Galois and Cartan Cohomology of Real Groups

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January 26, 2017

Abstract

Suppose G is a complex, reductive algebraic group. A real form of G is an antiholomorphic involutive automorphism σ, so $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ is a real Lie group. Write $H^1(\sigma, G)$ for the Galois cohomology (pointed) set $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G)$. A Cartan involution for σ is an involutive holomorphic automorphism θ of G, commuting with σ, so that $\theta\sigma$ is a compact real form of G. Let $H^1(\theta, G)$ be the set $H^1(\mathbb{Z}_2, G)$ where the action of the nontrivial element of $\mathbb{Z}_2$ is by θ. By analogy with the Galois group we refer to $H^1(\theta, G)$ as Cartan cohomology of G with respect to θ. Cartan's classification of real forms of a connected group, in terms of their maximal compact subgroups, amounts to an isomorphism $H^1(\sigma, G_{\mathrm{ad}}) \simeq H^1(\theta, G_{\mathrm{ad}})$ where G_{ad} is the adjoint group. Our main result is a generalization of this: there is a canonical isomorphism $H^1(\sigma, G) \simeq H^1(\theta, G)$.

We apply this result to give simple proofs of some well known structural results: the Kostant-Sekiguchi correspondence of nilpotent orbits; Matsuki duality of orbits on the flag variety; conjugacy classes of Cartan subgroups; and structure of the Weyl group. We also use it to compute $H^1(\sigma, G)$ for all simple, simply connected groups, and to give a cohomological interpretation of strong real forms. For the applications it is important that we do not assume G is connected.

1 Introduction

Suppose G is a complex, reductive algebraic group. A real form of G is an antiholomorphic involutive automorphism σ of G, so $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ is a real Lie group. See Section 3 for more details. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and write $H^i(\Gamma, G)$ for the Galois cohomology of G (if G is nonabelian $i \leq 1$). If we want to specify how the nontrivial element of Γ acts we will write $H^i(\sigma, G)$. The equivalence

2000 Mathematics Subject Classification: 11E72 (Primary), 20G10, 20G20
Jeffrey Adams is supported in part by National Science Foundation Grant #DMS-1317523
Olivier Taïbi is supported by ERC Starting Grant 306326.

(i.e. conjugacy) classes of real forms of G, which are inner to σ (see Section 3), are parametrized by $H^1(\sigma, G_{\mathrm{ad}})$ where G_{ad} is the adjoint group.

On the other hand, at least for G connected, Cartan classified the real forms of G in terms of holomorphic involutions as follows. We say a Cartan involution for σ is a holomorphic involutive automorphism θ, commuting with σ, so that $\sigma^c = \theta\sigma$ is a compact real form. If G is connected then θ exists, and is unique up to conjugacy by G^σ. Following Mostow we prove a similar result in general. See Section 3.

Let $H^i(\mathbb{Z}_2, G)$ be the group cohomology of G where the nontrivial element of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts by θ. As above we denote this $H^i(\theta, G)$, and we refer to this as Cartan cohomology of G. Conjugacy classes of involutions which are inner to θ are parametrized by $H^1(\theta, G_{\mathrm{ad}})$.

Thus the equivalence of the two classifications of real forms amounts to an isomorphism (for connected G) of the first Galois and Cartan cohomology spaces $H^1(\sigma, G_{\mathrm{ad}}) \simeq H^1(\theta, G_{\mathrm{ad}})$. It is natural to ask if the same isomorphism holds with G in place of G_{ad}. For our applications it is helpful to know the result for disconnected groups as well.

Theorem 1.1 *Suppose G is a complex, reductive algebraic group (not necessarily connected), and σ is a real form of G. Let θ be a Cartan involution for σ. Then there is a canonical isomorphism $H^1(\sigma, G) \simeq H^1(\theta, G)$.*

The interplay between the σ and θ pictures plays a fundamental role in the structure and representation theory of real groups, going back at least to Harish Chandra's formulation of the representation theory of $G(\mathbb{R})$ in terms of $(\mathfrak{g}, K)$-modules. The theorem is an aspect of this, and we give several applications.

Suppose X is a homogeneous space for G, equipped with a real structure σ_X which is compatible with σ_G. Then the space of $G(\mathbb{R})$ orbits on $X(\mathbb{R})$ can be understood in terms of the Galois cohomology of the stabilizer of a point in X. Similar remarks apply to computing G^θ-orbits. Note that these stabilizers may be disconnected, even if G is connected. See Proposition 5.4.

We use this principle to give simple proofs of several well known results, including the Kostant-Sekiguchi correspondence and Matsuki duality. Let $G(\mathbb{C})$ be a connected complex reductive group, with real form σ and corresponding Cartan involution θ. Let $G(\mathbb{R}) = G(\mathbb{C})^\sigma$, and $K(\mathbb{C}) = G(\mathbb{C})^\theta$. Let $\mathfrak{g}_0 = \mathfrak{g}^\sigma$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$. The Kostant-Sekiguchi correspondence is a bijection between the nilpotent $G(\mathbb{R})$-orbits on $\mathfrak{g}_0$ and the nilpotent $K(\mathbb{C})$-orbits on $\mathfrak{p}$. Matsuki duality is a bijection between the $G(\mathbb{R})$ and $K(\mathbb{C})$ orbits on the flag variety of G. See Propositions 6.1.5 and 6.2.8.

On the other hand Proposition 5.8 applied to the space of Cartan subgroups gives a simple proof of another result of Matsuki: there is a bijection between $G(\mathbb{R})$-conjugacy classes of Cartan subgroups of $G(\mathbb{R})$ and K-conjugacy classes of θ-stable Cartan subgroups of G [18]. Also a well known result about two versions of the rational Weyl group (Proposition 6.3.2) follows.

If G is connected Borovoi proved $H^1(\sigma, G) \simeq H^1(\sigma, H_f)/W_i$ where H_f is a fundamental Cartan subgroup, and W_i is a certain subgroup of the Weyl group

[8]. Essentially the same proof carries over to give $H^1(\theta, G) \simeq H^1(\theta, H_f)/W_i$. We prove this as a consequence of Theorem 1.1 (Proposition 7.5).

Let Z be the center of G and let Z_{tor} be its torsion subgroup. Associated to a real form σ is its central invariant, denoted $\text{inv}(\sigma) \in Z^\sigma_{\text{tor}}/(1+\sigma)Z_{\text{tor}}$. The formulation of a precise version of the Langlands correspondence requires the notion of strong real form. See Section 8 for this definition, and for the notion of central invariant of a strong real form, which is an element of Z^σ_{tor}.

Theorem 1.2 (Proposition 8.17) *Suppose σ is a real form of G. Choose a representative $z \in Z^\sigma_{\text{tor}}$ of $\text{inv}(\sigma) \in Z^\sigma_{\text{tor}}/(1+\sigma)Z_{\text{tor}}$. Then there is a bijection*

$$H^1(\Gamma, G) \stackrel{1-1}{\longleftrightarrow} \textit{the set of strong real forms with central invariant } z.$$

This bijection is useful in both directions. On the one hand it is not difficult to compute the right hand side, thereby computing $H^1(\sigma, G)$. Over a p-adic field $H^1(\sigma, G) = 1$ if G is simply connected. Over $\mathbb{R}$ this is not the case, and we use Theorem 1.1 to compute $H^1(\sigma, G)$ for all such groups. See Section 3 and the tables in Section 10. We used the Atlas of Lie Groups and Representations software for some of these calculations. See [9] for another approach.

On the other hand the notion of strong real form is important in formulating a precise version of the local Langlands conjecture. In that context it would be more natural if strong real forms were described in terms of classical Galois cohomology. The Theorem provides such an interpretation. See Corollary 8.18.

The authors would like to thank Michael Rapoport for asking about the interpretation of strong real forms in terms of Galois cohomology, and apologize it took so long to get back to him. We are also grateful to Tasho Kaletha for several helpful discussions during the writing of this paper and of [13], and Skip Garibaldi for a discussion of the Galois cohomology of the spin groups.

2 Preliminaries on Group Cohomology

See [22] for an overview of group cohomology.

For now suppose τ is an involutive automorphism of an abstract group G. Define $H^i(\mathbb{Z}_2, G)$ to be the group cohomology space where the nontrivial element of $\mathbb{Z}_2$ acts by τ. We will also denote this group $H^i(\tau, G)$ [1]. If G is abelian these are groups and are defined for all $i \geq 0$. Otherwise these are pointed sets, and defined only for $i = 0, 1$. Let

$$Z^1(\tau, G) = G^{-\tau} = \{g \in G \mid g\tau(g) = 1\}.$$

Then we have the standard identifications

$$H^0(\tau, G) = G^\tau,\ H^1(\tau, G) = Z^1(\tau, G)/\{g \mapsto xg\tau(x^{-1})\}$$

[1] There is a small notational issue here. If $\tau = 1$ (the identity automorphisms of G), $H^1(1, G)$ denotes the group $H^1(\mathbb{Z}_2, G)$ with $\mathbb{Z}_2$ acting trivially.

where $G^{-\tau}=\{g\in G\mid g\tau(g)=1\}$. For $g\in G^{-\tau}$ let $cl(g)$ be the corresponding class in $H^1(\tau,G)$.

If G is abelian we also have the Tate cohomology groups $\widehat{H}^i(\tau,G)$ $(i\in\mathbb{Z})$. These satisfy

$$\widehat{H}^0(\tau,G)=G^\tau/(1+\tau)G,\quad \widehat{H}^1(\tau,G)=H^1(\tau,G),$$

and (since τ is cyclic), $\widehat{H}^i(\tau,G)\simeq\widehat{H}^{i+2}(\tau,G)$ for all i.

Suppose $1\to A\to B\to C\to 1$ is an exact sequence of groups with an involutive automorphism τ. Then there is an exact sequence

(2.1)
$$1\to H^0(\tau,A)\to H^0(\tau,B)\to H^0(\tau,C)\to H^1(\tau,A)\to H^1(\tau,B)\to H^1(\tau,C)$$

Furthermore if $A\subset Z(B)$ ($Z(*)$ denotes the center of a group) then there is one further step $\to H^2(\tau,A)=A^\tau/(1+\tau)A$.

We will need the following generalization of $H^1(\tau,G)$.

Definition 2.2 *Suppose τ is an involutive automorphism of G, and A is a subset of $Z(G)$. Define*

(2.3)(a)
$$Z^1(\tau,G;A)=\{g\in G\mid g\tau(g)\in A\}$$

and

(2.3)(b)
$$H^1(\tau,G;A)=Z^1(\tau,G;A)/[g\sim tg\tau(t^{-1})\ (t\in G)].$$

These are pointed sets if $1\in A$. The map $g\mapsto g\tau(g)$ factors to a map from $H^1(\tau,G;A)$ to A.

Taking $A=\{1\}$ gives ordinary cohomology $H^1(\tau,G)$. Write $cl(g)$ for the image of $g\in Z^1(\tau,G;A)$ in $H^1(\tau,G;A)$.

We make use of twisting in nonabelian cohomology [22, Section III.4.5]. Let $Z=Z(G)$. For $g\in G$ let $\mathrm{int}(g)$ be the inner automorphism $\mathrm{int}(g)(h)=ghg^{-1}$. Fix an involutive automorphism τ of G, and $z\in Z$. Note that $\mathrm{int}(g)\circ\tau$ is an involution if and only if $g\in Z^1(\tau,G;Z)$.

Lemma 2.4 *Suppose $\tau'=\mathrm{int}(g)\circ\tau$ for some $g\in Z^1(\tau,G;Z)$. Let $w=g\tau(g)\in Z$. Then the map $h\mapsto hg^{-1}$ induces an isomorphism*

$$H^1(\tau,G;z)\to H^1(\tau',G;zw^{-1}).$$

If $H^1(\tau,Z)=1$, this isomorphism is independent of the choice of $g\in Z^1(\tau,G;w)$ satisfying $\tau'=\mathrm{int}(g)\circ\tau$.

In particular $H^1(\tau,G)\simeq H^1(\tau',G)$ if $\tau'=\mathrm{int}(g)\circ\tau$, where $g\in Z^1(\tau,G)$, and this isomorphism is canonical if $H^1(\tau,Z)=1$.

Finally suppose τ' is conjugate to τ by an inner automorphism of G. Then $H^1(\tau,G)\simeq H^1(\tau',G)$, and this isomorphism is canonical if

$$\ker\big(H^1(\tau,Z)\to H^1(\tau,G)\big)=1.$$

We omit the elementary proof.

Write $[\tau]$ for the G-conjugacy class of τ.

Definition 2.5 *Assume* $\ker\big(H^1(\tau,Z)\to H^1(\tau,G)\big)=1$. *Given a G-conjugacy class* $[\tau]$ *of involutive automorphisms of* G*, define* $H^1([\tau],G)=H^1(\tau,G)$.

This is well-defined by the Lemma.

3 Real Forms and Cartan involutions

In the rest of the paper, unless otherwise noted, G will denote a complex, reductive algebraic group. Except in a few places we do not assume G is connected. Write G^0 for the identity component.

We identify G with its complex points $G(\mathbb{C})$ and use these interchangeably. We may view G either as an algebraic group or as a complex Lie group. The identity component of G as an algebraic group is the same as the topological identity component when viewed as a Lie group, the component group G/G^0 is finite.

A real form of G is a real algebraic group H endowed with an isomorphism $\phi: H_{\mathbb{C}}\simeq G$, where $H_{\mathbb{C}}$ denotes the base change of H from $\mathbb{R}$ to $\mathbb{C}$. By an *algebraic, conjugate linear, involutive automorphism* of $H_{\mathbb{C}}$ we mean an algebraic, involutive automorphism of $H_{\mathbb{C}}$ (considered as a scheme over $\mathbb{R}$) such that the induced morphism between rings of polynomial functions on H is conjugate linear, and compatible with the morphisms defining the group structure on H. Naturally associated to a real form H is an algebraic, conjugate linear, involutive automorphism σ_H of $H_{\mathbb{C}}$. Transporting σ_H to G via ϕ this is equivalent to having an algebraic, conjugate linear, involutive automorphism σ of G. Conversely, by Galois descent any such automorphism of G comes from a real form (H,ϕ), which is unique up to unique isomorphism. See [10, §6.2, Example B and §6.5] for details in a much more general situation.

It is convenient to work with a more elementary notion of real form, using only the structure of G as a complex Lie group. Any algebraic, conjugate linear, involutive automorphism of G induces an antiholomorphic involutive automorphism of G. In fact every antiholomorphic automorphism arises this way:

Lemma 3.1 *Let* G *be a complex reductive algebraic group. Then any antiholomorphic involutive automorphism of* G *is induced by a unique algebraic conjugate linear involutive automorphism of* $G(\mathbb{C})$.

Proof. Fix a representation $\rho: G\to \mathrm{GL}(V)$, where V is a complex vector space of finite dimension, such that ρ is a closed immersion [7, Proposition 1.10]. Suppose that $\varphi: G\to G$ is an antiholomorphic involutive automorphism. Choose an arbitrary real structure on V, and let σ_V denote complex conjugation $\mathrm{GL}(V)\to\mathrm{GL}(V)$ with respect to this real structure. Then $\sigma_V\circ\rho\circ\varphi$ is a holomorphic representation of G, so it is algebraic and φ is algebraic conjugate linear. □ The Lemma justifies the following elementary definition of real forms.

Definition 3.2 *A real form of G is an antiholomorphic involutive automorphism σ of G. Two real forms are equivalent if they are conjugate by an inner automorphism. Write $[\sigma]$ for the equivalence class of σ.*

We say two real forms σ_1, σ_2 are inner to each other, or in the same inner class, if $\sigma_1\sigma_2^{-1}$ is an inner automorphism of G. This is well defined on the level of equivalence classes.

See Remark 8.2 for a subtle point regarding this notion of equivalence.

If σ is a real form of G, let $G(\mathbb{R}) = G^\sigma$ be the fixed points of σ. This is a real Lie group, with finitely many connected components.

We turn now to compact real forms and Cartan involutions. If G is connected these results are well known. The general case is due to Mostow [19].

Definition 3.3 *A real form σ of G is said to be a* compact real form *if G^σ is compact and meets every component of G.*

Mostow's definition [19, Section 2] of compact real form refers to the subgroup G^σ, rather than the automorphism σ. Let us check that our definition is equivalent to this.

Lemma 3.4 *For any complex reductive group G, the map $\sigma \mapsto G^\sigma$ is a bijection between the set of compact real forms of G, in the sense of Definition 3.3, to the set of compact real forms of G, in the sense of* [19].

Proof. If σ is any real form of G, then $\dim_{\mathbb{R}} G^\sigma = \dim_{\mathbb{C}} G$, by Hilbert's Theorem 90 applied to the action of σ on $\mathrm{Lie}(G)$. Choose a faithful algebraic representation $\rho : G \hookrightarrow \mathrm{GL}(V)$. If K is any compact subgroup of G, then V admits a hermitian form for which $\rho(K)$ is unitary. In particular we see that $\mathrm{Lie}(K) \cap i\mathrm{Lie}(K) = 0$. These two facts imply that for any compact real form σ of G, G^σ is a compact real form of G in the sense of [19].

Let us now check that $\sigma \mapsto G^\sigma$ is injective. The action of σ on G^0 is determined by its action on $\mathrm{Lie}(G) = \mathrm{Lie}(G^\sigma) \oplus i\mathrm{Lie}(G^\sigma)$. Once $\sigma|_{G^0}$ is determined, σ is determined by the requirement that it fixes G^σ pointwise, since G^σ meets every connected component of G.

Finally we show that $\sigma \mapsto G^\sigma$ is surjective. Suppose K is a compact real form of G in the sense of [19]. Choose ρ and a hermitian form on V as above. Choosing an orthonormal basis for V, we can view ρ as a closed embedding $G \to \mathrm{GL}_n(\mathbb{C})$ such that $\rho(K) \subset U(n)$. Let $\tau(g) = {}^t\bar{g}^{-1}$ ($g \in GL_n(\mathbb{C})$). Then $\rho(G^0)$ is stable under τ, since $\mathrm{Lie}(\rho(G)) = \mathrm{Lie}(\rho(K)) \oplus i\mathrm{Lie}(\rho(K))$, and $d\tau$ fixes $\mathrm{Lie}(\rho(K)) \subset \mathfrak{u}(n)$ pointwise. Furthermore $\rho(G)$ is stable under τ since τ fixes $\rho(K)$ pointwise, and $G = G^0K$. Pull back τ to G to define $\sigma = \rho^{-1} \circ \tau \circ \rho$. This is a compact real form of G, and $K \subset G^\sigma$. By the Cartan decomposition [19, Lemma 2.1] $G^\sigma \cap G^0 = K \cap G^0$, and this implies $G^\sigma = K$. □

Using the Lemma we will refer to σ or $K = G^\sigma$ as a compact real form of G.

The *Cartan decomposition* holds in our setting (see [19, Lemma 2.1]).

Lemma 3.5 (Mostow) *Suppose σ is a compact real form of G. Let $K = G^\sigma$ and $\mathfrak{p} = \mathrm{Lie}(G)^{-\sigma} = i\mathrm{Lie}(K)$. Then the map $(k, X) \mapsto k\exp(X)$ is a diffeomorphism from $K \times \mathfrak{p}$ onto G considered as a real Lie group. Furthermore K is a maximal compact subgroup of G.*

Although we will not use it, it is not difficult to check that the complexification functor [22, Section III.4.5], from the category of compact Lie groups to that of complex reductive groups endowed with a compact real form, induces a bijection on the level of isomorphism classes.

It is important to know the existence of compact real forms. See [19, Lemma 6.1].

Theorem 3.6 (Weyl, Chevalley, Mostow) *Every complex reductive group has a compact real form.*

We turn next to uniquess of the compact form. See [19, Theorem 3.1], and [12, Ch. XV] for a proof which handles one case overlooked in [19].

Theorem 3.7 (Cartan, Hochschild, Mostow) *Let σ be a compact real form of a complex reductive group G, and set $K = G^\sigma$. Let L be a compact subgroup of G. Then there exists $g \in G^0$ such that $gLg^{-1} \subset K$. The compact real forms of G are unique up to conjugation by G^0.*

Fix a compact real form K of G. The center $Z(G^0)$ of G^0 is a normal subgroup of G. If G is connected it is well known that $Z(G) = Z(K)A$ where $A = \exp(i\mathrm{Lie}(Z(G^0))) \subset \exp(\mathfrak{p})$ is a vector group. Therefore in general we have

$$(3.8)(a) \qquad Z(G^0) = Z(K^0)A.$$

Since $G = KG^0$ we have (writing superscript for invariants): $Z(G^0)^K = Z(G^0)^G$, independent of the choice of K. Also $K/K^0 \simeq G/G^0$ acts on $Z(G^0)$, normalizing A, and

$$(3.8)(b) \quad Z(G) \cap G^0 = Z(G^0)^{G/G^0} = Z(K^0)^{K/K^0} A^{G/G^0} = (Z(K) \cap K^0)A^{G/G^0}$$

Lemma 3.9 *Suppose K is a compact real form of G. Then the Cartan decomposition of $\mathrm{Norm}_G(K)$ is $\mathrm{Norm}_G(K) = KA^{G/G^0}$.*

Proof. Since $G = K\exp(\mathfrak{p})$, it suffices to show that $\mathrm{Norm}_G(K) \cap \exp(\mathfrak{p}) = A^{G/G^0}$. Let $X \in \mathfrak{p}$ be such that $\exp(X)$ normalizes K. For $k \in K$, there exists $k' \in K$ such that $\exp(X)k\exp(-X) = k'$. This can be rewritten as

$$k\exp(-X) = k'\exp(-\mathrm{Ad}(k')^{-1}(X))$$

so by uniqueness of the Cartan decomposition, $k' = k$ and $\mathrm{Ad}(k)(X) = X$, so X is invariant under K. The fact that X is invariant under K^0 means that $X \in \mathrm{Lie}(A)$, and since K meets every connected component of G, $X \in \mathrm{Lie}(A)^{G/G^0}$. □

Lemma 3.10 *Let σ be a compact real form of a real reductive group G. Let H be a σ-stable algebraic subgroup of G. Then H is reductive and $\sigma|_H$ is a compact real form of H.*

Proof. The algebraic group H is clearly linear. The unipotent radical U of H is stable under σ and connected, and so U^σ is Zariski-dense in U. Any unipotent element of G^σ is trivial, thus $U = \{1\}$ and H is reductive. Clearly H^σ is compact, and we are left to show that H^σ meets every connected component of H. For $h \in H$ write $h = k\exp(X)$ where $k \in G^\sigma$ and $X \in \mathfrak{p}$. Then $\exp(2X) = \sigma(h)^{-1}h \in H$, and thus $\exp(2nX) \in H$ for all $n \in \mathbb{Z}$. Since H is Zariski-closed in G this implies $\exp(tX) \in H$ for all $t \in \mathbb{C}$, which implies $X \in \mathfrak{h}^{-\sigma}$, $k \in H^\sigma$, and H^σ meets every component of H. This argument is classical. □

Definition 3.11 *Suppose σ is a real form of a complex reductive group G. A* Cartan involution *for σ is a holomorphic involutive automorphism θ of G, commuting with σ, such that $\theta\sigma$ is a compact real form of G.*

By Lemma 3.1 applied to σ and $\theta\sigma$, any Cartan involution is algebraic. In fact a simple variant of the proof of Lemma 3.1 shows directly that any holomorphic automorphism of a complex reductive group is automatically algebraic.

Theorem 3.12 *Let G be a complex reductive group, possibly disconnected.*

(1) Suppose σ is a real form of G.

  (a) There exists a Cartan involution θ for σ, unique up to conjugation by an inner automorphism from $(G^\sigma)^0$.

  (b) Suppose (H, θ_H) is a pair consisting of a σ-stable reductive subgroup of G and a Cartan involution θ_H for $\sigma|_H$. Then there exists a Cartan involution θ for G such that $\theta(H) = H$ and $\theta|_H = \theta_H$.

(2) Suppose θ is a holomorphic, involutive automorphism of G.

  (a) There is a real form σ of G such that θ is a Cartan involution for σ, unique up to conjugation by an inner automorphism from $(G^\theta)^0$.

  (b) Suppose (H, σ_H) is a pair consisting of a θ-stable reductive subgroup of G and a real form σ_H such that $\theta|_H$ is a Cartan involution for σ_H. Then there exists a real form σ of G such that $\sigma(H) = H$ and $\sigma|_H = \sigma_H$.

For applications to the classification of real forms and to homogeneous spaces, the fact that the statement of Theorem 3.12 is symmetric in σ and θ is crucial.

We will deduce (1) and (2) from the next Lemma, whose proof is adapted from [19, Theorem 4.1].

Lemma 3.13 *Suppose τ is an involutive automorphism of G, either holomorphic or anti-holomorphic.*

(1) There exists a compact real form σ^c of G which commutes with τ.

(2) Suppose H is a τ-stable reductive subgroup of G, σ^c_H is a compact real form of H, and τ commutes with σ^c_H. Then we can find σ^c satisfying (1) so that σ^c restricted to H equals σ^c_H.

Proof. Choose any compact real form σ^c_1 of G and set $K_1 = G^{\sigma^c_1}$, $\mathfrak{p}_1 = \mathrm{Lie}(G)^{-\sigma^c_1}$, and $P_1 = \exp(\mathfrak{p}_1)$. Then $\tau(K_1)$ is another compact real form of G, so by Theorem 3.7 there exists $g \in G^0$ so that

$$\text{(3.14)(a)} \qquad \tau(K_1) = gK_1g^{-1}.$$

Applying τ to both sides we see $\tau(g)g \in \mathrm{Norm}_G(K_1)$. By Lemma 3.9 we can write

$$\text{(3.14)(b)} \qquad \tau(g)g = ak \quad (a \in A^{G/G^0}, k \in K_1).$$

By (a) $g^{-1}\tau(K_1)g = K_1$, i.e. $\mathrm{int}(g^{-1}) \circ \tau$ stabilizes K_1. Since this isomorphism is holomorphic or antiholomorphic and $\mathfrak{p}_1 = i\mathrm{Lie}(K_1)$, this implies $g^{-1}\tau(P_1)g = P_1$. By the Cartan decomposition $G = K_1P_1$ we may assume $g \in P_1$, in which case $g^{-1}\tau(g)g \in P_1$. Plugging in (b) we conclude $g^{-1}ak \in P_1$, which by uniquess of the Cartan decomposition implies $k = 1$, so

$$\text{(3.14)(c)} \qquad \tau(g)g \in A^{G/G_1}.$$

Set $a = \tau(g)g \in Z(G)$. Then $\tau(a) = g\tau(g) = gag^{-1} = a$. After replacing g with $ga^{-\frac{1}{2}}$ we may assume $\tau(g) = g^{-1}$ (we are writing $\frac{1}{2}$ for the square root in the vector group P_1). We observe that $g^{-1}\tau(g^{\frac{1}{2}})g$ is an element of P_1 and its square equals $g^{-1}\tau(g)g = g$, therefore $\tau(g^{\frac{1}{2}}) = g^{\frac{1}{2}}$.

Now let $\sigma^c = \mathrm{int}(g^{\frac{1}{2}}) \circ \sigma^c_1 \circ \mathrm{int}(g^{-\frac{1}{2}})$, $K = G^{\sigma^c} = g^{\frac{1}{2}}K_1g^{-\frac{1}{2}}$, and $\mathfrak{p} = \mathrm{Lie}(G)^{-\sigma^c}$. Then

$$\tau(K) = \tau(g^{\frac{1}{2}})\tau(K)\tau(g^{-\frac{1}{2}}) = g^{-\frac{1}{2}}gKg^{-1}g^{\frac{1}{2}} = K.$$

This also implies $\tau(\mathfrak{p}) = \mathfrak{p}$, and τ commutes with σ^c, as one can check using the Cartan decomposition.

Now suppose we are given (H, σ^c_H) as in (2), and set $K_H = H^{\sigma^c_H}$. In the first step of the preceding argument choose σ^c_1 so that $K_H \subset K_1$ (then $K_H = K_1 \cap H$ since K_H is a maximal compact subgroup of H). Suppose $h \in K_H$. Choosing

$g \in P_1$ as above, recall $(\mathrm{int}(g^{-1}) \circ \tau)(K_1) = K_1$, so let $k = g^{-1}\tau(h)g \in K_1$. Since τ commutes with σ^c_H, $\tau(K_H) = K_H \subset K_1$, so $\tau(h) \in K_1$. Write

$$(3.14)(\mathrm{d}) \qquad kg^{-1} = \tau(h^{-1}) \cdot \tau(h^{-1})g^{-1}\tau(h).$$

By uniqueness of the Cartan decomposition we conclude $g\tau(h) = \tau(h)g$ for all $h \in K_H$. Since τ is an automorphism of K_H we see $gh = hg$ for all $h \in K_H$. Since $\mathrm{int}(K_H) \subset \mathrm{int}(K_1)$ acts on P_1, this implies that $g^{\frac{1}{2}}h = hg^{\frac{1}{2}}$ for all $h \in K_H$ as well. Define σ^c, K and P as before. Then $K_H = K \cap H$ and $\sigma^c(h) = h$ for all $h \in K_H$. Now $(\sigma^c)^{-1} \circ \sigma^c_H : H \to G$ is a holomorphic automorphism which is the identity on K_H, thus it is the identity on H (recall that $\mathrm{Lie}(H) = \mathrm{Lie}(K_H) \oplus i\mathrm{Lie}(K_H)$ and that K_H meets every connected component of H).

□

Proof of Theorem 3.12. For existence in (1)(a) apply the Lemma to $\tau = \sigma$ to construct a compact real form σ^c, commuting with σ, and set $\theta = \sigma\sigma^c$. For (1)(b) apply Lemma 3.13(2) with $\tau = \sigma$, $\sigma^c_H = \sigma|_H\theta_H$ to construct σ^c, commuting with σ, and let $\theta = \sigma\sigma^c$.

We now prove the uniqueness statement in (1)(a). Suppose θ, θ_1 commute with σ, and $\sigma^c = \sigma\theta$ and $\sigma^c_1 = \sigma\theta_1$ are compact real forms. By Theorem 3.7 there exists $g \in G^0$ so that

$$\sigma^c_1 = \mathrm{int}(g) \circ \sigma^c \circ \mathrm{int}(g^{-1}) = \mathrm{int}(g\sigma^c(g^{-1})) \circ \sigma^c.$$

Let $G = K\exp(\mathfrak{p})$ be the Cartan decomposition with respect to σ^c. Then we can take $g = \exp(X)$ for $X \in \mathfrak{p}$, so $g\sigma^c(g^{-1}) = \exp(2X)$. Since σ^c and σ^c_1 commute with σ, so does $\mathrm{int}(g\sigma^c(g^{-1})) = \mathrm{int}(\exp(2X))$, so by (3.8)(b)

$$\exp(2\sigma(X))\exp(-2X) \in Z(G) \cap G^0 = (Z(K) \cap K^0)A^{G/G^0}.$$

Applying the Cartan decomposition for σ^c again we conclude

$$\exp(2\sigma(X))\exp(X) \in A^{G/G^0},$$

so $\sigma(X) - X \in A^{G/G^0}$. We are free to multiply g by an element of $Z(G) \cap G^0$, which contains A^{G/G^0}. In particular we can replace X with $X + (\sigma(X) - X)/2 \in \mathfrak{p}^\sigma$. Then $g \in \exp(\mathfrak{p}^\sigma) \in (G^\sigma)^0$.

The proof of (2) is similar. We apply Lemma 3.13 with $\tau = \theta$. For existence in (2)(a) apply part (1) of the Lemma to construct σ^c, commuting with θ, and let $\sigma = \theta\sigma^c$. For (2)(b) apply part (2) of the Lemma with $\sigma^c_H = \sigma_H\theta|_H$ to construct σ^c, commuting with θ, and let $\sigma = \theta\sigma^c$. We omit the proof of the conjugacy statement, which is similar to case (1)(a). □

Let $\mathrm{Int}(G)$ be the group of inner automorphisms of G, $\mathrm{Aut}(G)$ the (holomorphic) automorphisms, and set $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$. Let $\mathrm{Int}^0(G)$ be the subgroup of $\mathrm{Int}(G)$ consisting of automorphisms induced by elements of G^0, so that $\mathrm{Int}^0(G) \simeq G^0/(Z(G) \cap G^0)$.

Corollary 3.15 *The correspondence between real forms and Cartan involutions induces a bijection between*

(3.16)(a) $\{$*antiholomorphic involutive automorphisms of* $G\}/\mathrm{Int}^0(G)$

and

(3.16)(b) $\{$*holomorphic involutive automorphisms of* $G\}/\mathrm{Int}^0(G)$.

Both quotients are by the conjugation action of inner automorphisms coming from G^0. *The same statement holds with* $\mathrm{Int}^0(G)$ *replaced by any group* $\mathcal{A}$ *satisfying* $\mathrm{Int}^0(G) \subset \mathcal{A} \subset \mathrm{Aut}(G)$.

If $\mathrm{Int}^0(G)$ is replaced by $\mathrm{Int}(G) = G_{\mathrm{ad}}$ then (a) is the set of equivalence classes of real forms of G (Definition 3.2). We use this bijection to identify an equivalence class of real forms with an equivalence class of Cartan involutions as in (b).

4 Borel-Serre's Theorem

In this section only G denotes a *real* Lie group. Since it requires no extra effort we work in the following generality.

Definition 4.1 *We say a real Lie group* G *has a Cartan decomposition* $(K, \mathfrak{p})$ *if* K *is a compact subgroup of* G, $\mathfrak{p}$ *is a subspace of* $\mathrm{Lie}(G)$ *stable under* $\mathrm{Ad}(K)$, *and the map* $(k, X) \mapsto k\exp(X)$ *is a diffeomorphism from* $K \times \mathfrak{p}$ *onto* G.

It is easy to see that K is necessarily a *maximal* compact subgroup of G.

Recall we have a Cartan decomposition in the case that G is the group $H(\mathbb{C})$ of complex points of a reductive group H viewed as a real group (Lemma 3.5): for any compact real form σ^c of H, we have $H = H^{\sigma^c}\exp(\mathrm{Lie}(H)^{-\sigma^c})$. Although we will not use this fact, it is easy to deduce that if σ is a real form of a complex reductive group H, then for any Cartan involution θ of (H, σ), the Lie group $H(\mathbb{R}) = H^\sigma$ has a Cartan decomposition $H(\mathbb{R}) = H(\mathbb{R})^\theta \exp(\mathrm{Lie}(H(\mathbb{R}))^{-\theta})$.

More general real Lie groups G admit a Cartan decomposition, including many non-linear ones (for example the *finite* covers of $\mathrm{SL}_2(\mathbb{R})$) or non-reductive ones (for example $G = H(\mathbb{R})$ where H is a real linear algebraic group). On the other hand the universal cover $\widetilde{G}$ of $\mathrm{SL}_2(\mathbb{R})$ has a decomposition $\widetilde{G} = L\exp(\mathfrak{p})$ where $L \simeq \mathbb{R}$ is the universal cover of the circle, hence noncompact. For a generalization of the Cartan decomposition to any real Lie group having finitely many connected components see [12, Ch. XV] or [19, Theorem 3.2].

Proposition 4.2 *Suppose* G *is a real Lie group admitting a Cartan decomposition* $(K, \mathfrak{p})$. *Let* τ *be an involutive automorphism of* G *which preserves* K *and* $\mathfrak{p}$. *Let* $Z_K = Z(G) \cap K$. *The inclusion map* $K \to G$ *induces an isomorphism*

(4.3) $$H^1(\tau, K; Z_K) \simeq H^1(\tau, G; Z_K)$$

which respects the maps to Z_K.

The proof is adapted from [6, Théorème 6.8] (see also [22, Section III.4.5]). This specializes to Borel-Serre's Theorem (see (4.7)).

Proof. It is enough to prove this when Z_K is replaced by $\{z\} \subset Z_K$ where z is any single element of Z_K. The left hand side of (4.3) is

$$\text{(4.4)(a)} \qquad \{k \in K \mid k\tau(k) = z\}/[k \sim tg\tau(t^{-1}) \quad (t \in K)]$$

and the right hand side is

$$\text{(4.4)(b)} \qquad \{g \in G \mid g\tau(g) = z\}/[g \sim tg\tau(t^{-1}) \quad (t \in G)].$$

Consider the map ϕ from (a) to (b) induced by inclusion.

We first show that ϕ is surjective. Suppose $g \in G$ satisfies $g\tau(g) = z$. Let $P = \exp(\mathfrak{p})$, and write $g = kp$ with $k \in K, p \in P$. Then $kp\tau(kp) = z$, which can be written

$$k\tau(k) \cdot \tau(k^{-1})p\tau(k) = z \cdot \tau(p^{-1}).$$

By uniqueness of the Cartan decomposition we conclude $k\tau(k) = z$ and $\tau(k^{-1})p\tau(k) = \tau(p^{-1})$. The latter condition is equivalent to $kpk^{-1} = \tau(p^{-1})$. The set of $p \in P$ satisfying this condition is the exponential of the subspace $\{Y \in \mathfrak{p} \mid \mathrm{Ad}(k)Y = -\tau(Y)\}$. Therefore $p = q^2$ for some $q \in P$ satisfying $kq = \tau(q^{-1})k$. Then $g = kq^2 = (kq)q = \tau(q^{-1})kq$. Therefore ϕ takes $cl(k)$ in (a) to $cl(g)$ in (b).

We now show that ϕ is injective. Suppose $k, k' \in K$, $k\tau(k) = k'\tau(k') = z$, and $k' = tk\tau(t^{-1})$ for some $t \in G$. Write $t^{-1} = xp$ with $x \in K, p \in P$. Then $k' = p^{-1}x^{-1}k\tau(x)\tau(p)$, i.e.

$$k' \cdot (k')^{-1}pk' = x^{-1}k\tau(x) \cdot \tau(p)$$

By uniqueness of the Cartan decomposition we conclude $k' = x^{-1}k\tau(x)$ with $x \in K$, i.e. k and k' are equivalent in (a). $\square$

Corollary 4.5 *Suppose G is a real Lie group admitting a Cartan decomposition $(K, \mathfrak{p})$, and as before let $Z_K = Z(G) \cap K$. Let τ, μ be involutive automorphisms of G which preserve K and $\mathfrak{p}$, and assume that $\tau|_K = \mu|_K$. Then there are canonical isomorphisms*

$$H^1(\tau, G; Z_K) \simeq H^1(\tau|_K, K, Z_K) \simeq H^1(\mu, G; Z_K)$$

compatible with the maps to Z_K. In particular there is a canonical isomorphism of pointed sets

$$\text{(4.6)} \qquad H^1(\tau, G) \simeq H^1(\mu, G).$$

Now let G be a complex reductive group, viewed as a real group. Recall (Section 3) G has a compact real form σ^c, and a Cartan decomposition $G = K\exp(\mathfrak{p})$. Hence Proposition 4.2 applies. Taking $\tau = \sigma^c$ and restricting to the fibres of $\{1\} \subset Z_K$ gives Borel-Serre's Theorem [6, Théorème 6.8], [22, Section III.4.5]

$$\text{(4.7)} \qquad H^1(\sigma^c, K) \simeq H^1(\sigma^c, G).$$

This admits the following natural generalization to arbitrary real forms.

Corollary 4.8 *Suppose G is a complex, reductive algebraic group G, σ is a real form of G, and θ is a Cartan involution for σ. Let $\sigma^c = \sigma\theta$.*

There are canonical isomorphisms

$$H^1(\theta, G; Z^{\sigma^c}) \simeq H^1(\theta, G^{\sigma^c}; Z^{\sigma^c}) \simeq H^1(\sigma, G; Z^{\sigma^c}).$$

In particular there is a canonical isomorphism of pointed sets:

$$H^1(\theta, G) \simeq H^1(\sigma, G).$$

This follows from Corollary 4.5 for the Cartan decomposition of G induced by σ^c, using the fact that σ and θ agree on $K = G^{\sigma^c}$.

Example 4.9 If θ is the identity, $H^1(\theta, G)$ is the set of conjugacy classes of involutions in G. If G is connected this is in bijection with H_2/W, where H is a Cartan subgroup, H_2 is the group of involutions in H and W is the Weyl group (see Example 8.6).

On the other hand $H^1(\theta, G(\mathbb{R}))$ is the set of conjugacy classes of involutions in $G(\mathbb{R})$, i.e. $H(\mathbb{R})_2/W$. Since $H(\mathbb{R})$ is compact this is equal to H_2/W. So we recover [22, Theorem 6.1]: $H^1(\sigma, G) \simeq H^1(\theta, G(\mathbb{R})) = H(\mathbb{R})_2/W$.

Example 4.10 Suppose $G = PSL(2, \mathbb{C})$. This has two real forms, $PGL(2, \mathbb{R}) \simeq SO(2, 1)$ and $SO(3)$. Since G is adjoint $|H^1(\sigma, G)| = 2$ for either real form.

Now let $G = SL(2, \mathbb{C})$. From Example 4.9 if $G(\mathbb{R}) = SU(2)$ then $|H^1(\sigma, G)| = 2$. On the other hand if $G(\mathbb{R}) = SL(2, \mathbb{R})$ then it is well known that $H^1(\sigma, G) = 1$. Thus in contrast to the adjoint case, although $SL(2, \mathbb{R})$ and $SU(2)$ are inner forms of each other, their cohomology is different. See Lemma 8.11.

5 Rational Orbits

We use the results of the previous section to study rational orbits of G-actions for real reductive groups.

Write

(5.1)(a) $$(G, \tau_G, X, \tau_X)$$

to indicate the following situation, which occurs repeatedly. First of all G is an abstract group equipped with an involutive automorphism τ_G, and X is a set equipped with an involutive automorphism τ_X. Furthermore there is a left action of $g : x \mapsto g \cdot x$ of G on X. We assume (τ_G, τ_X) are *compatible*:

(5.1)(b) $$\tau_X(g \cdot X) = \tau_G(g) \cdot \tau_X(x) \quad (g \in G, x \in X).$$

We will apply this with G a complex group, X a complex variety, and τ_G and τ_X each acting holomorphically or anti-holomorphically.

When X is a homogeneous space the following description of the set of orbits for the action of G^{τ_G} on X^{τ_X} is well known.

Lemma 5.2 *In the setting of* (5.1) *suppose X is a homogenous space for G. Assume that $X^{\tau_X} \neq \emptyset$, choose $x \in X^{\tau_X}$ and denote by G^x the stabilizer of x. Then we have a bijection*

$$\begin{aligned} X^{\tau_X}/G^{\tau_G} &\to \ker\left(H^1(\tau_G, G^x) \to H^1(\tau_G, G)\right) \\ g \cdot x &\mapsto cl(g^{-1}\tau_G(g)) \end{aligned}$$

If σ_G is a compact real form of G then X^{σ_X} is a homogeneous space for G^{σ_G}:

Lemma 5.3 *In the setting of* (5.1), *suppose G is a complex reductive algebraic group, X is a homogeneous space for G, and σ_G is a compact real form of G. Let $K = G^{\sigma_G}$.*

(1) K acts transitively on X^{σ_X}.

(2) Suppose H is a σ_G-stable subgroup of G, and $H = G^x$ for some $x \in X$. Assume $X^{\sigma_X} \neq \emptyset$. Then $H = G^y$ for some $y \in X^{\sigma_X}$.

Proof.
For (1), if X^{σ_X} is empty there is nothing to prove, so choose $x \in X^{\sigma_X}$. By the previous lemma we have to show that

$$\text{(a)} \qquad \ker\left(H^1(\sigma_G, G^x) \to H^1(\sigma_G, G)\right)$$

is trivial. By Lemma 3.10 σ_G restricts to a compact real form of G^x, so Proposition 4.2 implies (a) is isomorphic to

$$\text{(b)} \qquad \ker\left(H^1(\sigma_G, (G^x)^{\sigma_G}) \to H^1(\sigma_G, G^{\sigma_G})\right)$$

which is clearly trivial, proving (1).

For (2) choose $x \in X^{\sigma_X}$. The set of subgroups H in (2) is identified with the set of σ_G-fixed elements of the homogeneous space $G/\mathrm{Norm}_G(G^x)$. By (1) G^{σ_G} acts transitively on this set. Thus for any such H there exists $g \in G^{\sigma_G}$ such that $H = gG^xg^{-1}$. Then $g \cdot x \in X^{\sigma_X}$ and $H = G^{g \cdot x}$. □

We next consider homogeneous spaces for noncompact groups.

Proposition 5.4 *Suppose G is a complex, reductive algebraic group, possibly disconnected, acting transitively on a complex algebraic variety X. Suppose we are given:*

(1) a pair (σ_G, θ_G) consisting of a real form, and a corresponding Cartan involution, of G;

(2) a pair (σ_X, θ_X) of commuting involutions of X, with σ_X antiholomorphic and θ_X holomorphic.

Assume (σ_G, σ_X) *are compatible, and so are* (θ_G, θ_X) *(see* (5.1)(b)*).*

Assume $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$*. Then the two natural maps*

$$X^{\sigma_X}/G^{\sigma_G} \leftarrow (X^{\sigma_X} \cap X^{\theta_X})/(G^{\sigma_G} \cap G^{\theta_G}) \rightarrow X^{\theta_X}/G^{\theta_G}$$

are bijective.

Proof. Choose $x \in X^{\sigma_X} \cap X^{\theta_X}$. Lemma 5.2 applied to $(G, \sigma_G, X, \sigma_X)$ provides an identification

$$X^{\sigma_X}/G^{\sigma_G} \simeq \ker\left(H^1(\sigma_G, G^x) \rightarrow H^1(\sigma_G, G)\right)$$

Similarly, Lemma 5.2 applied to $(G, \theta_G, X, \theta_X)$ gives

$$X^{\theta_X}/G^{\theta_G} \simeq \ker\left(H^1(\theta_G, G^x) \rightarrow H^1(\theta_G, G)\right)$$

Let $\sigma_G^c = \sigma_G \theta_G$. By Lemma 5.3, $G^{\sigma_G^c}$ acts transitively on $X^{\sigma_X \theta_X}$, so that we can also apply Lemma 5.2 to $(G^{\sigma_G^c}, \sigma_G, X^{\sigma_X \theta_X}, \sigma_X)$:

$$(X^{\sigma_X} \cap X^{\theta_X})/(G^{\sigma_G} \cap G^{\theta_G}) \simeq \ker\left(H^1(\sigma_G, (G^x)^{\sigma_G^c}) \rightarrow H^1(\sigma_G, G^{\sigma_G^c})\right).$$

By Corollary 4.8 we have the following commutative diagram:

$$\begin{array}{ccccc}
H^1(\sigma_G, G^x) & \xleftarrow{\simeq} & H^1(\sigma_G, (G^x)^{\sigma_G^c}) & \xrightarrow{\simeq} & H^1(\theta_G, G^x) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(\sigma_G, G) & \xleftarrow{\simeq} & H^1(\sigma_G, G^{\sigma_G^c}) & \xrightarrow{\simeq} & H^1(\theta_G, G)
\end{array}$$

Note that σ_G and θ_G coincide on $G^{\sigma_G^c}$ so in the middle term we can replace $H^1(\sigma_G, *)$ with $H^1(\theta_G, *)$. This gives the two bijections of the Proposition.

These bijections (which involve the choice of x) agree with those of the Proposition (which are canonical). This comes down to: if $g \in G^{\sigma_G^c}$ then $g^{-1}\sigma_G(g) = g^{-1}\theta_G(g)$. This completes the proof. $\square$

Remark 5.5 In Proposition 5.8, the hypothesis $X^\sigma \cap X^\theta \neq \emptyset$ is necessary. Consider for example $G = X = \mathbb{C}^\times$, with G acting by multiplication, and $\sigma_G(z) = 1/\overline{z}, \sigma_X(z) = -1/\overline{z}, \theta_G(z) = \theta_X(z) = z$. Then $X^{\sigma_X} = \emptyset$ but $X^{\theta_X} = X$.

To apply the result it would be good to know that $X^{\sigma_X} \neq \emptyset$ or $X^{\theta_X} \neq \emptyset$ implies that $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$. As the Remark shows, this isn't always the case, but it holds under a weak additional assumption.

Lemma 5.6 *In the setting of the Proposition, assume that* $X^{\sigma_X \theta_X} \neq \emptyset$*. Then the following conditions are equivalent:* $X^{\sigma_X} \neq \emptyset$*,* $X^{\theta_X} \neq \emptyset$*, and* $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$*.*

Proof. If $x \in X^{\sigma_X\theta_X}$ then G^x is σ^c-stable so G^x is reductive by Lemma 3.10. Since these groups are all conjugate this holds for all $x \in X$.

Let us now show that if $X^{\sigma_X} \neq \emptyset$ then $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$. Fix $x \in X^{\sigma_X}$. Then G^x is a reductive group stable under σ_G, and thus it admits a Cartan involution θ'_x. By Theorem 3.12 it extends to a Cartan involution θ'_G of G, and there exists $g \in G^{\sigma_G}$ such that $\theta_G = \mathrm{int}(g) \circ \theta'_G \circ \mathrm{int}(g^{-1})$, so that $g \cdot x \in X^{\sigma_X}$ has the property that $G^{g\cdot x}$ is θ_G-stable. In other words, after replacing x by $g \cdot x$, we may assume G^x is σ^c-stable, and $\sigma_c|_{G^x}$ is a compact real form of G^x. By Lemma 5.3 we can find $y \in X^{\sigma_X\theta_X}$ so that $G^y = G^x$.

Let $N_y = \mathrm{Norm}_G(G^y)$, and set $M_y = N_y/G^y$, By [23, Proposition 5.5.10] M_y is a linear algebraic group. Both N_y and M_y are σ^c-stable, and therefore reductive by Lemma 3.10 again.

Since $G^{\sigma_X(y)} = \sigma_G(G^y) = G^y$ there exists unique $m \in M_y$ such that

(5.7)(a) $$\sigma_X(y) = m \cdot y.$$

Similarly since $G^x = G^y$ there exists unique $n \in M_y$ such that

(5.7)(b) $$x = n \cdot y$$

Since $\sigma_X\theta_X$ fixes both y and $\sigma_X(y)$, applying this to both sides of (a) gives $\sigma_X(y) = \sigma^c(m)\cdot y$, and comparing this with (a) gives $m \in (M_y)^{\sigma^c}$. On the other hand applying σ_X to both sides of (a) gives $y = \sigma_G(m) \cdot \sigma_X(y) = \sigma_G(m)m \cdot y$, so $\sigma_G(m)m = 1$. Finally apply σ_X to both sides of (b) to give $\sigma_X(x) = \sigma_G(n) \cdot \sigma_X(y)$. Using $\sigma_X(x) = x$ and (a) gives $x = \sigma_G(n)m \cdot y$, and comparing this with (b) gives $\sigma_G(n)^{-1}n = m$.

These three facts imply that m defines an element of

$$\ker\Big(H^1(\sigma_G, (M_y)^{\sigma^c}) \to H^1(\sigma_G, M_y)\Big).$$

By Corollary 4.8 this kernel is trivial, so there exists $u \in (M_y)^{\sigma^c}$ such that $h = \sigma_G(u)^{-1}u$. Then $u \cdot y \in X^{\sigma_X\theta_X} \cap X^{\sigma_X} = X^{\sigma_X} \cap X^{\theta_X}$.

A similar argument, substituting θ for σ, shows that $X^{\theta_X} \neq \emptyset$ implies that $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$. □

We can now formulate our result in its most useful form.

Proposition 5.8 *Suppose G is a complex, reductive algebraic group, possibly disconnected, and X is a complex algebraic variety, equipped with an action of G. Suppose we are given:*

(1) a pair (σ_G, θ_G) consisting of a real form and a corresponding Cartan involution of G.

(2) a pair (σ_X, θ_X) of commuting involutions, with σ_X antiholomorphic and θ_X holomorphic.

Assume (σ_G, σ_X) *are compatible, as are* (θ_G, θ_X) (5.1)(b).

Assume that for all $x \in X^{\sigma_X} \cup X^{\theta_X}$ *the* G*-orbit of* x *intersects* $X^{\sigma_X \theta_X}$. *Then the two natural maps*

$$X^{\sigma_X}/G^{\sigma_G} \leftarrow (X^{\sigma_X} \cap X^{\theta_X})/(G^{\sigma_G} \cap G^{\theta_G}) \rightarrow X^{\theta_X}/G^{\theta_G}$$

are bijective.

Proof. It is enough to prove this with X replaced by the G-orbit $G \cdot x$ of any $x \in X^{\sigma_X} \cup X^{\theta_X}$. By Lemma 5.6 we can apply Proposition 5.8 to $G \cdot x$, which gives the conclusion. □

6 Applications

Throughout this section we fix a *connected* complex reductive group G, a real form σ of G, and a corresponding Cartan involution θ. Set $G(\mathbb{R}) = G^\sigma$ and $K = G^\theta$.

6.1 Kostant-Sekiguchi correspondence

Let $\mathfrak{g} = \mathrm{Lie}(G)$. The Jacobson-Morozov theorem (see [11, ch. VIII, §11]) gives a bijection between the nilpotent orbits of G on $\mathfrak{g}$ and G-conjugacy classes of homomorphisms from $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{g}$:

(6.1.1)(a) $$\{\phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}\}/G.$$

Let $\mathfrak{g}_0 = \mathrm{Lie}(G(\mathbb{R})) = \mathfrak{g}^\sigma$. Then the same result applies to $G(\mathbb{R})$, and gives a bijection between the $G(\mathbb{R})$ conjugacy classes of nilpotent elements of $\mathfrak{g}_0$ and

(6.1.1)(b) $$\{\phi : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g}_0\}/G(\mathbb{R}).$$

Equivalently if σ_0 denotes complex conjugation on $\mathfrak{sl}(2, \mathbb{C})$ with respect to $\mathfrak{sl}(2, \mathbb{R})$, then (b) can be replaced with

(6.1.1)(c) $$\{\phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g} \mid \phi(\sigma_0 X) = \sigma(\phi(X))\}/G(\mathbb{R}).$$

Now write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g}^\theta = \mathrm{Lie}(K)$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$. For $X \in \mathfrak{sl}(2, \mathbb{C})$ define $\theta_0(X) = -{}^tX$; this is a Cartan involution for σ_0. Kostant and Rallis [15] showed that the nilpotent K-orbits on $\mathfrak{p}$ are in bijection with

(6.1.1)(d) $$\{\phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g} \mid \phi(\theta_0(X)) = \theta(\phi(X))\}/K.$$

The Kostant-Sekiguchi correspondence is a bijection between the nilpotent orbits of $G(\mathbb{R})$ on $\mathfrak{g}_0$ and the nilpotent K-orbits on $\mathfrak{p}$ [21].

Let X be the set of morphisms $\mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}$. This has a natural structure of complex algebraic variety. Define an antiholomorphic involution σ_X of X by

(6.1.2)(a) $$\sigma_X(\psi)(A) = \sigma(\psi(\sigma_0(A))) \quad (A \in \mathfrak{sl}(2, \mathbb{C}), \psi \in X).$$

Also define a holomorphic involution θ_X by

(6.1.3) $$\theta_X(\psi)(A) = \theta(\psi(\theta_0(A))) \quad (A \in \mathfrak{sl}(2,\mathbb{C}), \psi \in X).$$

It is straightforward to check that (σ_G, σ_X) and (θ_G, θ_X) are compatible.

Lemma 6.1.4 *Every orbit of G on X contains a $\sigma_X\theta_X$-invariant point. In particular, $\sigma_X\theta_X$ acts trivially on X/G, and an orbit of G on X is σ_X-stable if and only if it is θ_X-stable.*

Proof. We need to show that for any morphism $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$, there exists $g \in G$ such that the morphism $\mathrm{Ad}(g)\circ\phi$ is $\sigma\theta$-equivariant. Any such ϕ integrates to an algebraic morphism $\psi : \mathrm{SL}_2(\mathbb{C}) \to G^0$. Let $SU(2) = SL(2,\mathbb{C})^{\sigma_0\theta_0}$, with Lie algebra $\mathfrak{su}(2)$. Since $SU(2)$ is compact, so is its image in G^0, so by Theorem 3.7 there exists $g \in G^0$ such that $g\psi(\mathrm{SU}(2))g^{-1} \subset (G^0)^{\sigma\theta}$. Since $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes_{\mathbb{R}} (\mathbb{C})$ this implies that $\mathrm{Ad}(g)\cdot\phi$ is $\sigma\theta$-equivariant. □

The Kostant-Sekiguchi correspondence is now an immediate consequence of Proposition 5.8.

Proposition 6.1.5 *For any nilpotent orbit $\mathcal{O}$ of G on $\mathfrak{g}$, there is a canonical bijection between $(\mathcal{O} \cap \mathfrak{g}_0)/G(\mathbb{R})$ and $(\mathcal{O} \cap \mathfrak{p})/K$.*

Proof. Let $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ be a morphism corresponding to an element of $\mathcal{O}$ as in (a). Let $Y \subset X$ be the G-orbit of ϕ, which only depends on $\mathcal{O}$ and not on the choice of a particular morphism. By Lemma 6.1.4, Y is σ_X-stable if and only if it is θ_X-stable. If it it not the case, both quotient sets are empty.

If it is the case we can apply Proposition 5.8 to X, and by the Jacobson-Morozov theorem over $\mathbb{R}$ and the result of Kostant and Rallis recalled above, we obtain:

$$(\mathcal{O} \cap \mathfrak{g}_0)/G(\mathbb{R}) \simeq X^{\sigma_X}/G^{\sigma} \simeq X^{\theta_X}/G^{\theta} \simeq (\mathcal{O} \cap \mathfrak{p})/K.$$

□

Remark 6.1.6 The set of orbits $(X^{\sigma_X} \cap X^{\theta_X})/(G^{\sigma} \cap G^{\theta})$ that appears as a middle term in Proposition 5.8, that is the set of $K(\mathbb{R})$-conjugacy classes of morphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ equivariant under σ and θ, does *not* have an obvious link to nilpotent orbits, since $\mathfrak{p}_0$ has no non-zero nilpotent elements.

6.2 Matsuki Duality

Matsuki duality is a bijection between the $G(\mathbb{R})$ and K orbits on the space $\mathcal{B}$ of Borel subgroups of G [18].

Unlike in the case of Kostant-Sekiguchi duality, $G(\mathbb{R})$ and K are acting on the same space $\mathcal{B}$. So to derive this from Proposition 5.8 we need to find (X, σ_X, θ_X) so that $X^{\sigma_X} \simeq X^{\theta_X} \simeq \mathcal{B}$. This holds if we take $X = \mathcal{B} \times \mathcal{B}$, and define $\sigma_X(B_1, B_2) = (\sigma(B_1), \sigma(B_2))$, $\theta_X(B_1, B_2) = (\theta(B_1), \theta(B_2))$. However with this definition the condition $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$ of Proposition 5.8 does not hold. Also note that the stabilizer of a point in $\mathcal{B}$ is the intersection of two Borel subgroups, which is typically not reductive. Instead we use a variant of X.

Write $\sigma_G = \sigma, \theta_G = \theta$.

Definition 6.2.1 *Let*

(6.2.2) $$X = \{(B_1, B_2, T) \mid B_1, B_2 \in \mathcal{B}, T \subset B_1 \cap B_2 \textit{ is a Cartan subgroup}\}$$

Let G act on X by conjugation on each factor. Define involutive automorphisms σ_X and θ_X of X as follows:

(6.2.3) $$\sigma_X(B_1, B_2, T) = (\sigma_G(B_2\textit{-opp}), \sigma_G(B_1\textit{-opp}), \sigma_G(T))$$

where -opp denotes the opposite Borel with respect to T, and

(6.2.4) $$\theta_X(B_1, B_2, T) = (\theta_G(B_2), \theta_G(B_1), \theta_G(T)).$$

Thanks to the Bruhat decomposition [7, §14.12], for any $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$ the algebraic subgroup $B_1 \cap B_2$ of G is connected and solvable and contains a maximal torus of G. In particular the natural map $X \to \mathcal{B} \times \mathcal{B}$ is surjective.

Lemma 6.2.5 *The conditions of Proposition 5.8 hold.*

Proof. The fact that σ_X, θ_X commute, and the facts that (σ_G, σ_X) and (θ_G, θ_X) are compatible is immediate. Let us check that each G-orbit in X contains a $\sigma_X\theta_X$-fixed point. Let $(B_1, B_2, T) \in X$. Since the real reductive group $(G, \sigma_G\theta_G)$ has a maximal torus defined over $\mathbb{R}$ [7, Theorem 18.2], up to conjugating by an element of G we can assume that T is $\sigma_G\theta_G$-stable. Since $(T, \sigma_G\theta_G)$ is anisotropic we have $\sigma_G\theta_G(B_i) = B_i$-opp for $i \in \{1, 2\}$, and (B_1, B_2, T) is automatically fixed by $\sigma_X\theta_X$. □

Proposition 5.8 now applies to give a bijection

(6.2.6) $$X/G(\mathbb{R}) \longleftrightarrow X/K.$$

Lemma 6.2.7 *Consider the projection p on the first factor, taking X to $\mathcal{B}$.*

(1) p restricted to X^{σ_X} is equivariant with respect to $G(\mathbb{R})$ and induces a bijection $X^{\sigma_X}/G(\mathbb{R}) \simeq \mathcal{B}/G(\mathbb{R})$.

(2) p restricted to X^{θ_X} is equivariant with respect to K and induces a bijection $X^{\theta_X}/K \simeq \mathcal{B}/K$.

Proof. The fact that p is G-equivariant, and $p|_{X^{\sigma_X}}$ is $G(\mathbb{R})$-equivariant, are immediate. Let B be a Borel subgroup of G. Then $B \cap \sigma_G(B)$ is an algebraic subgroup of G defined over $\mathbb{R}$, and so it contains a maximal torus T which is defined over $\mathbb{R}$. The Bruhat decomposition implies that T is also a maximal torus of G. This shows that $B \in p(X^{\sigma_X})$.

Moreover the unipotent radical U of B acts transitively on the set of maximal tori of B [7, Theorem 10.6], and since G is reductive this action is also free. Therefore U^{σ_G} acts simply transitively on the set of σ_G-stable maximal tori in B. This implies that p induces a bijection $X^{\sigma_X}/G(\mathbb{R}) \simeq \mathcal{B}/G(\mathbb{R})$.

The proof of (2) is similar, except for the fact that $B \cap \theta_G(B)$ contains a maximal torus which is θ_G-stable, which follows from [24, 7.6] applied to θ_G acting on $B \cap \theta_G(B)$. □

Together with (6.2.6) this proves:

Proposition 6.2.8 *There is a canonical bijection $\mathcal{B}/G(\mathbb{R}) \leftrightarrow \mathcal{B}/K$.*

6.3 Weyl groups and conjugacy of Cartan subgroups

We next give short proofs of two well known facts about Weyl groups and conjugacy of Cartan subgroups.

Let X be the set of Cartan subgroups of G. This is a homogeneous space for the conjugation action of G, with σ_X, θ_X coming from σ and θ. It is well known that G has a σ-stable Cartan subgroup, that is $X^{\sigma_X} \neq \emptyset$. This also applies to G equipped with its real form $\sigma\theta$, so that $X^{\sigma_X\theta_X} \neq \emptyset$.

Matsuki's result on Cartan subgroups ([18],[4, Proposition 6.18]) now follows from Proposition 5.8.

Proposition 6.3.1 *There are canonical bijections between*

- *$G(\mathbb{R})$-conjugacy classes of σ-stable Cartan subgroups of G,*
- *$K(\mathbb{R})$-conjugacy classes of σ- and θ-stable Cartan subgroups of G,*
- *K-conjugacy classes of θ-stable Cartan subgroups of G.*

In particular we recover the fact that G admits a θ-stable Cartan subgroup H in every $G(\mathbb{R})$-conjugacy class of σ-stable Cartan subgroups.

Next, we recover the following description of the *real* or *rational* Weyl group of H. See also [28, Proposition 1.4.2.1], [25, Definition 0.2.6].

Proposition 6.3.2 *Let H be a Cartan subgroup of G which is both σ and θ stable. Then the two natural morphisms*

$$\mathrm{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R}) \leftarrow \mathrm{Norm}_{K(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})^{\theta} \rightarrow \mathrm{Norm}_K(H)/(H \cap K)$$

are isomorphisms.

Proof. Naturally σ and θ act on $N = \mathrm{Norm}_G(H)$ and on the Weyl group $W = N/H$. Note that the three quotients in the Proposition are N^σ/H^σ, (resp. $(N^\sigma \cap N^\theta)/(H^\sigma \cap H^\theta)$, N^θ/H^θ), and thus are naturally subgroups of W^σ (resp. $W^\sigma \cap W^\theta$, W^θ). Denote by π the canonical surjective morphism $N \to W$. By Lemma 3.10, $N^{\sigma\theta}$ meets every connected component of N. Since $H = N^0$, this means that $N^{\sigma\theta}$ maps surjectively to W. In particular, $\sigma\theta$ acts trivially on W, and so $W^\sigma = W^\theta$.

For $w \in W^\sigma$ let $N_w = \pi^{-1}(\{w\})$. This is a torsor under H that contains a $\sigma\theta$-invariant point. By Corollary 4.8 the following conditions are equivalent: $(N_w)^\sigma \neq \emptyset$,$(N_w)^\theta \neq \emptyset$, and $(N_w)^{\sigma\theta} \neq \emptyset$. □

7 Relation with Cohomology of Cartan subgroups

We continue to assume G is a connected complex reductive group. Suppose σ is a real form of G, and θ is a Cartan involution for σ.

We say a σ-stable Cartan subgroup H_f of G is fundamental if $H_f(\mathbb{R})$ is of minimal split rank. Borovoi computes $H^1(\sigma, G)$ in terms of $H^1(\sigma, H_f)$ as follows. Before stating his result we make a few remarks about Weyl groups.

Lemma 7.1 *Suppose H is a σ-stable Cartan subgroup. There is an action of W^σ on $H^1(\sigma, H)$ defined as follows. Suppose $w \in W^\sigma$ and $h \in H^{-\sigma}$. Choose $n \in N$ mapping to w. Then the action of w on $H^1(\sigma, H)$ is $w : cl(h) \to cl(nh\sigma(n^{-1}))$; this is well defined, independent of the choices involved.*

The image of $H^1(\sigma, H)$ in $H^1(\sigma, N)$ is isomorphic to $H^1(\sigma, H)/W^\sigma$.

This is immediate. See [22, I.5.5, Corollary 1].

Suppose a Cartan subgroup H is σ-stable. Then σ acts on the roots of H in G. We say a root α of H in G is imaginary, real, or complex if $\sigma(\alpha) = -\alpha$, $\sigma(\alpha) = \alpha$, or $\sigma(\alpha) \neq \pm\alpha$, respectively. The set of imaginary roots is a root system. Let W_i denote its Weyl group.

Lemma 7.2 $H^1(\sigma, H)/W^\sigma = H^1(\sigma, H)/W_i$.

Proof. Write $W^\sigma = (W_C)^\sigma \ltimes [W_i \times W_r]$ as in [26, Proposition 4.16]. Here W_r is Weyl group of the real roots, and $(W_C)^\sigma$ is a certain Weyl group, generated by terms of the form $s_\alpha s_{\sigma\alpha}$ where $\alpha, \sigma\alpha$ are orthogonal. It is easy to see that W_r acts trivially on $H^1(\sigma, H)$, and $(W_C)^\sigma$ does as well [3, Proposition 12.16]. □

Proposition 7.3 (Borovoi [8]**)** *Suppose H_f is a fundamental σ-stable Cartan subgroup. The natural map $H^1(\sigma, H_f) \to H^1(\sigma, G)$ induces an isomorphism $H^1(\sigma, H_f)/W_i \simeq H^1(\sigma, G)$.*

The Theorem in [8] is stated in terms of W^σ, so we have used the preceding Lemma to replace this with W_i.

Remark 7.4 Borovoi has pointed out that we can replace W_i with another group, which is much smaller in the unequal rank case. Fix a pinning $(H_f, B, \{X_\alpha\})$. The inner class of σ corresponds to an involution $\delta \in \mathrm{Aut}(G)$ which preserves the pinning (see Section 3). Thus δ defines an involution of the simple roots, which is trivial if and only if the derived group is equal rank.

Let W_0 be the Weyl group generated by the δ-fixed simple roots. For example in type A_n, if δ is nontrivial, then W_0 is trivial if n is even, or $\mathbb{Z}_2$ if n is odd. Borovoi proves that $H^1(\sigma, H_f)/W_i \simeq H^1(\sigma, H_f)/W_0$.

Proposition 7.5 *There is a canonical isomorphism $\phi : H^1(\theta, G) \simeq H^1(\theta, H_f)/W_i$ making the following diagram commute:*

$$\begin{array}{ccc} H^1(\sigma, G) & \xrightarrow{\simeq} & H^1(\sigma, H_f)/W_i \\ \simeq \downarrow & & \downarrow \simeq \\ H^1(\theta, G) & \xrightarrow[\simeq]{\phi} & H^1(\theta, H_f)/W_i \end{array}$$

The top isomorphism is Borovoi's result and the two vertical arrows are from Theorem 1.1 applied to G and H, respectively.

This is immediate.

Remark 7.6 In an earlier version of this paper we proved the isomorphism $H^1(\sigma, G) \simeq H^1(\theta, G)$ using this diagram. It is simpler to prove this isomorphism directly as we have done in Section 4 and deduce this as a consequence.

For later use we note that, in the unequal rank case, the cohomology is captured by a proper subgroup.

Suppose H is a θ-stable Cartan subgroup. Then $H = TA$ where T and A are connected complex tori, T is the identity component of H^θ, and A is the identity component of $H^{-\theta}$.

Corollary 7.7 *Suppose H_f is a σ and θ-stable fundamental Cartan subgroup. Let A_f be the identity component of $H_f^{-\theta}$, and let $M_f = \mathrm{Cent}_G(A_f)$. Then*

$$H^1(\sigma, G) \simeq H^1(\sigma, M_f) \simeq H^1(\theta, M_f) \simeq H^1(\theta, G).$$

Note that $A_f \subset Z \Leftrightarrow M_f = G \Leftrightarrow$ the derived group of G is of equal rank.

This follows from Proposition 7.5, and the fact that the imaginary Weyl groups of H_f in G and M_f are the same.

8 Strong real forms

In this section we assume G is connected complex reductive group.

Lemma 8.1 *Fix a real form σ of G. The set of equivalence classes of real forms in the inner class of σ is parametrized by $H^1(\sigma, G_{\mathrm{ad}})$.*

Explicitly the map is $cl(h) \mapsto [\mathrm{int}(h) \circ \sigma]$ where $h\sigma(h) = 1$.

By Lemma 2.4 it makes sense to define $H^1([\sigma], G_{\mathrm{ad}}) = H^1(\sigma, G_{\mathrm{ad}})$.

Remark 8.2 Our definition of equivalence of real forms (Definition 3.2) is by conjugation by an inner automorphism of G. The standard definition, for example see [22, III.1], allows conjugation by $\mathrm{Aut}(G)$. With the standard definition the Lemma would hold with $H^1(\sigma, G_{\mathrm{ad}})$ replaced by the image of the map to $H^1(\sigma, \mathrm{Aut}(G))$.

For example suppose $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. In the inner class of the split real form of G, there are four equivalence classes of real forms according to our definition: split, compact, split $\times$ compact and compact $\times$ split. If one allows conjugation by outer automorphisms there are only three real forms, since the last two are equivalent.

For simple groups these two notions agree except in type D_{2n}. See [3, Section 3], [5, Example 3.3] and Section 10.3.

It is easy to see that two real forms σ_1, σ_2 are in the same inner class (Definition 3.2) if and only if θ_1, θ_2 have the same image in $\mathrm{Out}(G)$, where θ_i is a Cartan involution for σ_i. So let $\mathrm{Out}(G)_2$ be the set of elements δ of $\mathrm{Out}(G)$ such that $\delta^2 = 1$, and write p for the natural map $\mathrm{Aut}(G) \to \mathrm{Out}(G)$. We say a (holomorphic) involution θ is in the inner class of δ if $p(\theta) = \delta$. We say a real form σ is in the inner class of δ if this holds for a Cartan involution for σ.

For example the inner class corresponding to $\delta = 1$ is called the *compact* or *equal rank* inner class; a real form is in this class if and only if its Cartan involution is an inner automorphism.

Suppose σ is a real form in the inner class of δ. There are two natural choices of a basepoint for the set $H^1(\sigma, G_{\mathrm{ad}})$ of real forms in this inner class. One is the quasisplit (most split) real form. Because of our focus on θ, rather than σ, we prefer to choose the quasicompact (most compact) form, which we now define.

We say a real form is *quasicompact* if its Cartan involution preserves a pinning datum $(H, B, \{X_\alpha\}_{\alpha\in\Delta})$. Every inner class contains a unique distinguished involution, which is unique up to conjugation by an inner automorphism. See [3, Chapter 3].

Definition 8.3 *Suppose $\delta \in \mathrm{Out}(G)_2$. Let $\theta_{qc}(\delta)$ be a distinguished automorphism in the inner class of δ, and let $\sigma_{qc}(\delta)$ be a corresponding real form by Corollary 3.15. We refer to $[\sigma_{qc}(\delta)]$ or $[\theta_{qc}(\delta)]$ as the equivalence class of quasicompact real forms in the inner class of δ.*

If δ is fixed we will write θ_{qc} and σ_{qc}.

Since any two choices of $\theta_{\mathrm{qc}}(\delta)$ are conjugate by an inner automorphism $[\theta_{\mathrm{qc}}(\delta)]$ and $[\sigma_{\mathrm{qc}}(\delta)]$ are well defined.

Lemma 8.4 *There is a canonical isomorphism*

$$H^1([\sigma_{qc}], G_{\mathrm{ad}}) \simeq H^1([\theta_{qc}], G_{\mathrm{ad}}). \tag{8.5}$$

These pointed sets canonically parametrize the equivalence classes of real forms in the inner class of δ, with the distinguished class going to the equivalence class of quasicompact real forms.

Example 8.6 The group $G(\mathbb{R}) = G^\sigma$ is compact if and only if $\theta = 1$. Then $H^1(\theta, G) = \{g \in G \mid g^2 = 1\}/\{g \mapsto xgx^{-1}\}$, i.e. $H^1(\theta, G)$ is the set of conjugacy classes of involutions of G. Therefore, if we fix a Cartan subgroup H, with Weyl group W, then

$$H^1(\theta, G) \simeq H_2/W$$

where $H_2 = \{h \in H \mid h^2 = 1\}$. See Example 4.9.

Let $Z = Z(G)$. The action of $\mathrm{Aut}(G)$ on Z factors to an action of $\mathrm{Out}(G)$ on Z. Let Z_{tor} be the subgroup of Z consisting of all elements of finite order.

Lemma 8.7 *Fix $\delta \in \mathrm{Out}(G)_2$, and suppose σ is a real form in the inner class of δ. Let θ be a Cartan involution for σ. Note that the action of θ on Z coincides with δ.*

Then $Z^\sigma_{\mathrm{tor}} = Z^\theta_{\mathrm{tor}}$ and there is a canonical isomorphism

$$Z^\sigma/(1+\sigma)Z \simeq Z^\delta/(1+\delta)Z.$$

Proof. The closure of Z_{tor} is compact, so by Theorem 3.7 Z_{tor} is a subgroup of every compact real form of G. Therefore $\sigma^c = \theta\sigma$ acts trivially on Z_{tor}, i.e. θ, σ and δ all have the same action on Z_{tor}. Also since $Z^\sigma/(1+\sigma)Z$ and $Z^\delta/(1+\delta)Z$ are two-groups, the quotients Z/Z^0, $Z^\sigma/(Z^\sigma)^0$ and $Z^\delta/(Z^\delta)^0$ are finite and Z^0 is divisible, it is easy to see $Z^\sigma/(1+\sigma)Z \simeq Z^\sigma_{\mathrm{tor}}/(1+\sigma)Z_{\mathrm{tor}} \simeq Z^\delta_{\mathrm{tor}}/(1+\delta)Z_{\mathrm{tor}} \simeq Z^\delta/(1+\delta)Z$. $\square$

Definition 8.8 *Fix $\delta \in \mathrm{Out}(G)_2$ and a real form σ in the inner class of δ. Identify $[\sigma]$ with a class in $H^1([\sigma_{qc}], G_{\mathrm{ad}})$, and define the central invariant*

$$\mathrm{inv}([\sigma]) \in Z^\delta/(1+\delta)Z \tag{8.9}$$

by the composition of maps:

$$H^1([\sigma_{qc}], G_{\mathrm{ad}}) \to H^2(\sigma_{qc}, Z) \xrightarrow{\simeq} \widehat{H}^0(\sigma_{qc}, Z)) \xrightarrow{\simeq} Z^\sigma/(1+\sigma)Z \xrightarrow{\simeq} Z^\delta/(1+\delta)Z$$

The first map is from the connecting homomorphism in (2.1) *coming from the exact sequence $1 \to Z \to G \to G_{\mathrm{ad}} \to 1$. The second and third arrows are from properties of Tate cohomology (see Section 2), and the last one is from the preceding Lemma.*

Remark 8.10 Alternatively we could define $\mathrm{inv} : H^1([\theta_{\mathrm{qc}}], G_{\mathrm{ad}}) \to Z^\delta/(1+\delta)Z$ similarly, with $\theta, \theta_{\mathrm{qc}}$ in place of $\sigma, \sigma_{\mathrm{qc}}$. It is clear from the Lemma and the Definition that the following diagram commutes:

$$\begin{array}{ccc} H^1([\sigma_{\mathrm{qc}}], G_{\mathrm{ad}}) & \longrightarrow & Z^\delta/(1+\delta)Z \\ \simeq\downarrow & \nearrow & \\ H^1([\theta_{\mathrm{qc}}], G_{\mathrm{ad}}) & & \end{array}$$

The central invariant allows us to see how $H^1(\sigma, G)$ varies in a given inner class, as in Example 4.10. See [22, Section I.5.7, Remark 1].

Lemma 8.11 *Suppose σ_1, σ_2 are inner forms of G. If* $\mathrm{inv}([\sigma_1]) = \mathrm{inv}([\sigma_2])$ *then $H^1(\sigma_1, G) \simeq H^1(\sigma_2, G)$.*

Proof. Write $\sigma_i = \mathrm{int}(g_i) \circ \sigma_{\mathrm{qc}}$, where $g_i\sigma_{\mathrm{qc}}(g_i) \in Z$ $(i = 1, 2)$. A straightforward calculation shows that the map $h \to hg_1g_2^{-1}$ induces the desired isomorphism, provided $g_1\sigma_{\mathrm{qc}}(g_1) = g_2\sigma_{\mathrm{qc}}(g_2)$. Unwinding Definition 8.8 we see this condition is equivalent to $\mathrm{inv}(\sigma_1) = \mathrm{inv}(\sigma_2)$. We leave the details to the reader. □

The map $H^1(\sigma, G) \to H^1(\sigma, G_{\mathrm{ad}})$ is not necessarily surjective. This failure of surjectivity causes some difficulties in precise statements of the local Langlands conjecture. See [2], [27], and for the p-adic case [13]. This leads to the notion of *strong real form* of G.

Definition 8.12 *Fix $\delta \in \mathrm{Out}(G)_2$ and a distinguished involution θ_{qc} in the inner class of δ. A strong real form, in the inner class of θ_{qc}, is an element $g \in G$ satisfying $g\theta_{qc}(g) \in Z_{\mathrm{tor}}$, i.e. an element of $Z^1(\theta_{qc}, G; Z_{\mathrm{tor}})$. Two strong real forms g, h are said to be equivalent if $h = tg\theta_{qc}(t^{-1})$ for some $t \in G$. Write $[g]$ for the equivalence class of g, and let $\mathrm{SRF}_{\theta_{qc}}(G) = H^1(\theta_{qc}, G; Z_{\mathrm{tor}})$ be the set of equivalence classes of strong real forms in the inner class of θ_{qc}.*

If g is a strong real form define $\mathrm{inv}(g) = g\theta_{qc}(g) \in Z^\delta_{\mathrm{tor}}$. We refer to inv *as the central invariant of a strong real form. This factors to a well defined map* $\mathrm{inv} : \mathrm{SRF}_{\theta_{qc}}(G) \to Z^\delta_{\mathrm{tor}}$.

Remark 8.13 In [3] strong real forms are defined as elements of the non-identity component of extended group ${}^{\theta_{\mathrm{qc}}}G = G \rtimes \langle\theta_{\mathrm{qc}}\rangle$, with equivalence being conjugation by G. The map taking a strong real form g of our definition to $g\theta_{\mathrm{qc}} \in G\theta_{\mathrm{qc}}$ is a bijection between the two notions.

We want to eliminate the dependence of $\mathrm{SRF}_{\theta_{\mathrm{qc}}}(G)$ on the choice of θ_{qc}.

Lemma 8.14 *Fix $\delta \in \mathrm{Out}(G)_2$ and distinguished involutions $\theta_{qc}, \theta'_{qc}$ in the inner class of δ.*

(1) There exists $h \in G$ such that $\theta'_{qc} = \mathrm{int}(h) \circ \theta_{qc} \circ \mathrm{int}(h)^{-1}$, and for any such h we have a bijection

$$\begin{aligned} Z^1(\theta'_{qc}, G; Z_{\mathrm{tor}}) &\longrightarrow Z^1(\theta_{qc}, G; Z_{\mathrm{tor}}) \\ g &\longmapsto gh\theta_{qc}(h)^{-1} \end{aligned}$$

which is compatible with the maps inv *to Z^δ_{tor}.*

(2) The induced map

$$Z^1(\theta'_{qc}, G; Z_{\mathrm{tor}})/(1-\delta)Z \to Z^1(\theta_{qc}, G; Z_{\mathrm{tor}})/(1-\delta)Z$$

does not depend on the choice of h. In particular we get a canonical *bijection* $\mathrm{SRF}_{\theta'_{qc}}(G) \simeq \mathrm{SRF}_{\theta_{qc}}(G)$.

Proof.

(1) This is an elementary computation.

(2) The element h is well defined up to multiplication on the right by an element of the preimage of $(G_{\mathrm{ad}})^{\theta_{\mathrm{qc}}}$ in G. By [16, Lemma 1.6], this preimage is $Z(G)G^{\theta_{\mathrm{qc}}}$, and the result follows.

□

Definition 8.15 *Fix* $\delta \in \mathrm{Out}(G)_2$. *Let*

$$\mathrm{SRF}_\delta(G) = \lim_{\theta_{qc}} \mathrm{SRF}_{\theta_{qc}}(G)$$

where the (projective or injective) limit is taken over all quasicompact involutions θ_{qc} *in the inner class, using Lemma 8.14.*

We have a map $g \mapsto \mathrm{int}(g) \circ \theta_{\mathrm{qc}}$ from $Z^1(\theta_{\mathrm{qc}}, G; Z_{\mathrm{tor}})/(1-\delta)Z$ to the set of holomorphic involutions of G in the inner class of δ, and it is easy to show that it is surjective. Moreover as θ_{qc} varies in the set of distinguished involutions in the inner class of δ, these maps commute with the maps defined in Lemma 8.14 (1). We obtain a natural *surjective* map from $\mathrm{SRF}_\delta(G)$ to the set of equivalence classes of holomorphic involutions of G in the inner class of δ.

Remark 8.16 In [2] and [27] strong real forms are defined in terms of the Galois action, as opposed to the Cartan involution as in [3] (and elsewhere, including [1]). The preceding discussion together with Corollary 4.8 show that these two theories are indeed equivalent. However the choices of basepoints in the two theories are different. In the Galois setting we choose the quasisplit form, and in the algebraic setting we use the quasicompact one.

The invariant of a Galois strong real form is defined [27, (2.8)(c)]. This differs from the normalization here by multiplication by $\exp(2\pi i\rho^\vee) \in Z$. Note that the "pure" rational forms, which are parametrized by $H^1(\sigma, G)$, include the quasisplit one [27, Proposition 2.7(c)], rather than the quasicompact one.

We can now describe strong real forms in terms of Galois cohomology sets $H^1(\sigma, G)$. Recall if σ is a real form in the inner class of δ, then the central invariant $\mathrm{inv}([\sigma])$ is an element of $Z^\delta/(1+\delta)Z$ (Definition 8.8).

Proposition 8.17 *Suppose* σ *is a real form of* G*, in the inner class of* δ. *Choose a representative* $z \in Z^\delta_{\mathrm{tor}}$ *of* $\mathrm{inv}([\sigma]) \in Z^\delta/(1+\delta)Z$. *Then there is a bijection*

$$H^1(\sigma, G) \longleftrightarrow \text{the set of strong real forms of central invariant } z.$$

Proof. Fix a distinguished involution $\theta_{\rm qc}$ of G in the inner class of δ. There exists $g \in G$ such that $\mathrm{int}(g) \circ \theta_{\rm qc}$ is a Cartan involution for σ and $g\theta_{\rm qc}(g) = z$. Fix such a g and let $\theta = \mathrm{int}(g) \circ \theta_{\rm qc}$. Then we have a bijection

$$\begin{aligned} H^1(\theta_{\rm qc}, G; \{z\}) &\longrightarrow H^1(\theta, G) \\ h &\longmapsto hg^{-1} \end{aligned}$$

and composing with the isomorphism $H^1(\theta, G) \simeq H^1(\sigma, G)$ of Corollary 4.8 gives the result. $\square$

Note that the bijection not only depends on the choice of representative $z \in Z^\delta_{\rm tor}$ of $\mathrm{inv}([\sigma]) \in Z^\delta_{\rm tor}/(1+\delta)Z_{\rm tor}$, but also on the choice of g in the proof: g could be replaced by gx, where $x \in Z$ is such that $x\delta(x) = 1$.

Corollary 8.18 *Choose representatives $\{z_i \mid i \in I\}$ for the image of* $\mathrm{inv} : \mathrm{SRF}_\delta(G) \to Z^\delta_{\rm tor}$. *For each $i \in I$ choose a real form σ_i of G such that* $\mathrm{inv}([\sigma_i]) = z_i \mod (1+\delta)Z_{\rm tor}$. *Then there is a bijection*

$$\mathrm{SRF}_\delta(G) \longleftrightarrow \bigcup_i H^1(\sigma_i, G).$$

This gives an interpretation of $\mathrm{SRF}_\delta(G)$ in classical cohomological terms. A similar statement holds in the p-adic case [13].

The set I is finite if and only if the connected center of G is split (this condition only depends on δ). As in [13] or [3, Section 13] the theory can be modified to replace this with a finite set even when this condition is not satisfied. In any case the group $Z^\delta_{\rm tor}/(1+\delta)Z_{\rm tor}$ is finite, and for $z \in Z^\delta_{\rm tor}$ and $x \in Z_{\rm tor}$ there is an obvious isomorphism

$$H^1(\theta_{\rm qc}, G; \{z\}) \simeq H^1(\theta_{\rm qc}, G; \{zx\delta(x)\}).$$

Corollary 8.19 *Suppose σ is an equal rank real form of G. Choose $x \in G$ so that* $\mathrm{int}(x)$ *is a Cartan involution for σ, and let $z = x^2 \in Z$. Then*

$$H^1(\sigma, G) \longleftrightarrow \textit{the set of conjugacy classes of } G \textit{ with square equal to } z$$

If H is a Cartan subgroup, with Weyl group W, then this is equal to

$$\{h \in H \mid h^2 = z\}/W \tag{8.20}$$

Example 8.21 Taking $x = z = I$ gives $G(\mathbb{R})$ compact and recovers [22, III.4.5]: $H^1(\sigma, G)$ is the set of conjugacy classes of involutions in G. See Example 4.9.

Example 8.22 Let $G(\mathbb{R}) = Sp(2n, \mathbb{R})$. We can take $x = \mathrm{diag}(iI_n, -iI_n)$, $z = -I$. It is easy to see that every element of G whose square is $-I$ is conjugate to x. This gives the classical result $H^1(\sigma, G) = 1$, which is equivalent to the classification of nondegenerate symplectic forms [20, Chapter 2].

Example 8.23 Suppose $G(\mathbb{R}) = SO(Q)$, the isometry group of a nondegenerate real quadratic form. Suppose Q has signature (p, q). If pq is even we can take $z = I$, Corollary 8.19 applies, and the set (8.20) is equal to $\{\mathrm{diag}(I_r, -I_s) \mid r + s = p + q;\ s \text{ even}\}$.

Suppose p and q are odd. Apply Corollary 7.7 with $M_f(\mathbb{R}) = SO(p-1, q-1) \times GL(1, \mathbb{R})$. By the previous case we conclude $H^1(\sigma, G)$ is parametrized by $\{\mathrm{diag}(I_r, I_s) \mid r + s = p + q - 2;\ r, s \text{ even}\}$. Adding $(1, 1)$ this is the same as $\{\mathrm{diag}(I_r, -I_s) \mid r + s = p + q;\ s \text{ odd}\}$.

In all cases we recover the classical fact that $H^1(\sigma, G)$ parametrizes the set of equivalence classes of quadratic forms of the same dimension and discriminant as Q [20, Chapter 2], [22, III.3.2].

Example 8.24 Now suppose $G(\mathbb{R}) = \mathrm{Spin}(p, q)$, which is a (connected) two-fold cover of the identity component of $\mathrm{SO}(p, q)$. A calculation similar to that in the previous example shows that $|H^1(\sigma, \mathrm{Spin}(p, q))| = \lfloor \frac{p+q}{4} \rfloor + \delta(p, q)$ where $0 \le \delta(p, q) \le 3$ depends on $p, q \pmod 4$. See Section 10.2.

Skip Garibaldi pointed out this result can also be derived from the exact cohomology sequence associated to the exact sequence $1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n, \mathbb{C}) \to SO(n, \mathbb{C}) \to 1$; the preceding result; the fact that $SO(p, q)$ is connected if $pq = 0$ and otherwise has two connected components; and a calculation of the image of the map from $H^1(\sigma, \mathrm{Spin}(n, \mathbb{C})) \to H^1(\sigma, SO(n, \mathbb{C}))$. See [14, after (31.41)], [22, III.3.2] and also section 9. The result is:

$|H^1(\sigma, \mathrm{Spin}(Q))|$ equals the number of quadratic forms having the same dimension, discriminant, and Hasse invariant as Q with each (positive or negative) definite form counted twice.

Remark 8.25 Kottwitz relates $H^1(\sigma, G)$ to the center of the dual group [17, Theorem 1.2]. This is a somewhat different type of result. It describes a certain quotient $H^1_{\mathrm{sc}}(\sigma, G)$ of $H^1(\sigma, G)$ (see [13, 3.4]), but if G is simply connected this gives no information.

9 Fibers of $H^1(\sigma, G) \to H^1(\sigma, \overline{G})$

In this section G is a connected complex reductive group, and σ is a real form of G. Suppose $A \subset Z(G)$ is σ-stable and let $\overline{G} = G/A$. It is helpful to analyze the fibers of the map $\psi : H^1(\sigma, G) \to H^1(\sigma, \overline{G})$. In particular taking $G = G_{\mathrm{sc}}, \overline{G} = G_{\mathrm{ad}}$, and summing over $H^1(\sigma, G_{\mathrm{ad}})$, we obtain a description of $H^1(\sigma, G_{\mathrm{sc}})$, complementary to that of Proposition 8.17.

Write $G(\mathbb{R}, \sigma) = G^\sigma$ and $\overline{G}(\mathbb{R}, \sigma) = \overline{G}^\sigma$. Write p for the projection map $G \to \overline{G}$. This restricts to a map $G(\mathbb{R}, \sigma) \to \overline{G}(\mathbb{R}, \sigma)$, taking the identity component of $G(\mathbb{R}, \sigma)$ to that of $\overline{G}(\mathbb{R}, \sigma)$. Therefore p factors to a map (not necessarily an injection):

(9.1)(a) $$p^* : \pi_0(G(\mathbb{R}, \sigma)) \to \pi_0(\overline{G}(\mathbb{R}, \sigma)).$$

Define

$$\text{(9.1)(b)} \qquad \pi_0(G,\overline{G},\sigma)=\pi_0(\overline{G}(\mathbb{R},\sigma))/p^*(\pi_0(G(\mathbb{R},\sigma))).$$

There is a natural action of $\overline{G}(\mathbb{R},\sigma)$ on $H^1(\sigma,A)$ defined as follows. Suppose $g\in\overline{G}(\mathbb{R},\sigma)$. Choose $h\in G$ satisfying $p(h)=g$. Then $g:a\to ha\sigma(h^{-1})$ factors to a well defined action of $\overline{G}(\mathbb{R},\sigma)$ on $H^1(\sigma,A)$. Furthermore the image of $G(\mathbb{R},\sigma)$, which includes the identity component, acts trivially, so this factors to an action of $\pi_0(G,\overline{G},\sigma)$.

Proposition 9.2 *Suppose $\gamma\in H^1(\sigma,G)$, and write $\gamma=cl(g)$ $(g\in G^{-\sigma})$. Let $\sigma_\gamma=\mathrm{int}(g)\circ\sigma$. Then there is a bijection*

$$H^1(\sigma,G)\supset\psi^{-1}(\psi(\gamma))\longleftrightarrow H^1(\sigma,A)/\pi_0(G,\overline{G},\sigma_\gamma).$$

Proof. First assume γ is trivial, and take $g=1$. Consider the exact sequence

$$H^0(\sigma,G)\to H^0(\sigma,\overline{G})\to H^1(\sigma,A)\overset{\phi}{\to}H^1(\sigma,G)\overset{\psi}{\to}H^1(\sigma,\overline{G}).$$

This says $\psi^{-1}(\psi((\gamma))=\phi(H^1(\sigma,A))$, i.e. the orbit of the group $H^1(\sigma,A)$ acting on the identity coset. This is $H^1(\sigma,A)$, modulo the action of $H^0(\sigma,\overline{G})$, and this action factors through the image of $H^0(\sigma,G)$. The general case follows from an easy twisting argument. □

We specialize to the case $G=G_{\mathrm{sc}}$ is simply connected and $\overline{G}=G_{\mathrm{ad}}=G_{\mathrm{sc}}/Z_{\mathrm{sc}}$ is the adjoint group.

Corollary 9.3 *Suppose σ is a real form of G_{sc} and consider the map $\psi:H^1(\sigma,G_{\mathrm{sc}})\to H^1(\sigma,G_{\mathrm{ad}})$.*

Suppose $\gamma\in H^1(\sigma,G_{\mathrm{ad}})$, and write $\gamma=cl(g)$ $(g\in G_{\mathrm{ad}}^{-\sigma})$. Let $\sigma_\gamma=\mathrm{int}(g)\circ\sigma$, viewed as an involution of G_{sc}.

$$\text{(9.4)(a)} \qquad \gamma \text{ is in the image of } \psi\Leftrightarrow\mathrm{inv}([\sigma_\gamma])=\mathrm{inv}([\sigma]),$$

in which case

$$\text{(9.4)(b)} \qquad |\psi^{-1}(\gamma)|=|H^1(\sigma,Z_{\mathrm{sc}})|/|\pi_0(G_{\mathrm{ad}}(\mathbb{R},\sigma_\gamma))|.$$

Furthermore

$$\text{(9.4)(c)} \qquad |H^1(\sigma,G_{\mathrm{sc}})|=|H^1(\sigma,Z_{\mathrm{sc}})|\sum_{\substack{\gamma\in H^1(\sigma,G_{\mathrm{ad}})\\ \mathrm{inv}([\sigma_\gamma])=\mathrm{inv}([\sigma])}}|\pi_0(G_{\mathrm{ad}}(\mathbb{R}),\sigma_\gamma)|^{-1}$$

Proof. Statements (b) and (c) follow from the Proposition. For (a), when $\sigma=\sigma_{\mathrm{qc}}$ and $\gamma=1$ the proof is immediate, and the general case follows by twisting. We leave the details to the reader. □

10 Tables

Most of these results can be computed by hand from Theorem 1.2, or using Proposition 9.2 and the classification of real forms (i.e. the adjoint case).

By Theorem 1.2 the computation of $H^1(\Gamma, G)$ reduces to calculating the strong real forms of G and their central invariants. The Atlas of Lie Groups and Representations does this computation as part of its parametrization of (strong) real forms. This comes down to calculating the orbits of a finite group (a subgroup of the Weyl group) on a finite set (related to elements of order 2 in a Cartan subgroup). See [3, Proposition 12.9] and `www.liegroups.org/tables/galois`.

10.1 Classical groups

Group	$\|H^1(\sigma, G)\|$	
$SL(n,\mathbb{R}), GL(n,\mathbb{R})$	1	
$SU(p,q)$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$	Hermitian forms of rank $p+q$ and discriminant $(-1)^q$
$SL(n,\mathbb{H})$	2	$\mathbb{R}^*/\mathrm{Nrd}_{\mathbb{H}/\mathbb{R}}(\mathbb{H}^*)$
$Sp(2n,\mathbb{R})$	1	real symplectic forms of rank $2n$
$Sp(p,q)$	$p+q+1$	quaternionic Hermitian forms of rank $p+q$
$SO(p,q)$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$	real symmetric bilinear forms of rank n and discriminant $(-1)^q$
$SO^*(2n)$	2	

Here $\mathbb{H}$ is the quaternions, and $\mathrm{Nrd}_{\mathbb{H}/\mathbb{R}}$ is the reduced norm map from $\mathbb{H}^*$ to $\mathbb{R}^*$ (see [20, Lemma 2.9]). For more information on Galois cohomology of classical groups see [22], [20, Sections 2.3 and 6.6] and [14, Chapter VII].

10.2 Simply connected groups

The only simply connected groups with classical root system, which are not in the table in Section 10.1 are $\mathrm{Spin}(p,q)$ and $\mathrm{Spin}^*(2n)$.

Define $\delta(p,q)$ by the following table, depending on $p, q \pmod 4$.

	0	1	2	3
0	3	2	2	2
1	2	1	1	0
2	2	1	0	0
3	2	0	0	0

See Example 8.24 for an explanation of these numbers.

Group	$\lvert H^1(\sigma, G)\rvert$
Spin(p,q)	$\lfloor \frac{p+q}{4} \rfloor + \delta(p,q)$
Spin$^*(2n)$	2

Simply connected exceptional groups					
inner class	group	K	real rank	name	$\lvert H^1(\sigma, G)\rvert$
compact	E_6	A_5A_1	4	quasisplit' quaternionic	3
	E_6	D_5T	2	Hermitian	3
	E_6	E_6	0	compact	3
split	E_6	C_4	6	split	2
	E_6	F_4	2	quasicompact	2
compact	E_7	A_7	7	split	2
	E_7	D_6A_1	4	quaternionic	4
	E_7	E_6T	3	Hermitian	2
	E_7	E_7	0	compact	4
compact	E_8	D_8	8	split	3
	E_8	E_7A_1	4	quaternionic	3
	E_8	E_8	0	compact	3
compact	F_4	C_3A_1	4	split	3
	F_4	B_4	1		3
	F_4	F_4	0	compact	3
compact	G_2	A_1A_1	2	split	2
	G_2	G_2	0	compact	2

10.3 Adjoint groups

If G is adjoint $|H^1(\sigma, G)|$ is the number of real forms in the given inner class, which is well known. We also include the component group, which is useful in connection with Corollary 9.3.

One technical point arises in the case of $PSO^*(2n)$. If n is even there are two real forms which are related by an outer, but not an inner, automorphism. See Remark 8.2.

Adjoint classical groups		
Group	$\vert\pi_0(G(\mathbb{R}))\vert$	$\vert H^1(\sigma, G)\vert$
$PSL(n,\mathbb{R})$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$
$PSL(n,\mathbb{H})$	1	2
$PSU(p,q)$	$\begin{cases} 2 & p = q \\ 1 & \text{otherwise} \end{cases}$	$\lfloor \frac{p+q}{2} \rfloor + 1$
$PSO(p,q)$	$\begin{cases} 1 & pq = 0 \\ 1 & p, q \text{ odd and } p \neq q \\ 4 & p = q \text{ even} \\ 2 & \text{otherwise} \end{cases}$	$\begin{cases} \lfloor \frac{p+q+2}{4} \rfloor & p, q \text{ odd} \\ \frac{p+q}{4} + 3 & p, q \text{ even}, p + q = 0 \pmod 4 \\ \frac{p+q-2}{4} + 2 & p, q \text{ even}, p + q = 2 \pmod 4 \\ \frac{p+q+1}{2} & p, q \text{ opposite parity} \end{cases}$
$PSO^*(2n)$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$	$\begin{cases} \frac{n}{2} + 3 & n \text{ even} \\ \frac{n-1}{2} + 2 & n \text{ odd} \end{cases}$
$PSp(2n,\mathbb{R})$	2	$\lfloor \frac{n}{2} \rfloor + 2$
$PSp(p,q)$	$\begin{cases} 2 & p = q \\ 1 & \mathit{else} \end{cases}$	$\lfloor \frac{p+q}{2} \rfloor + 2$

The groups E_8, F_4 and G_2 are both simply connected and adjoint. Furthermore in type E_6 the center of the simply connected group G_{sc} has order 3, and it follows that $H^1(\sigma, G_{\mathrm{ad}}) = H^1(\sigma, G_{\mathrm{sc}})$ in these cases. So the only groups not covered by the table in Section 10.2 are adjoint groups of type E_7.

Adjoint exceptional groups						
inner class	group	K	real rank	name	$\pi_0(G(\mathbb{R}))$	$\vert H^1(G)\vert$
compact	E_7	A_7	7	split	2	4
	E_7	D_6A_1	4	quaternionic	1	4
	E_7	E_6T	3	Hermitian	2	4
	E_7	E_7	0	compact	1	4

References

[1] J. Adams, *Lifting of characters*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), 1991, pp. 1–50. MR1168475 (93d:22014)

[2] J. Adams, Dan Barbasch, and David A. Vogan Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104, Birkhäuser Boston Inc., Boston, MA, 1992. MR1162533 (93j:22001)

[3] J. Adams and Fokko du Cloux, *Algorithms for representation theory of real reductive groups*, J. Inst. Math. Jussieu **8** (2009), no. 2, 209–259. MR2485793

[4] J. Adams and David A. Vogan Jr., *L-groups, projective representations, and the Langlands classification*, Amer. J. Math. **114** (1992), no. 1, 45–138. MR1147719 (93c:22021)

[5] Jeffrey Adams, *Guide to the Atlas software: computational representation theory of real reductive groups*, Representation theory of real reductive Lie groups, 2008, pp. 1–37. MR2454331

[6] A. Borel and J.-P. Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. **39** (1964), 111–164. MR0181643

[7] Armand Borel, *Linear algebraic groups*, Second, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR1102012 (92d:20001)

[8] M. V. Borovoi, *Galois cohomology of real reductive groups and real forms of simple lie algebras*, Funct. Anal. Appl. **22** (1988), no. 2, 135–136.

[9] Mikhail Borovoi and Dmitry A. Timashev, *Galois cohomology of real semisimple groups*.

[10] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822

[11] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 7–9*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley. MR2109105

[12] G. Hochschild, *The structure of Lie groups*, Holden-Day, Inc., San Francisco-London-Amsterdam, 1965. MR0207883

[13] Tasho Kaletha, *Rigid inner forms of real and p-adic groups* (2013). arXiv:1304.3292.

[14] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits. MR1632779 (2000a:16031)

[15] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809. MR0311837

[16] Robert E. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), no. 3, 611–650. MR757954

[17] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR858284 (88d:22027)

[18] Toshihiko Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), no. 2, 331–357. MR527548 (81a:53049)

[19] G. D. Mostow, *Self-adjoint groups*, Ann. of Math. (2) **62** (1955), 44–55. MR0069830

[20] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. MR1278263 (95b:11039)

[21] Jirō Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138. MR867991

[22] Jean-Pierre Serre, *Galois cohomology*, English, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. Translated from the French by Patrick Ion and revised by the author. MR1867431 (2002i:12004)

[23] T. A. Springer, *Linear algebraic groups*, Second, Progress in Mathematics, vol. 9, Birkhäuser Boston Inc., Boston, MA, 1998. MR1642713 (99h:20075)

[24] Robert Steinberg, *Générateurs, relations et revêtements de groupes algébriques*, Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962), 1962, pp. 113–127. MR0153677 (27 #3638)

[25] David A. Vogan Jr., *Representations of real reductive Lie groups*, Progress in Mathematics, vol. 15, Birkhäuser Boston, Mass., 1981. MR632407 (83c:22022)

[26] ———, *Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality*, Duke Math. J. **49** (1982), no. 4, 943–1073. MR683010 (84h:22037)

[27] ———, *The local Langlands conjecture*, Representation theory of groups and algebras, 1993, pp. 305–379. MR1216197 (94e:22031)

[28] Garth Warner, *Harmonic analysis on semi-simple Lie groups. II*, Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 189. MR0499000 (58 #16980)