$SL(2,\mathbb{R})$ Reference Card

Lie Algebra

$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$: two-by-two trace 0 real matrices
$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$: two-by-two trace 0 complex matrices

$$E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$[H,E] = 2E,\ [H,F] = 2F\ [E,F] = H$$

$\sigma(X) = \overline{X}$, $\mathfrak{g}_0 = \mathfrak{g}^\sigma$,
$\theta(X) = X + X^t$

$$\mathfrak{k}_0 = \mathfrak{g}_0^\theta = \mathbb{R}(iH), \mathfrak{k} = \mathfrak{g}^\theta = \mathbb{C}(H),$$

Lie Group

$G(\mathbb{C}) = SL(2,\mathbb{C})$: two-by-two complex, determinant $= 1$
$\sigma(g) = \overline{g}$
$G = G(\mathbb{R}) = G(\mathbb{C})^\sigma = SL(2,\mathbb{R})$: two-by-two,real, det $= 1$
$\theta(g) = {}^t g^{-1}$ (Cartan involution)

$K = G^\theta = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a,b \in \mathbb{C}, a^2 + b^2 = 1\} \simeq \mathbb{C}^*$

$K(\mathbb{R}) = G(\mathbb{R})^\theta = \{\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\} = SO(2) \simeq S^1$

The representations $I(\nu,\epsilon)$

$\nu \in \mathbb{C}, \epsilon \in \mathbb{Z}_2 = \{0,1\}$
$I(\nu,\epsilon)$ has basis $\{v_j \mid j \in \epsilon + 2\mathbb{Z}\}$
$(\pi, I(\nu,\epsilon))$:

$$\pi(H)v_j = jv_j$$
$$\pi(E)v_j = 1/2(\nu + (j+1))v_{j+2}$$
$$\pi(F)v_j = 1/2(\nu - (j-1))v_{j-2}$$

Auxiliary formulas

$$\pi(E)v_{j-2} = 1/2(\nu + (j-1))v_j$$
$$\pi(F)v_{j+2} = 1/2(\nu - (j+1))v_j$$
$$\pi(EF)v_j = \frac{1}{4}(\nu^2 - (j-1)^2)v_j$$
$$\pi(FE)v_j = \frac{1}{4}(\nu^2 - (j+1)^2)v_j$$

Casimir element

$$\Omega = H^2 + 2EF + 2FE + 1$$
$$= (H+1)^2 + 4FE$$
$$= (H-1)^2 + 4EF$$

This is the usual Casimir element +1.
Ω acts on $I(\nu,\epsilon)$ by ν^2.
Infinitesimal character of $I(\nu,\epsilon)$: $\nu \sim -\nu$,

Reducibility

$I(\nu,\epsilon)$ is reducible if and only if $\nu \in 1 + \epsilon + 2\mathbb{Z}$.
$\nu = n \in 1 + \epsilon + 2\mathbb{Z}, n \geq 0$:

1. $I_0(n,(-1)^{n+1})$: finite dimensional quotient, dimension n, K-types $-n+1,\dots,n-1$ (0 if $n = 0$)
2. $I_+(n,(-1)^{n+1})$: summand, K-types $n+1, n+3, \dots$
3. $I_-(n,(-1)^{n+1})$: summand, K-types $-n-1, -n-3, \dots$

$\nu = n \in 1 + \epsilon + 2\mathbb{Z}, n \leq 0$:

1. $I_0(n,(-1)^{n+1})$: finite dimensional summand, dimension n, K-types $n+1,\dots,-n-1$ (0 if $n = 0$)
2. $I_+(n,(-1)^{n+1})$: quotient, K-types $-n+1, -n+3, \dots$
3. $I_-(n,(-1)^{n+1})$: quotient, K-types $n-1, n-3, \dots$

Discrete Series and limits

$$DS_\pm(n) := I_\pm(n,(-1)^{n+1}) \quad (n = 1,2,\dots):$$

discrete series representations.
$DS_\pm(n)$ has K-types $\pm\{n+1, n+3, \dots\}$. Infinitesimal character is n (regular).

$$LDS_\pm := I_\pm(0,-):$$

limits of discrete series, tempered, not discrete, with K-types $\pm\{1,3,\dots\}$. Infinitesimal character is 0 (singular).

Finite Dimensional Representations

$$F(n) := I_0(\pm n, (-1)^{n+1}) \quad (n = 1,2,\dots):$$

finite dimensional, dimension n, infinitesimal character n, K-types $\{-n+1, n+3, \dots, n-1\}$.

Irreducible representations

Every irreducible $(\mathfrak{g},K)$-module is isomorphic to exactly one of these:

1. $I(\nu,\epsilon) = I(-\nu,\epsilon) \quad (\nu \notin 1 + \epsilon + 2\mathbb{Z})$;
2. $F(n)$ $(n = 1,2,\dots)$
3. $DS_\pm(n) \quad (n = 1,2,\dots)$

Irreducible Tempered representations

(a) $I(iy,+)$ $(y \in \mathbb{R})$
(b) $I(iy,-)$ $(y \in \mathbb{R}^*)$
(b′) $LDS_\pm$ (two components of $I(0,-)$)
(c) $DS_\pm(n)$ $(n = 1,2,\dots)$

Real Infinitesimal character

$I(\nu,\epsilon)$ has real infinitesimal character if and only if $\nu \in \mathbb{R}$

Invariant Hermitian Form

If $I(\nu,\epsilon)$ is irreducible the unique (up to scalar) invariant Hermitian form satisfies:

$$(v_{j+2}, v_{j+2}) = \frac{(-\overline{\nu} + (j+1))}{(\nu + (j+1))}(v_j, v_j).$$

c-invariant Hermitian Form

If $I(\nu,\epsilon)$ is irreducible the unique (up to scalar) c-invariant Hermitian form satisfies:

$$(v_{j+2}, v_{j+2})_c = \frac{(\overline{\nu} - (j+1))}{(\nu + (j+1))}(v_j, v_j)_c.$$

Unitary dual

1. $I(iy,+)$ $(y \in \mathbb{R})$: spherical unitary principal series
2. $I(iy,-)$ $(y \in \mathbb{R}^*)$: nonspherical unitary (irreducible) princpal series
3. $LDS_\pm = I_\pm(0,-)$: limits of discrete series
4. $DS_\pm(n) = I_\pm(n,(-1)^{n+1})$ $(n = 1,2,3\dots)$: discrete series
5. $I(x,0) \simeq I(-x,0)$ $(0 < x < 1)$: complementary series
6. the trivial representation $F(1) = I_0(\pm 1, +)$

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