$ for $g \in GL(n, \mathbb{Z})$ and P and $m \times m$ permutation matrix.

There is a natural bijection between isomorphism classes of root data and:

$$\{(A, B) \mid \text{ integral matrices of the same size, } A^t B \text{ is a Cartan matrix}\}/\sim .$$

A *Cartan matrix* is the matrix of $\langle \alpha_i, \alpha_j^\vee \rangle$ where $\{\alpha_1, \dots, \alpha_n\}$ are the simple roots of a semisimple Lie algebra. It can be defined without reference to Lie algebras [1], [6, Proposition 2.52].

In the setting of Exercise 1.29 n is the rank and $m \le n$ is the semisimple rank.

Remark 1.30 The matrices A and B are given by the `rootdatum` command of the `atlas` software.

Exercise 1.31 Up to equivalence there are 3 root data of rank $n = 2$ and semisimple rank $m = 1$. The only 1×1 Cartan matrix is (2), and the possibilities are $(A, B) =$

$$(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}), (\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}), (\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}) \tag{1.32}$$

2 Complex Reductive Groups

We are interested in representations of reductive Lie groups. Examples of such groups are:

1. $GL(n,\mathbb{R})$: all $n \times n$ invertible real matrices,

2. $SL(n,\mathbb{R})$: all $n \times n$ real matrices with determinant 1,

3. $O(p,q)$: all real matrices satisfying $gJg^t = J$ where $J = \mathrm{diag}(I_p, I_q)$

4. $SO(p,q)$ the subgroup (of index 2) of $O(p,q)$ of matrices with determinant 1

5. $SO_0(p,q)$: the identity component of $SO(p,q)$ (of index 1 or 2),

6. $Sp(2n,\mathbb{R})$: the set of all $2n\times 2n$ real matrices preserving a non-degenerate symplectic form,

7. $\widetilde{Sp}(2n,\mathbb{R})$: the metaplectic group, the unique non-trivial two-fold cover of $Sp(2n,\mathbb{R})$

Examples 1,2,4 and 6 share a special property: G is a "real form" of a connected, complex Lie group $G(\mathbb{C})$: $GL(n,\mathbb{C}), SL(n,\mathbb{C}), SO(p+q,\mathbb{C})$ and $Sp(2n,\mathbb{C})$, respectively. (We discuss real forms in the next section). Examples 3 and 5 differ in trivial ways from such a group: $SO_0(p,q) \subset SO(p,q) \subset O(p,q)$ with each containment of index 2 (the first containment is equality if $pq = 0$). In particular these are all matrix groups, i.e. can be realized as a closed subgroup of $GL(n,\mathbb{R})$ for some n. The final example is more serious: $\widetilde{Sp}(2n,\mathbb{R})$ is not a matrix group.

Note that the complex Lie groups we just listed are in fact *algebraic* groups: given by a set of polynomial equations. It turns out this is the best class of groups to consider. So we begin by describing connected, complex, reductive, algebraic groups, and then their real points.

Definition 2.1 *A connected complex algebraic group is a subgroup G of $GL(n,\mathbb{C})$ satisfying the following conditions:*

1. G is the set of zeros of a finite set of polynomial functions on $M_{n\times n}(\mathbb{C})$,

2. G is connected (in the Zariski topology), *Such a group is reductive if:*

3. *the Lie algebra* $\mathfrak{g}$ *of* G*, a Lie subalgebra of* $\mathrm{Lie}(GL(n,\mathbb{C})) = M_{n\times n}(\mathbb{C})$*, is reductive, i.e. the direct sum of simple and abelian factors,*

4. *every element of the center of* $\mathfrak{g}$ *is diagonalizable.*

Remark 2.2

1. We may view G is a Lie group, i.e. as a complex manifold with smooth group operations.

2. Condition (2) is equivalent to: G is connected as a Lie group, i.e. as a complex manifold.

3. Conditions (3) and (4) are equivalent to: the only normal subgroup consisting of unipotent elements of $GL(n,\mathbb{C})$ is trivial.

Remark 2.3 A (connected, complex) Lie group is *reductive* if its Lie algebra is reductive. The relationship between (connected, complex) *reductive Lie* groups and *reductive algebraic* groups is a bit subtle. For example the additive group $\mathbb{C}$ is a reductive Lie group, but as an algebraic group $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ is nilpotent and not algebraic.

We attach root data to a complex connected reductive algebraic group G as follows. Choose a Cartan subgroup T of G: $T \simeq \mathbb{C}^{*n}$ and is maximal with respect to this property. There is a natural pairing between $X^*(T) = \mathrm{Hom}(T,\mathbb{C}^*)$ and $X_*(T) = \mathrm{Hom}(\mathbb{C}^*,T)$ (algebraic homomorphisms), given by $\langle \phi, \psi \rangle = k$ if $\phi(\psi(z)) = z^k$ for $z \in \mathbb{C}^*$.

Let $\Delta \subset X^*(T)$ be the set of non-zero eigencharacters of the adjoint action of T on $\mathfrak{g} = Lie(G)$. That is $\alpha \in \Delta$ if and only if there exists $X \in \mathfrak{g}$ such that $Ad(t)(X) = \alpha(t)X$ for all $t \in T$.

A tricky part of the theory is the construction of $\Delta^\vee \subset X_*(T)$. This reduces to $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$. We obtain $\Delta^\vee \subset X_*(T)$, and a bijection $\Delta \ni \alpha \to \alpha^\vee \in \Delta^\vee$ so that $(X^*(T), \Delta, X_*(T), \Delta^\vee)$ is a root datum. See [11, Section 7.4]. Let $D(G,T)$ be the root data defined by G and T.

The main theorem is that connected complex algebraic groups are classified by root data.

Theorem 2.4 *Suppose $D = (X, \Delta, X^\vee, \Delta^\vee)$ is a root datum. Then there exists a connected complex algebraic group G and Cartan subgroup T such that $D \simeq D(G,T)$.*

Suppose $\phi : D(G,T) \to D(G',T')$ is an isomorphism. Then there is an isomorphism $\psi : G \to G'$ such that $\psi(T) = T'$ which induces ϕ.

In the last statment, an isomorphism ψ taking G to G' and T to T' naturally induces an isomorphism $D(G,T) \simeq D(G',T')$.

It is very helpful to keep the following picture in mind. Associated to the root datum $D(G,T)$ are the following lattices:

1. $X^*(T), X_*(T)$, the character and cocharacter lattices,
2. the root lattice $R = \mathbb{Z}\langle\Delta\rangle$,
3. the coroot lattice $R^\vee = \mathbb{Z}\langle\Delta^\vee\rangle$,
4. the weight lattice $P = \{\gamma \in X^*(T) \otimes \mathbb{Q} \,|\, \langle\gamma, \alpha^\vee\rangle \in \mathbb{Z}$ for all $\alpha^\vee \in \Delta^\vee$,
5. the coweight lattice $P^\vee = \{\gamma^\vee \in X_*(T) \otimes \mathbb{Q} \,|\, \langle\alpha, \gamma^\vee\rangle \in \mathbb{Z}\}$ for all $\alpha \in \Delta$.

The weight and coweight "lattices" are in fact lattices only in the semisimple case. In any event we have the following diagram:

$$\begin{array}{cc} P & P^\vee \\ \cup & \cup \\ X^*(T) & X_*(T) \\ \cup & \cup \\ R & R^\vee \end{array} \tag{2.5}$$

The lattices $P, P^\vee, R$ and $R^\vee$ only depend on the Lie algebra (are independent of "isogeny"); the position of $X^*(T)$ and $X_*(T)$ within the diagram is more subtle. In particular:

1. G is (semisimple and) simply connected if and only if $X^*(T) = P$ (iff $X_*(T) = R^\vee$)
2. G is adjoint if and only if $X^*(T) = R$ (iff $X_*(T) = P^\vee$)

Remark 2.6 Because of the connection with root data, we will always view G as being equipped with a fixed Cartan subgroup T of G, and sometimes also a Borel subgroup B containing T.

2.1 Relation with compact groups

In the setting of Lie groups, the connected compact Lie groups play a special role. It turns out that the classification of compact connected Lie groups is equivalent to the classification of connected complex reductive groups and also root data.

If K is a compact Lie group, there is associated to it a connected, complex algebraic group, the *envelope* of K, denoted $E(K)$.

Conversely if G is a connected complex algebraic group, there is a unique connected, compact Lie group $C(G)$, the *compact real form of* G.

Theorem 2.7 *The maps E and C are inverses, and give a bijection between*

$$\{\textit{connected compact Lie groups}\}/\textit{isomorphism} \tag{2.8}$$

and

$$\{\textit{connected complex reductive groups}\}/\textit{isomorphism} \tag{2.9}$$

In particular compact connected Lie groups are parametrized up to isomorphism by root data.

See [8, Section 7.2].

2.2 Defining a complex group

A complex reductive Lie algebra is a direct sum of simple and abelian factors. The corresponding statement for groups is:

Lemma 2.10 *Suppose G is a connected, complex, reductive algebraic group. Then the derived group G_d is connected and semisimple, and G contains a connected, complex, central torus Z so that $G = G_dZ$.*

The simply connected cover of G_d is a product of simple, simply connected groups; the center of such a group is finite. The following theorem follows easily from the lemma.

Theorem 2.11 *Suppose G is a connected, complex, reductive algebraic group. Then there exist:*

1. *simple, simply connected, connected groups $G_1, \dots, G_n$,*

2. a connected torus T,

3. a finite central subgroup A of $G_1 \times \cdots \times G_n \times T$

such that

$$G \simeq G_1 \times \cdots \times G_n \times T/A. \tag{2.12}$$

We may as well assume $A \cap T = 1$. Here is how to define an arbitrary G:

1. choose n irreducible root systems, each specified by a type $A_n, \ldots, G_2$,
2. choose a non-negative integer m, giving a torus of rank m,
3. choose elements $a_1, \ldots, a_k$ of $Z(G_1 \times \cdots \times G_n \times T)$

Then G is defined to be $G_1 \times \cdots \times G_n \times T$ modulo the group generated by $a_1, \ldots, a_k$.

Example 2.13 The center of $Spin(2n, \mathbb{C})$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if n is even, or $\mathbb{Z}/4\mathbb{Z}$ if n is odd. Thus $SO(2n, \mathbb{C}) = Spin(2n, \mathbb{C})/A$ where (with a natural choice of coordinates) $A = \pm(1, 1)$ if n is even, or the unique subgroup of the center of order 2 is n is odd. For example if $n = 3$, $Spin(6, \mathbb{C}) \simeq SL(4, \mathbb{C})$, and $SO(6, \mathbb{C}) \simeq SL(4, \mathbb{C})/\pm I$.

The center of every simple, simply connected group is cyclic, except the center of $Spin(2n, \mathbb{C})$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if n is even. Therefore a central element of $G_1 \times \cdots \times G_n$ can be specified by choosing the order of a generator for each simple factor, or 2 such in type D_{even}. Similarly an element of $\mathbb{C}^*$ of finite order is given by n elements of Q/Z: p/q gives the element $\exp(2\pi i p/q) \in \mathbb{C}^*$ of finite order.

Example 2.14 $GL(n, \mathbb{C}) = SL(n, \mathbb{C})D$ where $D = \mathrm{diag}(z, \ldots, z) \simeq \mathbb{C}^*$. The intersection $SL(n, \mathbb{C}) \cap D$ is $\{\mathrm{diag}(z, \ldots, z) \mid z^n = 1\}$. Choose a primitive n^{th} root of unity ζ. Then

$$GL(n, \mathbb{C}) \simeq SL(n, \mathbb{C}) \times \mathbb{C}^*/\langle \zeta I_n, \zeta^{-1} \rangle. \tag{2.15}$$

In the notation of the previous paragraph the generator of A is written $(1, -1/n)$, meaning (generator of $Z(SL(n, \mathbb{C})))^1 \times \exp(-2\pi i/n)$.

Exercise 2.16 Suppose F is an arbitrary field. Show that

$$GL(n,F)/SL(n,F)D \simeq F^*/F^{*n}. \tag{2.17}$$

By Theorem 2.7 the analogue of Theorem 2.11 holds almost word for word for compact groups.

Theorem 2.18 *Suppose G is a connected, compact group. Then there exist:*

1. *simple, simply connected, connected compact groups $G_1, \ldots, G_n$,*
2. *a connected compact torus T,*
3. *a finite central subgroup A of $G_1 \times \cdots \times G_n \times T$*

such that

$$G \simeq G_1 \times \cdots \times G_n \times T/A. \tag{2.19}$$

Lecture II: Real Groups

3 Real Groups

We now turn to the study of real groups. As discussed in Section 2 we are interested in general Lie groups, but restrict ourselves to groups which appear as the "real form" or "real points" of a connected, complex algebraic group.

Rather than appeal to the general theory of rational forms of algebraic groups (see [11, Section 12]), we utilize some of the special properties of $\mathbb{R}$.

Example 3.1 Let $G(C) = GL(n, \mathbb{C})$. This is a complex manifold of dimension n^2. The group $GL(n, \mathbb{R})$ is a *real* manifold of dimension n^2. It is the fixed points of the automorphism $\sigma_0 : g \to \overline{g}$ where $\overline{(a_{i,j})} = (\overline{a}_{i,j})$. This is a *real form* of $GL(n, \mathbb{C})$.

Example 3.2 Fix $x \in GL(n, \mathbb{C})$ and let $H = xGL(n, \mathbb{R})x^{-1}$. Obviously $H \simeq GL(n, \mathbb{R})$, and $H = GL(n, \mathbb{C})^\sigma$ where $\sigma = \mathrm{int}(x) \circ \sigma_0 \circ \mathrm{int}(x^{-1})$ ($\mathrm{int}(x)$ is the automorhism $g \to xgx^{-1}$). There is no point in distinguishing this from the real form $GL(n, \mathbb{R})$.

Example 3.3 Fix $p + q = n$ and let $J_{p,q} = \mathrm{diag}(I_p, -I_q)$. Define an automorphism σ of $GL(n, \mathbb{C})$ by

(3.4) $$\sigma_{p,q}(g) = J_{p,q}({}^t\overline{g}^{-1})J_{p,q}^{-1}$$

where ${}^tg$ denotes the transpose of g.

Let $G = GL(n, \mathbb{C})^\sigma$, the fixed points of σ. It is easy to see this is a subgroup of $GL(n, \mathbb{C})$. It is a *real* manifold (because of the $\overline{g}$), of dimension n^2.

Exercise 3.5 Show that the group defined in Example 3.3 is $U(p, q)$, the group of complex linear automorphisms of $\mathbb{C}^n$ preserving a Hermitian form of signature (p, q). In particular if $p = n, q = 0$

(3.6) $$U(n) = \{g \in GL(n, \mathbb{C}) \mid g{}^t\overline{g} = I\}.$$

This is the well known compact unitary group.

Exercise 3.7 Let $\mathbb{H}$ be the quaternions, let $V = \mathbb{H}^n$, and let G be the set of invertible, left $\mathbb{H}$-linear maps from V to V. That is $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $v, w \in V, \lambda \in \mathbb{H}$.

Identify $\mathbb{H}$ with $\mathbb{C}^2$, V with $\mathbb{C}^{2n}$, and therefore G with a subgroup of $GL(2n, \mathbb{C})$. Show that G is the fixed points of an automorphism of $GL(2n, \mathbb{C})$, and G is a real manifold of dimension $(2n)^2$. This group is denoted $GL(n, \mathbb{H})$.

We say the real forms of $GL(n, \mathbb{C})$ are $GL(n, \mathbb{R})$, $U(p, q)$ ($p \geq q, p+q = n$) and (if n is even) $GL(n/2, \mathbb{H})$. Keeping in mind Example 3.2 we define:

Definition 3.8 *A real form of a connected complex reductive group G is a G-conjugacy classes of subgroups, each of which is the fixed points of an anti-holomorphic involution of G.*

We usually refer to a single group $G_{\mathbb{R}} = G^\sigma$ as a real form; we identify conjugate subgroups so the set of real forms is finite, and to avoid having to say "equivalance (or conjugacy) classes of real forms".

Remark 3.9 There is a subtle distinction between this notion of real form and the traditional one: it is standard to identify two involutions if they are conjugate by $\mathrm{Aut}(G)$, not just $\mathrm{Int}(G)$. For simple groups these two notions are the same in almost all cases (see the next example).

Example 3.10 Let $G = SO(2n, \mathbb{C})$ with n even. The real form $SO^*(2n)$ has maximal complexified compact subgroup $K = GL(n, \mathbb{C})$. Thus K is the fixed points of an involution θ of G. (In fact θ is inner).

The outer automorphism of G takes K to an isomorphic subgroup K', and θ to θ'. In fact K is *not* G-conjugate to K'. Thus $SO(2n, \mathbb{C})$ has *two* real forms, denoted $SO^*(2n)_\pm$. In terms of real groups these two real algebraic groups are isomorphic, but not conjugate in $SO(2n, \mathbb{C})$.

In the standard literature this distinction is not made, there is only a single real form $SO^*(2n)$.

Exercise 3.11 Let $G = SO(2n, \mathbb{C})$ with n odd. There exist inner automorphisms θ such that $G^\theta = GL(n, \mathbb{C})$, corresponding to the real form $SO^*(2n)$. Show that any two such automorphisms are conjugate by $\mathrm{Int}(G)$ (not just $\mathrm{Out}(G)$). So in this cases there is only one real form $SO^*(2n)$.

The use of the *antiholomorphic* involution σ takes us out of the world of complex groups. A special property of $\mathbb{R}$ is that, thanks to the compact real form (see Theorem 2.7) we can discuss real forms of complex groups in terms of *holomorphic* involutions. (In the language of algebraic groups, we can work with algebraic involutions in place of Galois cohomology.)

Suppose $G_{\mathbb{R}}$ is a real form of the connected, complex reductive group G. Let $K_{\mathbb{R}}$ be a *maximal compact* subgroup of $G_{\mathbb{R}}$ (any two such groups are conjugate by $G_{\mathbb{R}}$). Then $K_{\mathbb{R}}$ is the fixed points of an involution θ of $G_{\mathbb{R}}$. Furthermore θ extends to a *holomorphic* involution of G, so let $K = G^{\theta}$, and K is a *complex* reductive algebraic group (not necessarily connected).

Example 3.12 Let $G = GL(n,\mathbb{C})$ and $G_{\mathbb{R}} = GL(n,\mathbb{R})$. A maximal compact subgroup of $GL(n,\mathbb{R})$ is $O(n) = \{g \in GL(n,\mathbb{R})\,,\, g\,{}^tg = I\}$, This is the fixed point of the involution $\theta(g) = {}^tg^{-1}$ of $GL(n,\mathbb{R})$. The same formula defines an involution θ of $GL(n,\mathbb{C})$, with fixed point $O(n,\mathbb{C}) = \{g \in GL(n,\mathbb{C}) \,|\, g^tg = I\}$.

Example 3.13 Let $G = GL(n,\mathbb{C})$ and $G_{\mathbb{R}} = U(p,q) = GL(n,\mathbb{C})^{\sigma_{p,q}}$ (cf. Exercise 3.3). A convenient maximal compact subgroup of $G_{\mathbb{R}}$ is $\{\mathrm{diag}(A,B) \,|\, A \in U(p), B \in U(q)\} \simeq U(p) \times U(q)$. This is the fixed points of the automorphism $\theta_{p,q}(g) = J_{p,q} g J_{p,q}^{-1}$ of $U(p,q)$. The same formula defines an involution of $GL(n,\mathbb{C})$, with fixed points $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$.

This illustrates that a real form of $GL(n,\mathbb{C})$ may be described by any of the following:

1. an anti-holomorphic involution σ_0 (Example 3.1) or $\sigma_{p,q}$ (Example 3.3);
2. a maximal compact subgroup $K_{\mathbb{R}}$ $O(n)$ or $U(p) \times U(q)$;
3. a complexified maximal compact subgroup $K = O(n,\mathbb{C})$ or $GL(p) \times GL(q)$;
4. a holomorphic (Cartan) involution θ_0 or $\theta_{p,q}$.

(If n is even there is also the real form $GL(n/2,\mathbb{H})$.)

Real forms of $\mathbf{GL}(\mathbf{n},\mathbb{C})$

$G(\mathbb{R})$:	$GL(n,\mathbb{R})$	$U(p,q)$	$GL(n/2,\mathbb{H})$
σ:	$\sigma_0(g) = \overline{g}$	$\sigma_{p,q}(g) = J_{p,q}{}^t\overline{g}^{-1}J_{p,q}^{-1}$	*
θ:	$\theta_0(g) = {}^tg^{-1}$	$\theta_{p,q}(g) = J_{p,q}gJ_{p,q}^{-1}$	*
$K_{\mathbb{R}}$:	$O(n)$	$U(p) \times U(q)$	*
K:	$O(n,\mathbb{C})$	$GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$	*

Exercise 3.14 Fill in the last column of the table.

This is a special case of the general situation.

Definition 3.15 *By involution of a connected, complex reductive group G we mean a holomorphic involution.*

Suppose σ is an antiholomorphic involution of G and $G_{\mathbb{R}} = G^\sigma$. A Cartan involution for $G_{\mathbb{R}}$ is an involution θ of G such that $(G_{\mathbb{R}})^\theta$ is a maximal compact subgroup of $G_{\mathbb{R}}$.

Theorem 3.16 *The map taking an antiholomorphic involution σ to a Cartan involution θ for G^σ induces a bijection from*

$$\{\textit{antiholomorphic involutions } \sigma\}/G \longleftrightarrow \{\textit{involutions } \theta\}/G. \tag{3.17}$$

The left hand side parametrizes real forms of G, so there is a natural bijection

$$\{\textit{real forms of } G\} \longleftrightarrow \{\textit{involutions } \theta\}/G. \tag{3.18}$$

Using this result we almost entirely ignore anti-holomorphic involutions from now on, in favor of (holomorphic) involutions, and call an involution θ (or more precisely a conjugacy class of such involutions) a real form.

This shows that studying real forms is the same as studying *symmetric subgroups*:

Definition 3.19 *A symmetric subgroup of G is the fixed points of an involution.*

This is the point of view taken in Peter Trapa's lectures.

The conjugacy classes of involutions of G is a natural object of study, and not hard to compute in many examples.

An obvious place to look for involutions of G is the automorphisms $\mathrm{int}(g)$ where the *element* g of G is an involution, i.e. $g^2 = 1$. It is usually not hard to classify the conjugacy classes of these elements.

Exercise 3.20 Consider the conjugacy classes of elements g of $GL(n, \mathbb{C})$ satisfying $g^2 = 1$. The eigenvalues of any such conjugacy class are ± 1, and these determine it. There are n such conjugacy classes, represented by $J_{p,q} = \mathrm{diag}(I_p, -I_q)$ $(p + q = n)$.

Now consider *inner* involutions: those of the form $\mathrm{int}(g)$ for $g \in G$. Note that $\mathrm{int}(g)$ is an involution if and only if $g^2 \in Z(G)$.

Definition 3.21 *An involution of G is inner if it is of the form int(g) for some $g \in G$; necessarily $g \in Z(G)$.*

Remark 3.22 (Dangerous Bend 1) An involution in G is, of course, an element g satisfying $g^2 = 1$. There is an important difference between

$$\{\text{inner involutions of } G\}$$

and

$$\{\mathrm{int}(g) \,|\, g \in G \text{ is an involution}\}.$$

The first set may be bigger: it is equal to $\{\mathrm{int}(g) \,|\, g^2 \in Z(G)\}$, while the second set is $\{\mathrm{int}(g) \,|\, g^2 = 1\}$.

Exercise 3.23 Let $G = SL(2, \mathbb{C})$ act by conjugation on its inner involutions. Let $\theta_s = \mathrm{int}(\mathrm{diag}(i, -i))$, $\theta_c = Id$, and show that

$$\{\text{inner involutions of } G\}/G = \{\theta_c, \theta_s\}$$

Show that only Id is of the form $\mathrm{int}(g)$ for g an involution of G.

Remark 3.24 (Dangerous Bend 2) There is a difference between conjugacy classes of elements g with $g^2 \in Z(G)$, and conjugacy classes the corresponding involutions $\mathrm{int}(g)$. For example $\pm I \in GL(n, \mathbb{C})$ are not conjugate, but both give the trivial automorphism of $GL(n, \mathbb{C})$.

Example 3.25 Continuing with the previous example, $G = GL(n, \mathbb{C})$ has $n + 1$ conjugacy classes of involutions $g \in G$, $\{J_{p,q} \,|\, 0 \le p \le n\}$. Let $\theta_{p,q} = \mathrm{int}(J_{p,q})$, i.e. $\theta_{p,q}(g) = J_{p,q} g J_{p,q}^{-1}$. Then $J_{p,q}$ is conjugate to $-J_{q,p}$, so $\theta_{p,q}$ is conjugate to $\theta_{q,p}$. Therefore $GL(n, \mathbb{C})$ has $[\frac{n}{2}] + 1$ conjugacy classes of involutions: $\{\theta_{p,q} \,|\, p \ge q\} = \{\theta_{p,q} \,|\, 0 \le q \le [\frac{n}{2}]\}$. The corresponding real forms of $GL(n, \mathbb{C})$ are $U(p, q)$ with $p \ge q$.

Example 3.26 The following sets are closely related, but not equal.

$$\begin{aligned} A &= \{g \in G \,|\, g^2 = 1\}/G \\ B &= \{g \in G \,|\, g^2 =\in Z(G)\}/G \\ C &= \{\mathrm{int}(g) \,|\, g^2 \in Z(G)\}/G \end{aligned} \tag{3.27}$$

There are maps $A \overset{\alpha}{\twoheadrightarrow} B \overset{\beta}{\hookrightarrow} C$.

For example suppose $G = SL(2,\mathbb{C})$. Then

$$
\begin{aligned}
A &= \{I, -I)\} \\
B &= \{I, -I, \mathrm{diag}(t, -t)\} \\
C &= \{\theta_c, \theta_s\}
\end{aligned}
\tag{3.28}
$$

The map from A to B is not surjective, and the map from B to C takes $\pm I$ to θ_c, so is not injective. (cf. Example 3.23).

Dangerous Bends 1 and 2 only arise if $Z(G)$ is nontrivial. Here is an example where this doesn't arise.

Exercise 3.29 Let $G = SO(2n+1,\mathbb{C})$. Show that G has $n+1$ conjugacy classes of elements g satisfying $g^2 = 1$, and $n+1$ conjugacy classes of involutions. The corresponding real forms are

$$\{SO(2p, 2q+1) \,|\, p+q = n\} \tag{3.30}$$

with complexified maximal compact subgroups

$$\{SO(2p,\mathbb{C}) \times SO(2q+1) \,|\, p+q = n\} \tag{3.31}$$

These are all of the real forms of $SO(2n+1,\mathbb{C})$.

We return to $GL(n,\mathbb{C})$ for a moment, and consider all involutions of the form $\mathrm{int}(g)$ with $g^2 \in Z(G)$.

Exercise 3.32 For $G = GL(n,\mathbb{C})$, suppose θ is an inner involution, i.e. $\theta = \mathrm{int}(g)$ with $g^2 \in Z(G)$. Show that there exists h with $h^2 = 1$ so that $\theta = \mathrm{int}(h)$. Therefore all real forms appear in Example 3.25.

Example 3.33 Let $G = SL(2,\mathbb{C})$. Show that there are three conjugacy classes of elements g satisfying $g^2 \in Z(G)$: $I, -I$ and $\mathrm{diag}(i, -i)$. These give two conjugacy classes of involutions, i.e. real forms of G: $\pm I$ give $\theta = \mathrm{Id}$ and the compact real form $SU(2)$, and $\mathrm{diag}(i, -i)$ gives $K = \mathbb{C}^*$, $K_{\mathbb{R}} = S^1$, and $G_{\mathbb{R}} = SL(2,\mathbb{R})$. Note that only the compact real form comes from an element g with $g^2 = 1$.

Exercise 3.34 This generalizes the previous example. Let $G = Sp(2n, \mathbb{C})$. Show that G has $n+2$ conjugacy classes of elements g satisfying $g^2 \in Z(G)$:

$$\{K_{p,q} = \mathrm{diag}(I_p, -I_q, I_p, -I_q) \quad (p+q=n, 0 \le p \le n)\}$$

and

$$K_s = \mathrm{diag}(iI_n, -iI_n).$$

The fixed points of these involutions are $Sp(2n, \mathbb{C}) \times Sp(2q, \mathbb{C})$ and $GL(n, \mathbb{C})$. The corresponding real forms are $Sp(p, q)$ and $Sp(2n, \mathbb{R})$, and maximal compact subgroups are $Sp(p) \times Sp(q)$ and $U(n)$.

Note that $\mathrm{int}(K_{p,q}) = \mathrm{int}(K_{q,p})$, so $Sp(p, q)$ and $Sp(q, p)$ are the same real form, so the real forms can be written

$$\{Sp(p, q) \,|\, p \ge q\} \text{ and } Sp(2n, \mathbb{R}).$$

See Exercise 5.22.

This shows that to classify real forms of G it is natural to consider all element g satisfying $g^2 \in Z(G)$. In fact there is a bijection between the real forms of G and those of the adjoint group, so this isn't necessary when classifying real forms. However for our purposes we definitely do need to consider these elements: this leads to the notion of *strong real form* in Section 5.1.

Lecture III: Real Groups Continued

Mostly missing from the discussion in the preceding section are the groups $GL(n,\mathbb{R})$ and $GL(n/2,\mathbb{H})$. This is because their Cartan involutions are not inner.

Exercise 3.35 Show that the involution $g \to {}^tg^{-1}$ is not an inner automorphism of $GL(n,\mathbb{C})$. Show that it is an inner automorphism of $SL(n,\mathbb{C})$ if and only if $n=2$.

Example 3.36 Every inner involution of $GL(n,\mathbb{C})$ is conjugate to $\mathrm{int}(J_{p,q})$ for some p,q (see Example 3.25). Every non-inner involution of $GL(n,\mathbb{C})$ is conjuate to $g \to {}^tg^{-1}$, or the Cartan involution of $GL(n/2,\mathbb{H})$ if n is even. See Exercises 3.7 and 3.14.

One of the nice things about using holomorphic involutions is it makes some real forms "obvious". For example the compact real form is simply the identity. The *Chevalley involution* is another example, of which Example 3.35 is a special case.

Example 3.37 Fix a Cartan subgroup T of G. The involution $t \to t^{-1}$ of T extends to an involution of G (the *Chevalley involution*). The corresponding real form of G is the *split* real form. It contains the real form $\mathbb{R}^{*n}$ of T.

4 Inner Classes

Example 3.36 shows that the real forms of G can be grouped according to the *outer automorphisms*: the Cartan involutions of $U(p,q)$ are inner, while those of $GL(n,\mathbb{R})$ and $GL(n/2,\mathbb{H})$ are not. Here is the general situation.

Let $\mathrm{Aut}(G)$ be the group of (holomorphic) automorphisms of G. The map $g \to \mathrm{int}(g)$ embeds G in $\mathrm{Aut}(G)$, and the quotient is the group $\mathrm{Out}(G)$ of outer automorphisms.

$$1 \to Int(G) \to \mathrm{Aut}(G) \xrightarrow{p} \mathrm{Out}(G) \to 1 \tag{4.1}$$

Definition 4.2 *Two involutions θ_1,θ_2 are said to be inner to each other or in the same inner class if they have the same image in $Out(G)$. Explicitly if there exists $x \in G$ such that $\theta_2 = int(x)\circ\theta_1$, i.e.*

$$\theta_2(g) = x\theta_1(g)x^{-1} \quad \textit{for all } g \in G. \tag{4.3}$$

We say two real forms of G are inner to each other or in the same inner class, if their Cartan involutions are in the same inner class.

This decomposes the set of real forms into subsets (inner classes), which are parametrized by involutions in $\mathrm{Out}(G)$; associated to an involution $\gamma \in \mathrm{Out}(G)$ are the Cartan involutions θ such the $p(\theta) = \gamma$.

Example 4.4 Let $G = GL(n, \mathbb{C})$ with $n > 2$. See Example 3.36. The two inner classes are:

$$\begin{aligned} \gamma = 1 : &\quad \{U(p,q) \mid p+q = n, p \geq q\} \\ \gamma \neq 1 : &\quad \{GL(n,\mathbb{R}), GL(n/2,\mathbb{H})\}. \end{aligned} \tag{4.5}$$

Also note $\mathrm{Out}(G) = \mathbb{Z}/2\mathbb{Z}$ (cf. Exercise 4.11).

In practice the question of how to group the *real forms* of G into *inner classes* is made much easier by the following result.

Lemma 4.6 *A real form θ of G is inner if and only if rank(G^θ) = rank(G). This is known as the inner or equal rank inner class. The corresponding groups $G(\mathbb{R})$ are those which contain a compact Cartan subgroup. This includes the compact real form of G.*

In example 4.4 the groups $U(p, q)$ contain compact Cartan subgroups, while $GL(n, \mathbb{R})$ and $GL(n/2, \mathbb{H})$ do not.

This makes it clear it is important to understand the group $\mathrm{Out}(G)$, or at least the involutions in it.

Exercise 4.7 Suppose T is a torus of rank n, i.e. $T \simeq (\mathbb{C}^*)^n$. Show that $\mathrm{Aut}(T) \simeq \mathrm{Out}(T) \simeq GL(n, \mathbb{Z})$.

Exercise 4.8 Show that every involution in $GL(n, \mathbb{Z})$ is conjugate to a product of terms 1, -1 or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the diagonal (this is not trivial). Thus (up to outer automorphism) the real forms of $T = \mathbb{C}^{*n}$ are parametrized by ordered triples (a, b, c) with $a + b + 2c = n$, corresponding to the real group $S^{1a} \times \mathbb{R}^{*b} \times \mathbb{C}^{*c}$. (Strictly speaking we only identify real forms if they are conjugate by an inner automorphism; in practice we allow outer automorphisms in the case of tori.)

Here is the group $\mathrm{Out}(G)$ in the opposite case.

Lemma 4.9 *Suppose G is semisimple. Then Out(G) is a subgroup of the automorphism group of its Dynkin diagram, and these groups are equal if G is simply connected or adjoint.*

See Lemma 5.28.

For example $\mathrm{Out}(G) = 1$ for any simple group of type $A_1, B_n, C_n, E_7, E_8, F_4$ or G_2.

Exercise 4.10 Show that $\mathrm{Out}(Spin(2n, \mathbb{C}))$ is $\mathbb{Z}/2\mathbb{Z}$ for all $n \neq 4$, and S_3 for $n = 4$. Show that for $n \geq 5$

$$\mathrm{Out}(Spin(2n, \mathbb{C})) = \mathrm{Out}(SO(2n, \mathbb{C})) = \mathrm{Out}(PSO(2n, \mathbb{C})) = \mathbb{Z}/2\mathbb{Z}.$$

If n is even find a quotient G of $Spin(2n, \mathbb{C})$ so that $\mathrm{Out}(G) = 1$. (See the diagram after Theorem 2.4, and Section 4.2).

Exercise 4.11 Show that $\mathrm{Out}(SL(2, \mathbb{C})) = 1$ and $\mathrm{Out}(SL(n, \mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$. Show that $\mathrm{Out}(GL(n, \mathbb{C})) \simeq \mathbb{Z}/2\mathbb{Z}$ for all n. See Exercise 3.35. (Hint: use Exercise 2.14).

In general $\mathrm{Out}(G)$ is built out of Example 4.7 and Lemma 4.9. We discuss this in more detail in Section 5.2. In practice we can ignore the issue of $GL(n, \mathbb{Z})$ in Examples 4.7 and 4.8 and find essentially all involutions in $\mathrm{Out}(G)$ as follows.

4.1 Basic Inner Classes

Every complex group has several canonical inner classes, which may coincide:

1. The *compact* inner class: involutions in the same inner class as the Cartan involution of the compact group, i.e the identity. These are the *inner involutions.*
2. The *split* inner class of Example 3.37.
3. If $G = G_1 \times G_1$ then there is the *complex* inner class: this is the inner class of the involution $\theta(g, h) = (h, g)$. The corresponding real form of $G(\mathbb{C})$ is $G(\mathbb{R}) = G_1(\mathbb{C})$, the complex group $G_1(\mathbb{C})$ viewed as a real group.

This gives almost every inner class of a semisimple group. There is one exception:

Exercise 4.12 Let $G = SO(2n, \mathbb{C})$ with $n \geq 5$. Recall (Exercise 4.10) $\mathrm{Out}(G) \simeq \mathbb{Z}/2\mathbb{Z}$.

The group $SO(2n, \mathbb{C})$ has a family of real forms $SO(p, q)$ with $p+q = 2n$. (It also has a real form $SO^*(2n)$.) Two real forms $SO(p, q)$ and $SO(r, s)$ are inner to each other if and only if $p \equiv q \pmod 2$. The group $SO(n, n)$ is split and $SO(2n, 0)$ is compact.

Use these facts to show that $SO(2n, \mathbb{C})$ has two inner classes of real forms. If n is odd these are the compact and split inner classes. If n is even the compact and inner classes coincide, and there is one other inner class.

In addition to the inner *split, compact* and *complex* inner classes the atlas software also uses

4. The *unequal* rank inner class, which does not exists in all cases, and often coincides with the split or complex inner class.

Exercise 4.13 Suppose G is semisimple. Show that the split and compact inner classes coincide if and only if $-1 \in W$. If this holds show that $Z(G)$ is a two-group.

4.2 Defining an inner class of real forms

Suppose we've defined $G' = G_1 \times \cdots \times G_n \times T$, and $G = G'/A$ as in Section 2.2. Here is how to define an inner class of real forms of G, i.e. an involution in $\mathrm{Out}(G)$.

First assume $A = 1$. Then for each G_i we may specify the compact, split, or (in some cases the) unequal rank inner class. For any pair $G_i \times G_i$ we may also specify the complex inner class of the product. Similarly we choose the compact, split, or complex inner for each $\mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$ factor of T. This defines an involution γ in $\mathrm{Out}(G)$.

Example 4.14 $G = SL(n, \mathbb{C}) \times \mathbb{C}^*$ ($n \geq 2$). There are four inner classes: [split or compact]×[split or compact].

Now suppose A is not necessarily trivial. Choose $\theta' \in \mathrm{Aut}(G')$ mapping to $\gamma \in \mathrm{Out}(G')$ by the exact sequence (4.1). Then θ' gives an element

$\theta \in \mathrm{Aut}(G)$ if and only if $\theta'(A) = A$. If this holds define γ to be the image of θ in $\mathrm{Out}(G)$. This is independent of the choice of θ.

Example 4.15 Recall (Example 2.14) $GL(n,\mathbb{C}) = SL(n,\mathbb{C}) \times \mathbb{C}^*/\langle(\zeta I_n, \zeta^{-1})\rangle$. The split inner class (containing the real group $GL(n,\mathbb{R})$ and the compact inner class (containing $U(n)$) are well defined. However [split inner class of $SL(n,\mathbb{C})$] $\times$ [compact inner class of $\mathbb{C}^*$] does not preserve A, and so does not define a real form of $GL(n,\mathbb{C})$ (unless $n = 2$). See Exercise 4.11.

Lecture IV:
Extended Groups and Strong Real Forms

5 Extended Groups

When studying the representation theory of a real form of G, it is natural to consider other real forms at the same time. In fact the natural setting is a set of *inner forms* of G.

Definition 5.1 *Basic data is a pair* (G, γ) *where:*

1. *a complex, connected, reductive algebraic group* G*,*
2. *an involution* $\gamma \in Out(G)$*.*

Recall we have an exact sequence (4.1):

$$1 \to Int(G) \to \mathrm{Aut}(G) \xrightarrow{p} \mathrm{Out}(G) \to 1. \tag{5.2}$$

Associated to (G, γ) is the set of real forms in this inner class: i.e. the involutions $\theta \in \mathrm{Aut}(G)$ satisfying to $p(\theta) = \gamma$ in (5.2). In practice we specify G as in Section 2.2, and γ as in Section 4.2.

It is very helpful to package all of this information into a single *extended group*. For this we need a splitting of the exact sequence (5.2).

Lemma 5.3 *There is a canonical choice of* G*-conjugacy class of splittings of* (5.2)*.*

We refer to the splittings of the Lemma as *distinguished*.

Example 5.4 Obviously if $\gamma = 1$ then $s(\gamma) = 1$. The corresponding real form is the compact one, and this is the class of inner, or equal rank, forms of G. See Lemma 4.6.

To avoid too much terminology we defer a more careful explanation of these splittings to Section 5.2. In the mean time the following Remark is useful.

Remark 5.5 Suppose $\gamma \in \mathrm{Out}(G)$ is an involution. Associated to γ is the inner class of real forms: $\{\theta \in \mathrm{Out}(G) \,|\, p(\theta) = \gamma\}$. Then $s(\gamma) = \theta_0$ where θ_0 is the *most compact* real form in this inner class.

Exercise 5.6 Suppose $G = SO(2n, \mathbb{C})$ and $\gamma \neq 1$. Then $s(\gamma)(g) = \epsilon g \epsilon^{-1}$ where $\epsilon = \mathrm{diag}(1, 1, \dots, 1, -1) \in O(2n, \mathbb{C}) \backslash SO(2n, \mathbb{C})$.

Exercise 5.7 Let $G = GL(n, \mathbb{C})$ for n odd. Recall (4.11) $\mathrm{Out}(G) = \mathbb{Z}/2\mathbb{Z}$. Suppose $1 \neq \gamma \in \mathrm{Out}(G)$. Then we can take $s(\gamma)$ to be the Cartan involution of $GL(n, \mathbb{C})$, i.e. $s(\gamma)(g) = {}^t g^{-1}$. What is $s(\gamma)$ for n even?

Let $\Gamma = \{1, \delta\} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Definition 5.8 *Given basic data* (G, γ) *define*

(5.9)
$$G^\Gamma = G \rtimes \Gamma$$

where δ *acts by* $s(\gamma)$ *for some distinguished splitting* s.

By the Lemma the isomorphism class of G is independent of the choice of distinguished splitting s.

Here is another way to think of this definition. We can write

(5.10)
$$G^\Gamma = \langle G, \delta \rangle$$

with relations

(5.11)
$$\delta^2 = 1, \quad \delta g \delta^{-1} = s(\gamma)(g).$$

Example 5.12 Suppose $\gamma = 1$. Then $G^\Gamma = G \times \Gamma$ (direct product).

We encourage the reader to keep the case $\gamma = 1$ in mind.

5.1 Strong Real Forms

Fix basic data (G, γ) and define G^Γ. Let $\theta_\delta = \mathrm{int}(\delta) \in \mathrm{Aut}(G)$. In particular $p(\theta_\delta) = \gamma$.

Suppose $\theta \in \mathrm{Aut}(G)$ is an involution in the inner class of γ. By definition

(5.13)(a)
$$p(\theta) = p(\theta_\delta)$$

i.e.

$$\theta = \mathrm{int}(h)\theta_\delta \quad \text{for some } h \in G, \tag{5.13)(b}$$

i.e.

$$\theta(g) = h\theta_\delta(g)h^{-1} \text{ for all } g \in G. \tag{5.13)(c}$$

Using the fact that $\theta_\delta = \mathrm{int}(\delta)$ we can write this as

$$\theta(g) = (h\delta)g(h\delta)^{-1} \text{ for all } g \in G. \tag{5.13)(d}$$

or in other words

$$\theta = \mathrm{int}(h\delta). \tag{5.13)(e}$$

Thus every involution of G in this inner class is conjugation by an element x of $G^\Gamma\backslash G$. Note that $\mathrm{int}(x)$ is an involution if and only if $x^2 \in Z(G)$.

Exercise 5.14 Suppose G is adjoint. Then the map $x \to \mathrm{int}(x)$ is a bijection

$$\{x \in G^\Gamma\backslash G \,|\, x^2 = 1\}/G \leftrightarrow \{\theta \text{ in the inner class of } \gamma\}/G. \tag{5.15}$$

It turns out to be essential to keep track not just of θ, but of x giving rise to it.

Definition 5.16 *Given (G, γ), a strong involution of G in the inner class of γ is an element x of $G^\Gamma\backslash G$ satisfying $x^2 \in Z(G)$. Two strong involutions are equivalent if they are conjugacte by G. A strong real form of G is a conjugacy class of strong involutions.*

The map $x \to \mathrm{int}(x)$ takes strong involutions to involutions, and strong real forms (conjugacy classes of strong involutions) to real forms (conjugacy classes of involutions).

Definition 5.17 *If x is a strong involution let $\theta_x = int(x)$ be the corresponding involution. Let $K_x = G^{\theta_x}$.*

Lemma 5.18 *The map $x \to \theta_x$ induces a canonical surjection*

$$\{\textit{strong real forms in the inner class of } \gamma\} \tag{5.19)(a}$$

to

$$\{\textit{real forms in the inner class of } \gamma\}. \tag{5.19)(b}$$

Note that if we required $x^2 = 1$ then this map would not necessarily be a surjection, for example for $SL(2, \mathbb{C})$.

By Exercise 5.14 if G is adjoint the map (5.19) is a bijection.

The notion of *strong real form* is a refinement of that of real form: we count some real forms more than once.

Remark 5.20 Suppose $\gamma = 1$, so $G^\Gamma = G \times \Gamma$. Suppose $x_0 \in G$ satisfies $x_0^2 \in Z(G)$. Then $x_0\delta$ is a strong involution, and this is a bijection between strong involutions and involutions in G. Thus we may safely drop δ from the notation in this case.

Example 5.21 Let $G = SL(2, \mathbb{C})$, and (necessarily) $\gamma = 1$. The set (5.19)(b) has 2 elements θ_c, θ_s (cf. Exercise 3.23), and (5.19)(a) has three, $\pm I$ and $\mathrm{diag}(i, -i)$. The map from strong real forms to real forms is 2 to 1 for $SU(2)$, and 1 to 1 for $SL(2, \mathbb{R})$. We can think of these strong real forms as $SU(2, 0), SU(1, 1) = SL(2, \mathbb{R})$ and $SU(0, 2)$.

Exercise 5.22 Let $G = Sp(2n, \mathbb{C})$, which implies $\gamma = 1$. Recall (Exercise 3.34) the real forms of G are $Sp(p, q)$ with $p+q = n$ with $p \geq q$ and $Sp(2n, \mathbb{R})$. Also by Exercise 3.34 G has n strong real forms $\{K_{p,q} \,|\, p+q = n, 0 \leq p \leq n\}$ and K_s. The map from strong real forms to real forms is 2 to 1 for $Sp(p, q)$ with $p \neq q$, and $1-1$ for $Sp(p, p)$ and $Sp(2n, \mathbb{R})$. Thus we can informally think of the strong real forms as $Sp(p, q)$ where $Sp(p, q)$ and $Sp(q, p)$ are *distinct* strong real forms (for $p \neq q$).

Example 5.23 Let $G = SL(n, \mathbb{C})$, $\gamma = 1$. There is a unique compact real form, given by (the conjugacy class of) $\theta = 1$. The preimage in (5.19)(a) consists of the n elements of the center.

In general G has $|Z(G)|$ strong real forms mapping to the compact real form.

Remark 5.24 The fact that there is a difference between *strong real forms* and *real forms* is not a bug, it's a feature. In fact it is essential in the statement of the main result (Theorem 7.5). When working with a single real form it is possible to minimize the role of strong real forms, but even here they play a critical role for the dual group.

5.2 Splittings and Automorphisms

Here is a little more detail on Lemma 5.18 and the relationship with $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$.

To find a splitting of (5.2) we need a subgroup $\mathcal{A}$ of $\mathrm{Aut}(G)$, mapping surjectively to $\mathrm{Out}(G)$, containing no inner automorphisms. We want to take $\mathcal{A}$ to be the automorphisms of G fixing some extra data.

For starters fix $T \subset B \subset G$, Cartan and Borel subgroups of G and let $\mathcal{A} = \mathrm{Stab}_G(B,T)$. Fix $\tau \in \mathrm{Aut}(G)$. Any two such pairs (B,T) are conjugate by G, and therefore $\mathrm{int}(g) \circ \tau \in \mathcal{A}$ for some $g \in G$. Thus $\mathcal{A}$ maps onto $\mathrm{Out}(G)$.

On the other hand suppose $\tau = int(h) \in \mathrm{Int}(G)$. The fact that B is its own normalizer implies $h \in B$, and the normalizer of T in B is trivial, so $h \in T$. We would like to conclude $h \in Z(G)$, so $\tau = 1$. This is not necessarily the case; we need a little more data to force $h \in Z(G)$. The choice of B defines a set of positive roots, and a set of simple roots. If $\alpha(h) = 1$ for all simple roots then $h \in Z(G)$. This leads to the definition of *splitting data.*

Definition 5.25 *A set of splitting data for G or (more poetically) an épinglage, is a triple $(B,T,\{X_\alpha\})$ consisting of $T \subset B \subset G$, Cartan and Borel subgroups respectively, and a set $\{X_\alpha\}$ of simple root vectors for the action of T on Lie(G). (Here simple is with respect to the positive system defined by B).*

Let $\mathrm{Aut}(G,B,T,\{X_\alpha\})$ be the automorphisms of G preserving $(B,T,\{X_\alpha\})$. Any two splitting data are conjugate by G, and together with the preceding argument this proves:

Proposition 5.26 *Fix splitting data $(B,T,\{X_\alpha\})$ for G. Then there is a splitting s of* (5.2) *taking Out(G) isomorphically to Aut$(B,T,\{X_\alpha\}) \subset$ Aut(G). Any two such splittings are conjugate by Int(G).*

The group $\mathrm{Aut}(G,B,\{X_\alpha\})$ can be described in a different way.

Definition 5.27 *Suppose $(X,\Delta,X^\vee,\Delta^\vee)$ is a root datum. Choose a set of positive roots, with corresponding simple root Π and simple coroots $\Pi^\vee$. The set $(X,\Pi,X^\vee,\Pi^\vee)$ is called a based root datum.*

Given G, choose a Cartan subgroup T and use this to define the root data $D(G,T)$ of G. In addition choose a Borel subgroup B containing T. This *determines a set of positive roots, and therefore defines based root datum* $D_b(G, B, T)$.

Automorphisms of based root data are defined in the obvious way.

Lemma 5.28 *Choose pinning data* $(B, T, \{X_\alpha\})$. *Then*

$$Out(G) \simeq Aut(D_b(G, B, T)) \simeq Aut(G, B, T, \{X_\alpha\}).$$

See [10, 2.13].

Write $D_b(G, B, T) = (X, \Pi, X^\vee, \Pi^\vee)$. Restricting an automorphism ϕ of this to the second factor gives an automorphism of the Dynkin diagram $\mathcal{D}$. If G is semisimple Π is a basis of $X \otimes \mathbb{Q}$, and ϕ is determined by this restriction. An automorphism of Π extends to an automorphisms of the root lattice R and weight lattice P. Recall $X = R$ if G is adjoint, and P if G is simply connected (2.5). This proves

Lemma 5.29 *There is a surjective map*

$$Aut(D_b(G, B, T)) \to Aut(\mathcal{D}). \tag{5.30}$$

This is an injection if G *is semisimple, and a bijection if* G *is also simply connected or adjoint.*

5.3 K orbits on G/B

As discussed in Peter Trapa's lectures, a fundamental role is played by the space of K-orbits on the flag variety G/B. The extended group provides a convenient description of this space, for all (strong) real forms simultaneously.

Fix basis data (G, γ). As usual we encourage the reader to first consider the case $\gamma = 1$, in which case $G^\Gamma = G \times \Gamma$, and we can safely drop the extension.

We fix as usual Cartan and Borel subgroups $T \subset B$. We start with an elementary definition:

$$\mathcal{P}(G, \gamma) = \{(x, B')\} \tag{5.31}$$

consisting of a strong involution x (Definition 5.16) and a Borel subgroup B'. Let G act diagonally by conjuation on $\mathcal{P}(G, \gamma)$, and consider the space $\mathcal{P}(G, \gamma)/G$. When (G, γ) are understood we write $\mathcal{P} = \mathcal{P}(G, \gamma)$.

Recall a strong real form is a conjugacy class of strong involutions. We make repeated use of the following.

Definition 5.32 *Let*

$$\{x_i \mid i \in \mathcal{I}\} \tag{5.33}$$

be a set of representatives of strong real forms. Thus each $x_i \in G^\Gamma\backslash G$, $x_i^2 \in Z(G)$*, and every such element is* G*-conjugate to a unique* x_i*. For* $i \in \mathcal{I}$ *let* $\theta_i = int(x_i)$ *and* $K_i = G^{\theta_i}$.

Remark 5.34 If $Z(G)$ is finite then $\mathcal{I}$ is a finite set. In fact this holds if $Z(G)^\delta = \{z \in Z(G) \mid \delta z \delta^{-1} = z\}$ is finite. If the set is infinite it causes annoying but not serious book-keeping problems, which can be avoided by passing to the *reduced* parameter space [3, ?]. The reader is encouraged to think of this set as being finite.

On the other hand all Borel subgroups are G-conjugate. There are two ways to understand $\mathcal{P}/G$: by conjugating each x to some x_i, or B' to B. We do these one at a time.

Conjugate B' to B:

Given $(x, B') \in \mathcal{P}$, choose g so that $gB'g^{-1} = B$. A basic fact is that any strong involution, in particular gxg^{-1}, normalizes some Cartan subgroup of B [, ?], and after conjugating by $b \in B$ we may assume it normalizes T.

Therefore we can find $g \in G$ so that $g(x, B')g^{-1} = (gxg^{-1}, B)$ with $gxg^{-1} \in \mathrm{Norm}_{G^\Gamma\backslash G}(T)$. The only other such choice would be $(tg)(x, B')(tg)^{-1} = (t(gxg^{-1})t^{-1}, B)$ for some $t \in T$. This motivates the primary combinatorial definition of the atlas project.

Definition 5.35 *Given* (G, γ)*, and* T *fixed as usual, let*

$$\mathcal{X}(G, \gamma) = \{x \in Norm_{G^\Gamma\backslash G}(T) \mid x^2 \in Z(G)\}/T \tag{5.36}$$

where the quotient is by the conjugation action. When (G, γ) *are understood we write* $\mathcal{X} = \mathcal{X}(G, \gamma)$.

Here $\mathrm{Norm}_{G^\Gamma\backslash G}(T)$ are the elements of $G^\Gamma\backslash G = G\delta$ normalizing T.

Then $(x, B) \to gxg^{-1}$ as above gives well defined bijection

$$\mathcal{P}/G \longleftrightarrow \mathcal{X}. \tag{5.37}$$

Conjugate x to x_i:

Given $(x, B') \in \mathcal{P}$, choose g so that $gxg = x_i$ for $i \in \mathcal{I}$. Note that (x_i, B') and (x_j, B'') are G-conjugate if and only if $i = j$ and $kB'k^{-1} = B''$ for some $k \in K_i = G^{\theta_i}$ (Definition 5.32).

$$\mathcal{P} \simeq \prod_{i\in\mathcal{I}} \{(x_i, B')\}/K_i \tag{5.38}$$

Clearly $\{(x_i, B')\}/K_i$ is simply K_i conjugacy classes of classes of Borel subgroups. Every Borel subgroup is G-conjugate to B, and the map $gBg^{-1} \to K_i gB$ is a bijection between this space and $K_i\backslash G/B$:

$$\mathcal{P}/G \longleftrightarrow \coprod_{i\in\mathcal{I}} K_i\backslash G/B. \tag{5.39}$$

Putting together (5.37) and (5.39) we obtain the main result.

Theorem 5.40 *Given (G, γ), choose a set of representatives $\{x_i \,|\, i \in \mathcal{I}\}$ of the strong real forms (Defintion 5.32). Define $\mathcal{X} = \mathcal{X}(G, \gamma)$ as in Definition 5.35. There is a canonical bijection*

$$\coprod_{i\in\mathcal{I}} K_i\backslash G/B \longleftrightarrow \mathcal{X}. \tag{5.41}$$

Remark 5.42 A version of this, for one real form at a time, is in [9].

The set $\mathcal{X}$ is finite if $Z(G)^\delta$ is finite (cf. Remark 5.34). It is a combinatorial object, which may be computed explicitly using the *Tits group.* See [3, Section 15] for details.

Example 5.43 Let $G = SL(2, \mathbb{C})$. Then $\gamma = 1$, and as usual we can drop the extension. For $x \in \mathbb{R}^*$ let

$$w(x) = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ -\frac{1}{x} & 0 \end{pmatrix}. \tag{5.44}$$

Then

$$\begin{aligned} \{x \in \mathrm{Norm}_G(T) \,|\, x^2 \in Z(G)\}/T &= \{\pm I, \pm\mathrm{diag}(i, -i), w(x)\}/T \\ &= \{\pm I, \pm\mathrm{diag}(i, -i), w(1)\}. \end{aligned} \tag{5.45}$$

The elements $\pm I$ correspond to the compact form, i.e. $K = G$, so $K\backslash G/B$ is a point. This occurs twice, for the two corresponding strong real forms (cf. Example 5.21). The elements $\pm\mathrm{diag}(i, -i)$ and $w(1)$ are all G-conjugate, so they give the three K orbits on G/B for the split real form. The orbits given by $\pm(\mathrm{diag}(i, -i))$ are points, and the orbit corresponding to $w(1)$ is the open orbit.

5.4 Cross Action and Cayley Transforms

It is important to understand the structure of the set of K-orbits on G/B, which becomes a combinatorial problem about the set $\mathcal{X}$. Here are the basics, sweeping some details under the rug (cf. Remark 5.46).

There is an obvious action of W on $\mathcal{X}$: if $w \in W$, choose a representative $g_w \in \mathrm{Norm}_G(T)$, and define $w \times x = g_w x g_w^{-1}$. We refer to this as the *cross action*.

There is also a less obvious operation we can perform on $\mathcal{X}$. Fix an element $x \in \mathcal{X}$, and let σ_α be a representative in G of the reflection $s_\alpha \in W$. It is natural to ask if $\sigma_\alpha x \in \mathcal{X}$, and whether this is independent of all choices.

Suppose $\theta_x(\alpha) = \alpha$ and $\theta_x(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, where $\mathfrak{g}_{\pm\alpha}$ denotes the $\pm\alpha$ weight spaces for the action of T on $\mathfrak{g} = \mathrm{Lie}(G)$. In conventional terminology we say α is a *non-compact imaginary root*. Associated to α is a subgroup G_α of G isomorphic to $SL(2)$ or $PSL(2)$, and we can choose a representative σ_α of the reflection s_α in G_α. It turns out that with this choice $\sigma_\alpha x$ is a well defined element of $\mathcal{X}$, independent of the choices. See refer to the map $x \to \sigma_\alpha x$ as the *Cayley transform* of x via α.

Remark 5.46 We are glossing over some significant technical details in the preceding discussion. First of all $x \in \mathcal{X}$ is not an element of G, but a T-conjugacy class of such elements. Strictly speaking θ_x is not well defined, althought it is not hard to make sense of the equalities involving $\theta_x(\alpha)$ and $\theta_x(\mathfrak{g}_\alpha)$. Similarly to make the definition of $\sigma_\alpha x$ precise requires some care; see [3, Section 14] for details.

Remarkably, the cross action and Cayley transforms are enough to generate the subset of $\mathcal{X}$ corresponding to K orbits on G/B for a fixed K, starting with a single element $x \in X$. Here is the precise result.

Lemma 5.47 *Fix $\{x_i \,|\, i \in \mathcal{I}\}$ as in Definition 5.32. Recall $\theta_i = int(x_i)$ and $K_i = G^{\theta_i}$. Without loss of generality we may assume that for all i, $x_i = t_i\delta$ for some $t_i \in T$. Given i, let $\mathcal{X}_i$ be the subset of $\mathcal{X}$ generated by the cross action and Cayley transforms. Then $\mathcal{X}_i \simeq K_i\backslash G/B$.*

In other words $\mathcal{X}_i$ is the set of elements of the form $T_1T_2\dots T_n x_i$ where each operation T_i is either a cross action $x \to s_{\alpha_i} \times x$, for some root α_i or a Cayley transform $x \to \sigma_{\alpha_i} x$ (it is enough to take each α_i simple). The `atlas` software implements this computation using the Tits group. See [3, Section 15].

Lecture V
Parametrizing Admissible Representations

6 The Dual Group

Root data $D = (X, \Delta, X^\vee, \Delta^\vee)$ has an obvious symmetry; the *dual root datum* is $D^\vee = (X, \Delta, X^\vee, \Delta^\vee)$. This induces a map $G \to G^\vee$ on connected, complex reductive groups. In particular $G^\vee$ has a Cartan subgroup $T^\vee$, and identifications $X^*(T) = X_*(T^\vee)$ and $X_*(T) = X^*(T^\vee)$. Langlands brought the role of the dual group to the for in studying representation theory.

There is also a notion of duality for inner classes of real forms. Suppose (G, γ) is basic data, defining an inner class of real forms, and let $G^\Gamma = \langle G, \delta \rangle$ be the extended group. Then $\tau = \mathrm{int}(\delta)$ normalizes T, and can be viewed as an automorphism of $X^*(T)$, and of D.

Definition 6.1 *Let $\tau^\vee = -\tau^t$. This is an automorphism of $X_*(T)$, and induces an automorphism of $D^\vee$. By Theorem 2.4 we obtain an automorphism of $G^\vee$. Let $\gamma^\vee$ be the image of this automorphism in Out$(G^\vee)$.*

The minus sign in the definition is important. Because of it the map $\mathrm{Out}(G) \ni \gamma \to \gamma^\vee \in \mathrm{Out}(G^\vee)$ is not necessarily a group homomorphism.

Exercise 6.2 If $\gamma = 1$ then $\gamma^\vee$ is the image of $-w_0$ in $\mathrm{Out}(G)$ where w_0 is the long element of the Weyl group. This is trivial if and only if $-1 \in W$.

Example 6.3 Suppose $G = GL(n, \mathbb{C})$ and $\gamma = 1$. Recall $\mathrm{Out}(G) \simeq \mathbb{Z}/2\mathbb{Z}$. Then $G^\vee \simeq GL(n, \mathbb{C})$, and $\gamma^\vee \neq 1$.

Example 6.4 In general if $\gamma = 1$ then $\gamma^\vee$ is the inner class of the split real form of $G^\vee$, and vice versa.

Thus given basic data (G, γ) we obtain dual basic data $(G^\vee, \gamma^\vee)$, and the setup is entirely symmetric. Define the extended group $G^{\vee\Gamma} = G^\vee \rtimes \Gamma$ using $\gamma^\vee$.

Example 6.5 If γ is the inner class of the split real form, then by Example 6.4 $G^{\vee\Gamma} = G^\vee \times \Gamma$.

Let ${}^LG$ be the L-group of (G,γ) [5]. Recall ${}^LG = G^\vee \rtimes \Gamma$, and this is a direct product if γ is the inner class of the split real form. By Example 6.4 this suggests:

Lemma 6.6 *$G^{\vee\Gamma} \simeq {}^LG$.*

Because of our emphasis on the Cartan involution this is not entirely obvious; see [4, Definition 9.6 and (9.7)(a-d)].

We interrupt our program for a word from our sponsor.

6.1 Representations

As discussed in David Vogan's lectures we work in the context of $(\mathfrak{g},K)$-modules. We will be considering multiple real, and even strong real, forms simultaneously. So given basic data (G,γ), choose representatives $\{x_i \mid i \in \mathcal{I}\}$ of the strong real forms (cf. Definition 5.32) with associated Cartan involutions θ_i and subgroups $K_i = G^{\theta_i}$. Out basic object of study is the collection of $(\mathfrak{g},K_i)$ modules for $i \in \mathcal{I}$.

Also recall from Vogan's lectures that a basic invariant of an admissible representation is its infinitesimal character, which we can identify via the Harish-Chandra homomorphism with (the W-orbit of) an element λ of $\mathfrak{t}^*$. We say λ is *regular* (resp. *integral*) if $\langle\lambda,\alpha^\vee\rangle \neq 0$ (resp. is in $\mathbb{Z}$) for all roots. The set of irreducible admissible representations with a given infinitesimal character is finite.

An important special case is ρ, the infinitesimal character of the trivial representation.

The *Zuckerman translation principle* asserts that there is a bijection between the irreducible representations with infinitesimal character λ and λ' provided λ,λ' are regular and $\lambda-\lambda' \in X^*(T)$. For this reason it is natural to work with representations *modulo translation*. In order to avoid the extra machinery required to say things this way, we resort to the following simpler, but less natural, construction.

Lemma 6.7 *Let $P \subset \mathfrak{t}^*$ be the weight lattice: $P = \{\lambda \mid \langle\lambda,\alpha^\vee\rangle \in \mathbb{Z}\}$ for all roots α. Recall $\mathfrak{t}^* = X^*(T)\otimes\mathbb{C} = X_*(T^\vee)\otimes\mathbb{C} = \mathfrak{t}^\vee$. Using this idenification the map $\lambda \to \exp(2\pi i\lambda)$ is an isomorphism $P/X^*(T) \to Z(G^\vee)$.*

This is a slight variant on fairly standard facts in algebraic groups.

Definition 6.8 *Fix a set* $\Lambda \subset \mathfrak{t}^* = X^*(T) \otimes \mathbb{C}$ *of representatives of the quotient* $P/X^*(T)$*. If* G *is semisimple this is a finite set. We assume that the elements* λ *of* Λ *are regular:* $\langle \lambda, \alpha^\vee \rangle \neq 0$*. Write* $\lambda \to z^\vee(\lambda) = \exp(2\pi i \lambda)$ *for the isomorphism* $\Lambda \to Z(G^\vee)$*.*

For example suppose G is semisimple and simply connected. Then $Z(G^\vee) = 1$ and $P = X^*(T)$. The obvious choice of representatives of $P/X^*(T)$ is $\{0\}$, but 0 is not regular. Any regular weight will do, the standard choice is $\Lambda = \{\rho\}$.

Note that if λ is regular and integral then the representations with infinitesimal character are in bijection with those of infinitesimal character λ' for some $\lambda' \in \Lambda$. For this reason we consider only representation with infinitesimal character in Λ.

We can now define various set of representations. For K the fixed points of an involution let

$$\Pi(G, K, \lambda) \tag{6.9)(a}$$

be the set of irreducible admissible $(\mathfrak{g}, K)$-modules with infinitesimal character λ. This is a finite set. Let

$$\Pi(G, K, \Lambda) = \coprod_{\lambda \in \Lambda} \Pi(G, K, \lambda), \tag{6.9)(b}$$

the representations with infinitesimal character an element of Λ. With $\mathcal{I}$ as in Definition 5.32 let

$$\Pi(G, \gamma, \lambda) = \coprod_{i \in \mathcal{I}} \Pi(G, K_i, \lambda) \tag{6.9)(c}$$

and

$$\Pi(G, \gamma, \Lambda) = \coprod_{i \in \mathcal{I}} \Pi(G, K_i, \Lambda), \tag{6.9)(d}$$

the corresponding sets of representations of all strong real forms in the inner class γ.

Example 6.10 Suppose G is semisimple and simply connected, and as above take $\Lambda = \{\rho\}$, the infinitesimal character of the trivial representation. If G is also adjoint the strong real forms and real forms coincide, and (6.9)(d) becomes the collection of representations of real forms of G, in this innner class, with infinitesimal character ρ.

Remark 6.11 We are going to parametrize the set (6.9)(d). Roughly speaking you can think of this as representations of real forms of G in this inner class, with infinitesimal character ρ. It differs from this in two ways: we need *strong* real forms in place of real forms, and more than one (regular, integral) infinitesimal character. These technicalities are crucial for obtaining a natural bijection with an explicit combinatorial object.

6.2 L-homomorphisms

Returning to our regularly scheduled program, the Langlands classification describes representations of real forms of G in a given inner class in terms of homomorphisms of the Weil group in ${}^LG$. We briefly describe this in terms convenient for us.

Definition 6.12 *The Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ is $\langle \mathbb{C}^*, j\rangle$ with relations $jzj^{-1} = \overline{z}$ and $j^2 = -1$.*

An admissible homomorphism $\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$ is a continuous homomorphism such that $\phi(\mathbb{C}^*)$ consists of semisimple elements, and $\phi(j) \in G^{\vee\Gamma}\backslash G^\vee$.

Theorem 6.13 *(Langlands [7]) Fix a real form θ in the inner class of γ, and set $K = G^\theta$. Asociated to an admissible homomorphism ϕ is a finite set Π_ϕ of irreducible $(\mathfrak{g}, K)$-modules (possibly empty), called the L-packet of ϕ, depending only on the $G^\vee$-conjugacy class of ϕ. The irreducible admissible $(\mathfrak{g}, K)$-modules are a disjoint union of L-packets Π_ϕ, as ϕ runs over $G^\vee$-conjugacy classes of admissible homomorphisms.*

Here is how to define admissible homomorphisms explicitly. Suppose $\lambda \in X^*(T)\otimes\mathbb{C}$. Identifying this with $X_*(T^\vee)\otimes\mathbb{C}$ we can define $\exp(2\pi i\lambda) \in T^\vee$. Suppose $y \in G^{\vee\Gamma}\backslash G^\vee$ normalizes $T^\vee$, and satisfies $y^2 = \exp(2\pi i\lambda)$.

Exercise 6.14 Define

$$\begin{aligned} \phi(z) &= z^\lambda \overline{z}^{\mathrm{Ad}(y)\lambda} \\ \phi(j) &= \exp(-\pi i\lambda)y. \end{aligned} \tag{6.15}$$

The first line is shorthand for $\phi(e^z) = \exp(z\lambda + \overline{z}\mathrm{Ad}(y)\lambda)$. Show that ϕ is an admissible homomorphism, and every admissible homomorphism is G-conjugate to one of this form.

Suppose for the moment that $y^2 \in Z(G^\vee)$. Then y is in (the numerator of) the space of definition 5.35, applied to $G^\vee$, call it $\mathcal{X}^\vee$:

$$\mathcal{X}^\vee = \{y \in \mathrm{Norm}_{G^{\vee\Gamma}\backslash G^\vee}(T^\vee) \,|\, y^2 \in Z(G^\vee)\}/T^\vee. \tag{6.16}$$

Conversely suppose $y \in \mathcal{X}^\vee$. Then we can find $\lambda \in X^*(T) \otimes \mathbb{C}$ such that $\exp(2\pi i\lambda) = y^2$. The kernel of $X \to \exp(2\pi i X)$ from $X_*(T^\vee) \otimes \mathbb{C}$ to $T^\vee$ is precisely $X_*(T^\vee) = X^*(T)$, so λ is well defined up translation by $X^*(T)$.

The infinitesimal character of Π_ϕ is λ; the condition that $\exp(2\pi i\lambda) \in Z(G^\vee)$ is precisely that λ is *integral*.

Let

$$\mathrm{Hom}_{adm,int}(W_{\mathbb{R}}, G^{\vee\Gamma})/G^\vee$$

be the admissible Weil group homomorphisms such that the infinitesimal character of Π_ϕ is integral. Define an equivalence relation $\sim$ on this space, given by conjugation by $G^\vee$ and *translation*, replacing λ with $\lambda + \mu$ for $\mu \in X^*(T)$. Putting this all together, it is not hard to prove:

Lemma 6.17 *There is a canonical bijection between*

$$\mathcal{X}^\vee \longleftrightarrow Hom_{adm,int}(W_{\mathbb{R}}, G^{\vee\Gamma})/\sim . \tag{6.18}$$

7 Parametrizing the Admissible Dual

At the end of the previous Section we have observed a remarkable fact: $\mathcal{X} = X(G, \gamma)$ (Definition 5.35) parametrizes the space of K orbits on G/B, for all the different K simultaneously (Theorem 5.40), while the same construction when applied to $G^\vee$ parametrizes conjuacy class of integral, admissible Weil group homomorphisms, up to translation.

Of course the situation is completely symmetric, and we can view $\mathcal{X}^\vee$ as giving the $K^\vee$ orbits on $G^\vee/B^\vee$ (for various $K^\vee$). This is a very fruitful point of view, but for now we think of $\mathcal{X}$ and $\mathcal{X}^\vee$ differently.

7.1 R-packets and L-packets

We now have two ways of decomposing admissible representations into a disjoint union of finite sets.

On the one hand as discussed in Section 6.2 associated to an $\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$ is an L-packet Π_ϕ. Thus by Lemma 6.17 $\mathcal{X}^\vee$ parametrizes integral L-packets of representations (up to translation), simultaneously for all strong real forms of G.

On the other hand, fix a real form θ of G, with $K = G^\theta$. As described in the lectures of Peter Trapa, associated to a K-orbit $\mathcal{O}$ on G/B is a finite set of representations, all with trivial infinitesimal character, parametrized by K-equivariant local systems on the orbit.

By the translation principle (Section 6.1) we can view an R-packet as being a set of representations with infinitesimal character λ for any $\lambda \in \rho + X^*(T)$. This does not hold if λ is (integral but) not in $\rho + X^*(T)$. Suffice it to say that it is possible to generalize Trapa's construction, and associate to any orbit $\mathcal{O}$ and integral infinitesimal character λ a set of representations with infinitesimal character λ; this is a finte set, and may be empty.

We refer to this (possibly empty) set as an *R-packet*, defined by a K orbits on G/B and an infinitesimal character.

We would like to refine one or the other of these results to give data parametrizing individual representations. The key to this is:

Proposition 7.1 (Vogan [12] Propositioin 8.3) *The intersection of an R-packet and L-packet is at most one element.*

Roughly speaking, this says the admissible representation of all strong real forms of G, with integral infinitesimal character, up to translation, are parametrized by a subset of $\mathcal{X} \times \mathcal{X}^\vee$ as follows. Given $y \in \mathcal{X}^\vee$ recall $y^2 \in Z(G^\vee) \simeq \Lambda$ (Definition 6.8); choose $\lambda \in \Lambda$ so that $z(\lambda^\vee) = y^2$. Then take the intersection of the R-packet defined by $x \in \mathcal{X}$ and λ with the L-packet defined by $y \in \mathcal{X}^\vee$ (if non-empty).

The entire machinery was set up to make a very precise version of this statement true. To define the main parameter space we need one more definition.

Suppose $x \in \mathcal{X}$. Then $\theta_x = \text{int}(x)$ is a well defined involution of T or $\mathfrak{t} = Lie(T)$. (To be careful $\text{int}(x)$ is not necessarily a well-defined involution of G, since x is only defined up to conjugation by T). Its transpose θ_x^T can be viewed as an involution of $\mathfrak{t}^\vee = \text{Lie}(T^\vee)$.

Definition 7.2 *Given basic data (G, γ), define the dual data $(G^\vee, \gamma^\vee)$ as in*

Section 6, and define $\mathcal{X} = \mathcal{X}(G,\gamma)$*,* $\mathcal{X}^\vee = \mathcal{X}(G^\vee,\gamma^\vee)$*. let*

$$\mathcal{Z}(G,\gamma) = \{(x,y) \in \mathcal{X}\times\mathcal{X}^\vee \mid \theta_y = -(\theta_x)^t\}. \tag{7.3}$$

If (G,γ) *are understood we write* $\mathcal{Z} = \mathcal{Z}(G,\gamma)$*.*

Recall we have fixed a set Λ of infinitesimal characters in Section 6.1.

Definition 7.4 *Suppose* $(x,y) \in \mathcal{Z}$*. Choose* $\lambda \in \Lambda$ *so that* $z^\vee(\lambda) = y^2 \in Z(G^\vee)$ *(Definition 6.8). Define* $\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$ *by Exercise 6.14, and let* $\Pi_L(x,y)$ *be the L-packet of* $(\mathfrak{g},K_x)$*-modules defined by* ϕ*.*

Let $\Pi_R(x,y)$ *be the set of* $(\mathfrak{g},K_x)$*-modules obtained by taking all local systems on the* K_x *orbit on* G/B *corresponding to* x*, with infinitesimal character* λ *(see Peter Trapa's lectures).*

We're being cavalier about one technical point here. Each element $x \in \mathcal{X}$ is a T-conjugacy class of elements of G^Γ. In fact for each x we need to choose one such ξ, and K_x is really K_ξ.

Theorem 7.5 *The intersection* $\Pi_R(x,y)\cap\Pi_L(x,y)$ *is non-empty, and is therefore a single representation of this real form.*

This defines a bijection

$$\mathcal{Z} \longleftrightarrow \coprod_i \Pi(G,K_i,\Lambda), \tag{7.6}$$

from $\mathcal{Z}$ *to the set of the representations of strong real forms of* G*, with infinitesimal character contained in* Λ*.*

See [3, Theorem 10.3].

Recall by Theorem 5.40 we may view $\mathcal{Z}$ as a subset of

$$\coprod_{i\in\mathcal{I}} K_i\backslash G/B \times \coprod_{j\in\mathcal{I}^\vee} K_j^\vee\backslash G^\vee/B^\vee \tag{7.7}$$

where $\mathcal{I},\mathcal{I}^\vee$ are the strong real forms of G and $G^\vee$ as in Definition 5.32.

Example 7.8 Suppose G is semisimple and simply connected. Then

$$\mathcal{Z} \longleftrightarrow \coprod_i \Pi(G,K_i,\rho), \tag{7.9}$$

the representations of strong real forms of G with trivial central character.

Example 7.10 Representations of $SL(2)$ and $PGL(2)$.

Here is a table from [3, Section 12], discussed in more detail in [2]. For notation see the references; here is a sketch.

Note that $PGL(2,\mathbb{C}) = PSL(2,\mathbb{C})$, the adjoint group of type A_1. For both $SL(2)$ and $PGL(2)$ we can drop δ from the notation since $\gamma = 1$. Let $t = \mathrm{diag}(i, -i)$ and $n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, considered in $SL(2)$ or $PGL(2)$.

If $K = \mathbb{C}^* \subset SL(2,)$ there are three orbits of K on G/B, labelled $\mathcal{O}_0, \mathcal{O}_\infty, \mathcal{O}_*$ ($\mathcal{O}_*$ is open). For $PGL(2)$ there are two orbits, $\mathcal{O}'_+$ and $\mathcal{O}'_*$.

At infinitesimal character ρ $SL(2,\mathbb{R})$ has three irreducible representations: $DS_\pm$ (discrete series) and the trivial representation. It also has a unique irreducible principal series representation PS_{odd}.

At infinitesimal character ρ $PGL(2,\mathbb{R})$ has three irreducible representations: DS (discrete series), the trivial representation and the sgn representation. At infinitesimal character 2ρ it also has a two irreducible principal series representation $PS_\pm$.

Table of representations of $SL(2)$ and $PGL(2)$

Orbit	x	x^2	θ_x	$G_x(\mathbb{R})$	λ	rep	Orbit	y	y^2	θ_y	$G^\vee_y(\mathbb{R})$	λ	rep
$\mathcal{O}_{2,0}$	Id	Id	1	$SU(2,0)$	ρ	$\mathbb{C}$	$\mathcal{O}'_*$	n	Id	-1	$SO(2,1)$	2ρ	PS_+
$\mathcal{O}_{0,2}$	-Id	Id	1	$SU(0,2)$	ρ	$\mathbb{C}$	$\mathcal{O}'_*$	n	Id	-1	$SO(2,1)$	2ρ	PS_-
$\mathcal{O}_0$	t	-Id	1	$SU(1,1)$	ρ	DS_+	$\mathcal{O}'_*$	n	Id	-1	$SO(2,1)$	ρ	$\mathbb{C}$
$\mathcal{O}_\infty$	-t	-Id	1	$SU(1,1)$	ρ	DS_-	$\mathcal{O}'_*$	n	Id	-1	$SO(2,1)$	ρ	sgn
$\mathcal{O}_*$	n	-Id	-1	$SU(1,1)$	ρ	$\mathbb{C}$	$\mathcal{O}'_+$	t	Id	1	$SO(2,1)$	ρ	DS
$\mathcal{O}_*$	n	-Id	-1	$SU(1,1)$	ρ	PS_{odd}	$\mathcal{O}'_{3,0}$	Id	Id	1	$SO(3)$	ρ	$\mathbb{C}$

7.2 Vogan Duality

Recall basic data (G, γ) determine basic data $(G^\vee, \gamma^\vee)$. The parameter space $\mathcal{Z}(G, \gamma) \subset \mathcal{X} \times \mathcal{X}^\vee$ has an obvious symmetry. The map $(x, y) \to (y, x)$ takes

$\mathcal{Z}$ is a bijection

$$\mathcal{Z}(G,\gamma) \longleftrightarrow \mathcal{Z}(G^\vee,\gamma^\vee). \tag{7.11}$$

This is *Vogan duality* in our setting. As originally defined in [12] Vogan duality is a bijection between representation of real forms of G and representations of real forms of $G^\vee$. This is a stronger version of that result, in that it treats all (strong) real forms simultaneously.

7.3 Blocks

Theorem 7.5 says that $\mathcal{Z}$ parametrizes representations of all strong real forms of G simultaneously. Fix $\mathcal{I}$ as in Definition 5.32, $x_i \in \mathcal{I}$ with corresponding K_i. The subset of $\mathcal{Z}$ parametrizing $(\mathfrak{g}, K_i)$-modules is those (x,y) such that x is G-conjugate to x_i. This set of x is isomorphic to $K_i\backslash G/B$.

By symmetry it is natural to do the same on the dual side. So let $\{y_j \mid j \in \mathcal{I}^\vee\}$ be a choice of representatives of strong real forms of $G^\vee$, fix j and $K_j^\vee = \mathrm{Cent}_{G^\vee}(y_j)$. Then the $y \in \mathcal{X}^\vee$ which are $G^\vee$-conjugate to y_j is isomorphic to $K_j^\vee\backslash G^\vee/B^\vee$.

So fix i,j and consider

$$\{(x,y) \in \mathcal{Z} \,|, x \sim_G x_i, y \sim_{G^\vee} y_j\} \subset K_i\backslash G/B \times K_j^\vee\backslash G^\vee/B^\vee. \tag{7.12}$$

By Theorem 7.5 this parametrizes a set of $(\mathfrak{g}, K_i)$ modules: this set is a *block*. By the symmetry of the situation it also parametrizes the *dual block* of $(\mathfrak{g}^\vee, K_j^\vee)$-modules.

8 Appendix: Atlas Examples

This appendix is under construction as of July 14.

We illustrate some of the ideas of the notes using the `atlas` software.

8.1 Section 1: Root Data

Here are some examples of the matrices A and B of Remark 1.30.

Example 8.1 Here is the root datum for $SL(3,\mathbb{C})$:

```
main: type
Lie type: A2 sc s
main: rootdatum
Name an output file (return for stdout, ? to abandon):
cartan matrix :
   2  -1
  -1   2

root basis :
   2  -1
  -1   2

coroot basis :
   1   0
   0   1
```

For more on the `type` command see Section 8.2. The matrices A and B are the `root basis` and `coroot basis` matrices, respectively.

Example 8.2 Here is the root datum for $PSL(3,\mathbb{C})$:

```
empty: type
Lie type: A2 ad s
main: rootdatum
Name an output file (return for stdout, ? to abandon):
cartan matrix :
   2  -1
  -1   2
```

```
root basis :
   1   0
   0   1

coroot basis :
   2  -1
  -1   2
```

Note that A and B are switched.

Exercise 8.3 Show that the following three root data are isomorphic:
Case 1:

```
root basis :
   2  -3
  -1   2

coroot basis :
   1   0
   0   1
```

Case 2:

```
root basis :
   1   0
   0   1

coroot basis :
   2  -1
  -3   2
```

Case 3:

```
root basis :
  -1   2
   0   1

coroot basis :
  -2   1
   1   0
```

See Example 8.12.

8.2 Section 4: Inner Classes

Here are some examples of groups and inner forms defined using the `atlas` software.

The cyclic group of order n is specified as `Z/n`. We view this as the group

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z} = \{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n}\}$$

An element of finite order of $\mathbb{C}^*$ is given by an element of $\mathbb{Q}$.

The `type` command defines a complex group and an inner class. The user enters a product of simple groups and tori, and a finite subgroup of the center (cf. Section 2.2). The arguments `sc` and `ad` give the simply connected and adjoint groups. The arguments `s`, `c`, `C` and `u` give the split, compact, complex and unequal rank inner classes.

Example 8.4 (Example 1: unique inner class of $SL(2,\mathbb{C})$)

```
empty: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
```

Example 8.5 (Example 2: unique inner class of $PSL(2,\mathbb{C})$)

```
main: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2

enter inner class(es): s
```

Example 8.6 (Example 3: split inner class of $GL(6,\mathbb{C})$)

```
main: type
Lie type: A5.T1
elements of finite order in the center of the simply connected group:
Z/6.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/6,1/6

enter inner class(es): ss
main: showrealforms
(weak) real forms are:
0: sl(3,H).gl(1,R)
1: sl(6,R).gl(1,R)
```

Example 8.7 (Example 4: compact inner class of $GL(6,\mathbb{C})$)

```
main: type
Lie type: A5.T1
elements of finite order in the center of the simply connected group:
Z/6.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/6,1/6

enter inner class(es): cc
main: showrealforms
(weak) real forms are:
0: su(6).u(1)
1: su(5,1).u(1)
2: su(4,2).u(1)
3: su(3,3).u(1)
```

Example 8.8 (Example 5: compact=split inner class of $SO(10,\mathbb{C})$)

```
main: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
```

```
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): s
main: showrealforms
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
main: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): c
main: showrealforms
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
main: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): u
```

```
main: showrealforms
(weak) real forms are:
0: so(11,1)
1: so(9,3)
```

Example 8.9 (Example 6: illegal inner class)

```
main: type
Lie type: A1.A1
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

enter inner class(es): C
sorry, that inner class is not compatible with the weight lattice
```

Example 8.10 (Example 7: illegal inner class of type D_4)

```
main: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

enter inner class(es): u
sorry, that inner class is not compatible with the weight lattice
enter inner class(es):
```

Example 8.11 (Example 8: inner class of complex group)

```
main: type
Lie type: E8.E8
elements of finite order in the center of the simply connected group:

enter kernel generators, one per line
```

```
(ad for adjoint, ? to abort):

enter inner class(es): C
main: showrealforms
(weak) real forms are:
0: e8(C)
```

Example 8.12 There are three ways to define G_2 (and its unique inner class).

```
empty: type
Lie type: G2 sc s
```

or

```
main: type
Lie type: G2 ad s
```

or

```
main: type

Lie type: G2
elements of finite order in the center of the simply connected group:

enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
```

Since G_2 is both simply connected and adjoint, these are isomorphic. The software makes different choices for them, leading to the superficially different, but isomorphic, root data of Exercise 8.3.

8.3 Section 5.3: K orbits on G/B

Example 8.13 Here is $K\backslash G/B$ for $SL(2,\mathbb{R})$:

```
empty: kgb
Lie type: A1 sc s
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
kgbsize: 3
Name an output file (return for stdout, ? to abandon):
0:  0  0  [n]   1    2  e
1:  0  0  [n]   0    2  e
2:  1  1  [r]   2    *  1
```

There are 3 elements, two closed orbits and one open. The third column gives the cross action of the simple reflection, and the fourth the Cayley transform.

Example 8.14 Here is $K\backslash G/B$ for $PGL(2,\mathbb{R})$, which has two elements.

```
real: type
Lie type: A1 ad s
main: kgb
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
kgbsize: 2
Name an output file (return for stdout, ? to abandon):
0:  0  0  [n]   0    1  e
1:  1  1  [r]   1    *  1
```

If you just want the number of orbits of K on G/B use `kgborder`.

Example 8.15 `real: type`

```
Lie type: E8 sc s
main: kgborder
(weak) real forms are:
0: e8
1: e8(e7.su(2))
2: e8(R)
enter your choice: 2
kgbsize: 320206
```

Also `kgborder` gives the order relation on orbits.

Example 8.16 The Bruhat order for S_3.

If G is complex then $K\backslash G/B \simeq W$ and the order is the Bruhat order.

```
real: type
Lie type: A2.A2 sc C
main: kgb
there is a unique real form: sl(3,C)
kgbsize: 6
Name an output file (return for stdout, ? to abandon):
0:  0  0  [C,C,C,C]   2  1  2  1    *  *  *  *  e
1:  1  0  [C,C,C,C]   4  0  3  0    *  *  *  *  2,4
2:  1  0  [C,C,C,C]   0  3  0  4    *  *  *  *  1,3
3:  2  0  [C,C,C,C]   5  2  1  5    *  *  *  *  2,1,3,4
4:  2  0  [C,C,C,C]   1  5  5  2    *  *  *  *  1,2,4,3
5:  3  0  [C,C,C,C]   3  4  4  3    *  *  *  *  1,2,1,3,4,3
real: kgborder
kgbsize: 6
Name an output file (return for stdout, ? to abandon):
0:
1: 0
2: 0
3: 1,2
4: 1,2
5: 3,4
```

The last column gives an element of W embedded diagonally, so ignoring $3, 4$ gives $W = \{e, 2, 1, 21, 12, 121\}$ as products of simple reflections.

8.4 Section 7: Parametrizing the Admissible Dual

The `block` command gives the *block* of representations parametrized by pairs (x, y) corresponding to fixed strong real forms of G and $G^\vee$.

Example 8.17 Here is the block of representations of $SL(2,\mathbb{R})$ corresponding to the split real form $PGL(2,\mathbb{R})$ of $G^\vee$.

```
empty: type
```

```
Lie type: A1 sc s
main: block
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):
0(0,1):  0  0  [i1]  1   (2,*)   e
1(1,1):  0  0  [i1]  0   (2,*)   e
2(2,0):  1  1  [r1]  2   (0,1)   1
```

There are three representations, two discrete series and one principal series. See Example 7.10.

Example 8.18 Here is the block of $PGL(2,\mathbb{R})$, consisting of the trivial, sgn and discrete series representations at ρ, dual to the previous one.

```
block: type
Lie type: A1 ad s
main: block
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):
0(0,2):  0  0  [i2]  0   (1,2)   e
1(1,0):  1  1  [r2]  2   (0,*)   1
2(1,1):  1  1  [r2]  1   (0,*)   1
```

Example 8.19 Here is the unique block of the $SL(3,\mathbb{C})$ (viewed as a real group), parametrized by $W = S_3$.

```
block: type
Lie type: A2.A2 sc C
main: block
there is a unique real form: sl(3,C)
there is a unique dual real form choice: sl(3,C)
Name an output file (return for stdout, ? to abandon):
0(0,5):  0  0  [C+,C+,C+,C+]  2  1  2  1   (*,*)  (*,*)  (*,*)  (*,*)   e
1(1,4):  1  0  [C+,C-,C+,C-]  4  0  3  0   (*,*)  (*,*)  (*,*)  (*,*)   2,4
2(2,3):  1  0  [C-,C+,C-,C+]  0  3  0  4   (*,*)  (*,*)  (*,*)  (*,*)   1,3
3(3,2):  2  0  [C+,C-,C-,C+]  5  2  1  5   (*,*)  (*,*)  (*,*)  (*,*)   2,1,3,4
4(4,1):  2  0  [C-,C+,C+,C-]  1  5  5  2   (*,*)  (*,*)  (*,*)  (*,*)   1,2,4,3
5(5,0):  3  0  [C-,C-,C-,C-]  3  4  4  3   (*,*)  (*,*)  (*,*)  (*,*)   1,2,1,3,4,3
```

You can find out the sizes of all of the blocks using the `blocksizes` command. The rows and columns are parametrized by real forms of G and $G^\vee$, which are available from the `showrealforms` and `showdualrealforms` commands.

Example 8.20 Here are the blocks of real forms of E_8.

```
block: type
Lie type: E8 sc s
main: blocksizes
       0       0       1
       0    3150   73410
       1   73410  453060
main: showrealforms
(weak) real forms are:
0: e8
1: e8(e7.su(2))
2: e8(R)
main: showdualforms
(weak) dual real forms are:
0: e8
1: e8(e7.su(2))
2: e8(R)
```

The block of size $453,060$ is the block of the trivial representation of the split real form, which is self-dual. There are two blocks which are singletons: the trivial representation of the compact real form, and its dual, the unique irreducible principal series of the split real form with infinitesimal character ρ.

References

[1] http://en.wikipedia.org/wiki/Cartan_matrix.

[2] Jeffrey Adams. Guide to the atlas software: Computational representation theory of real reductive groups. In *Representation Theory of Real Reductive Groups, Proceedings of Conference at Snowbird, July 2006*, Contemp. Math. Amer. Math. Soc., 2008.

[3] Jeffrey Adams and Fokko du Cloux. Algorithms for representation theory of real reductive groups. *J. Inst. Math. Jussieu*, 8(2):209–259, 2009.

[4] Jeffrey Adams and David A. Vogan, Jr. L-groups, projective representations, and the Langlands classification. *Amer. J. Math.*, 114(1):45–138, 1992.

[5] A. Borel. *Automorphic L-Functions*, volume 33 of *Proc. Symp. Pure Math.* American Math. Soc., Providence, Rhode Island, 1979.

[6] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.

[7] R. P. Langlands. On the classification of irreducible representations of real algebraic groups. In *Representation theory and harmonic analysis on semisimple Lie groups*, volume 31 of *Math. Surveys Monogr.*, pages 101–170. Amer. Math. Soc., Providence, RI, 1989.

[8] Claudio Procesi. *Lie groups*. Universitext. Springer, New York, 2007. An approach through invariants and representations.

[9] R. W. Richardson and T. A. Springer. The Bruhat order on symmetric varieties. *Geom. Dedicata*, 35(1-3):389–436, 1990.

[10] T. A. Springer. Reductive groups. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 3–27. Amer. Math. Soc., Providence, R.I., 1979.

[11] T. A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1998.

[12] David A. Vogan, Jr. Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality. *Duke Math. J.*, 49(4):943–1073, 1982.