Annihilators and Associated Cycles of Harish-Chandra modules

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We will consider a real reductive Lie group and corresponding Harish-Chandra modules X.

Question: How does one compute the associated cycle of X??

Some notation

- $G_{\mathbb{R}}$: a real reductive Lie group.
- $K_{\mathbb{R}}$: a maximal compact subgroups of $G_{\mathbb{R}}$.
- G, K : the complexifications of $G_{\mathbb{R}}, K_{\mathbb{R}}$.
- $\mathfrak{g} := Lie(G)$.
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: the (complexified) Cartan decomposition of $\mathfrak{g}$.
- $\mathcal{N}$ the nilpotent cone in $\mathfrak{g}$ and $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{p}$.
- $\mathfrak{B}$ is the flag variety for G.

Brief Definitions

$\mathrm{gr}(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g}) \simeq P(\mathfrak{g}^*)$. Choose a good filtration $\{X_n\}$ of X and consider $gr(X)$, a finitely generated $(S(\mathfrak{g}), K)$-module.
Then
(a) $\mathrm{gr}(X)$ is supported in the nilpotent cone $\mathcal{N}$
(b) $\mathrm{supp}(\mathrm{gr}(X)) = \cup \overline{\mathcal{O}}_k$, K-orbits in $\mathcal{N}_\theta$
The support is called the *associated variety* of X. The multiplicity corresponding to each component $\overline{\mathcal{O}}_k$ is written as $m_{\mathcal{O}_k}$; the *associated cycle* of X is

Definition

The *associated cycle* of X is

$$AC(X) = \sum_{\mathcal{O}_k} m_{\mathcal{O}_k} \cdot \overline{\mathcal{O}}_k \quad \text{(formal sum)}$$

.

The AV is contained in the (affine) cone $\mathcal{N}_\theta$; $\mathrm{gr}(X)$ corresponds to a coherent sheaf. This is something like a vector bundle and the multiplicity is like the rank. Here are some properties of the AC.

- All $\mathcal{O}_k$ occurring in the AV have the same dimesion.
- The G-saturations $G \cdot \mathcal{O}_k$ are all the same and are equal to $AV(ann(X))$.
- $\dim(\mathcal{O}_k) = GKdim(X)$.
- If $\{X(\lambda)\}_{\lambda\in\lambda_0+\Lambda}$ is a coherent family (Λ the weight lattice the dual of a CSA $\mathfrak{h}$), then the corresponding multiplicities $m_{\mathcal{O}_k}(\lambda)$ are W-harmonic polynomials on $\mathfrak{h}$.
- The AC depends only on the K-types.
- Multiplicity is some measure of growth of K-types in X: If $AV(X) = \overline{(}\mathcal{O})$, then the multiplicity is

$$\lim_{N\to\infty} \frac{\sum_{|\mu|<N} \dim(X_\mu)}{\sum_{|\mu|<N} \dim(Reg(\overline{\mathcal{O}})_\mu)}$$

See David's 1991 paper for definitions and many properties.

An important formula for computation

The characteristic cycle of an HC module X is defined in a similar way, but for the $\mathcal{D}$-module associated to X. The characteristic variety is the support of the $\mathcal{D}$-module; it is a union of closures of conormal bundles to *kgb*'s. The characteristic cycle is

$$CC(X) = \sum_i n_i [T^*_{\mathcal{Q}_i}\mathfrak{B}] \in H^{BM}_{top}.$$

TWO FACTS:

- $AV(X) = \cup_i \mu(\overline{T^*_{\mathcal{Q}_i}\mathfrak{B}}) \ \ (n_i \neq 0)$.

Let $\mu_i = \mu|_{\overline{T^*_{\mathcal{Q}_i}\mathfrak{B}}}$ and set $\mu(\overline{T^*_{\mathcal{Q}_i}\mathfrak{B}}) = K \cdot f_i = \overline{\mathcal{O}_i}$, then for X irreducible

- $AC(X) = \sum_{\dim(im\mu_i)\ \text{max'l}} n_i \dim(H^0(\mu_i^{-1}(f_i), \mathcal{L}_{\tau_\lambda})) \cdot \overline{\mathcal{O}_i}$.

How do we compute AC's??

The d.s. reps correspond to closed K-orbits in the flag variety $\mathfrak{B}$. Since these orbits have smooth closure, the characteristic cycles are just $[T^*_{\mathfrak{Q}}\mathfrak{B}]$. So one needs to do the following:

1. Compute the moment map image $\mu(T^*_{\mathfrak{Q}}\mathfrak{B}) = \overline{K \cdot f}$, for some nilpotent $f \in \mathfrak{p}$.
2. Understand $\mu_{\mathfrak{Q}}^{-1}(f)$ (where $\mu_{\mathfrak{Q}} := \mu|_{\overline{T^*_{\mathfrak{Q}}\mathfrak{B}]}}$).
3. Compute sections, $H^0(\mu_{\mathfrak{Q}}^{-1}(f), \ldots)$.

Remark: $\mu^{-1}(f) \subset T^*\mathfrak{B}$ may be identified with its image in $\mathfrak{B}$, which is the 'Springer fiber' of f ($\mathfrak{B}^f = \{\mathfrak{b} : f \in \mathfrak{b}\}$). When $f \in \mathfrak{p}$, each component is contained in the in the closure of some conormal bundle to a KGB, and $\mu_{\mathfrak{Q}}^{-1}(f)$ is a single component of Springer fiber.

Let $\mathcal{Q} = K \cdot \mathfrak{b}$ be a closed KGB. One way to describe $\mu_{\mathcal{Q}}^{-1}(f)$ is as

$$\mu_{\mathcal{Q}}^{-1}(f) = N(K, f) \cdot \mathfrak{b}, \text{ where } N(K, f) = \{k \in K : k^{-1} \cdot f \in \mathfrak{b}\}.$$

Example: For h.d.s. $\mathfrak{b} \supset \mathfrak{p}_-$. Then $N(K, f) = K$, since $f \in \mathfrak{p}_-$ (say, $f = X_{-\gamma}$) and $\mathfrak{p}_-$ is a K stable space. So $\mu_{\mathcal{Q}}^{-1}(f) = K \cdot \mathfrak{b} = \mathcal{Q}$. So

$$AC(\text{h.d.s.}) = \prod_{\alpha \in \Delta_c^+} \frac{\langle \lambda, \alpha \rangle}{\langle \rho, \alpha \rangle} \cdot \overline{K} \cdot X_{-\gamma}$$

But $\mu_{\mathcal{Q}}^{-1}(f)$ is not typically homogeneous.

Work with L. Barchini

Let $G_{\mathbb{R}} = U(p,q)$. The d.s. reps are parametrized by regular integral λ that are dominant for some fixed Δ_c^+. Each gives a Borel $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$, a closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{b}$ and a coherent family $\{X_{\mathfrak{b}}(\lambda)\}$.

1. We gave an algorithm to find $f \in \mathfrak{n}^-$ so that $\mu(T_{\mathcal{Q}}^*\mathfrak{B}) = \overline{K \cdot f}$, thus determining the AV.
2. We gave an explicit expression (in terms of the algorithm) for $\mu_{\mathcal{Q}}^{-1}(f)$. This is an iterated bundle for which we may apply the Borel-Weil Thm to compute the sections.

We get

$$AC(X_{\mathfrak{b}}(\lambda)) = \dim(H^0(\mu_i^{-1}(f_i), \mathcal{L}_{\tau_\lambda})) \cdot \overline{K \cdot f}$$

with everything explicitly computable.

Example of algorithm to find f

Consider $G_{\mathbb{R}} = U(6,4)$ and $\lambda = (10,9,7,6,5,1 \mid 8,4,3,2)$ determining a positive system, a Borel $\mathfrak{b}$ and a KGB $\mathfrak{Q} - K \cdot \mathfrak{b}$.

Theorem (Barchini-Z)

For $G_{\mathbb{R}} = U(p,q)$, $\mathcal{Q}$ a closed orbit and f constructed as above,

$$\mu_{\mathcal{Q}}^{-1}(f) = L_m \ldots L_2 L_1 L \cdot \mathfrak{b},$$

an iterated bundle of flag varieties.

Example 1 Holomorphic discrete series again.

Example 2 Quaternionic discrete series of $U(p,2)$.

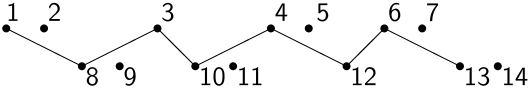
Example 3 Big discrete series of $U(p,p)$.

More Serious Example

Let $G_{\mathbb{R}} = SU(7,7)$. Consider the positive root system determined by

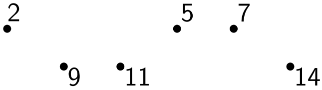
$$(14, 13, 10, 7, 6, 4, 3 \,|\, 12, 11, 9, 8, 5, 2, 1).$$

This gives the array



So
$L = GL(1) \times GL(2) \times GL(1) \times GL(2) \times GL(2) \times GL(2) \times GL(1) \times GL(2)$.

After deleting the first string the resulting array is



$L_1 = GL(2) \times GL(2) \times GL(2) \times GL(2) \times GL(1)$ and $L_2 = GL(1) \times GL(1)$

Example cont'd

So the Springer fiber is $L_2 L_1 L \cdot \mathfrak{b} = L_1 L \cdot \mathfrak{b}$ and one needs to compute $H^0(L_1 L \cdot B, \mathcal{L}_\tau)$. By Leray spectral sequence this is essentially

$$H(L_1/L_1 \cap Q, H(L/L \cap B, \mathcal{L}_\tau).$$

One gets
$\frac{1}{4}(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_5)(\lambda_6 - \lambda_7)(\lambda_8 - \lambda_9)(\lambda_{10} - \lambda_{11})(\lambda_{13} - \lambda_{14})(\lambda_4 + \lambda_5 - \lambda_6 - \lambda_7)(\lambda_8 + \lambda_9 - \lambda_{10} - \lambda_{11}).$

In carrying out the B-W calculation one needs to use the (simplest) branching rule: $GL(n) \to GL(1) \times GL(n-1)$.

Other groups

For $Sp(2n, R)$ and $SO_e(p, q)$ the situation is just about the same - there is an algorithm to find a good base point f in the image of the moment map and the corresponding component of the Springer fiber is an iterated bundle. The Borel-Weil theorem finds the multiplicity.

Now consider $SO^*(2n)$ and $Sp(p, q)$. There is an algorithm to find f, giving the AV, but one does not (quite) obtain the iterated bundle structure on the component of the Springer fiber, we are not able to find the multiplicity polynomial by a B-W calculation.

The multiplicity may be computed anyway. The method uses the coherent continuation representation on the cell of a discrete series repn - the atlas computes coherent continuation rep of W. We do the following: Let $\mathcal{C}$ be a cell containing some d.s. repn

1. Find a d.s. rep $X_{\mathcal{Q}}$ in $\mathcal{C}$ so that $\gamma_{\mathcal{Q}}^{-1}(f) = L \cdot f$ (L as before). Borel-Weil gives multiplity poly immediately.
2. Show $X_{\mathcal{Q}}$ generates the cell as W-repn

The map $X \mapsto m_{\mathcal{O}}(\lambda)$ is W-equivariant.

More Info on multiplicity polys

Suppose X is any irreducible HC module with regular integral infinitesimal character and suppose the annihilator is $ann(L_w)$ (L_w is the unique irreducible quotient of $M_w = \mathcal{U}(\mathfrak{g}) \underset{\mathfrak{b}}{\otimes} \mathbb{C}_{w\lambda-\rho}$, λ dominant regular integral). There are three W-harmonic polynomials on $\mathfrak{h}^*$:

$m_{\mathcal{O}}(\lambda)$, Goldie rank poly, $p_w(\lambda) = \sum_{y \in W} a_{y,w}(\lambda(y^{-1}\rho))^d$

($a_{y,w}$ give character of Vermas)

Each is a constant multiple of the other
(See J. T. Chang, A. Joseph, Barbasch-Vogan for these facts.)

Conclude: If we know
(a) The multiplicity $m_{\mathcal{O}}$ for one X in $\{X(\lambda)\}_{\lambda\in\Lambda}$,
(b) $ann(X)$ (i.e., know the w),
then we know the multiplicity poly $m_{\mathcal{O}}(\lambda)$ exactly (that is we may find the constant multiple of $p_w(\lambda)$). Note that atlas computes the $a_{y,w}$ appearing in the formula for $p_w(\lambda)$.

On the other hand: If we know $m_{\mathcal{O}}(\lambda)$ (or just a multiple) for some X, then we can find the $p_w(\lambda)$ that is a multiple. This tells us the annihilator of X is $ann(L_w)$.

Harish-Chandra Cells

Let $\mathcal{C}$ be an HC cell and $V_{\mathcal{C}}$ the corresponding cell repn of W - a subquotient of coherent continuation. $V_{\mathcal{C}}$ is a $\mathbb{C}$-vector space with basis $\{X : X \in \mathcal{C}\}$.

Facts:

- Each $V_{\mathcal{C}}$ contains a unique special repn of W.
- AV is constant on the cell.
- If $\mathcal{O}_i$ appears in the AV, then $\mathcal{O}^{\mathbb{C}} := G \cdot \mathcal{O}_i$ is $AV(ann(X))$, for each $X \in \mathcal{C}$.
- The special repn occurring in $V_{\mathcal{C}} \leftrightarrow (\mathcal{O}^{\mathbb{C}}, 1)$ by Springer correspondence.
- $X \mapsto m^X_{\mathcal{O}_i}(\lambda)$ is W- h.m. $V_{\mathcal{C}} \to P(\mathfrak{h}^*)$.

Atlas computation

Atlas computes coherent continuation repn of W.

This means one may compute:

- The decomposition of $V_{\mathcal{C}}$ into irreducibles - use a character table for W (which is available).
- Identify the special repn that occurs - known.
- Determine $\mathcal{O}^{\mathbb{C}}$ - use known tables (e.g., in Carter).

There are other ways to do this.

This information can be found at
http://lie.math.okstate.edu/atlas/data/
where B. Binegar has collected the information in useful and easy to read tables for many low rank groups.

Cells

```
atlas> set G=Sp(4,R)
atlas> set bl=block(G,dual_quasisplit_form (G))
atlas> print_W_cells (bl)
// Cells and their vertices.
  #0={0}
  #1={1}
  #2={2,5,7}
  #3={3,6,8}
  #4={4,9,11}
  #5={10}
// Individual cells.
// cell #0:
  0[0]:  {}
// cell #1:
  0[1]:  {}
// cell #2:
  0[2]:  {1} --> 1
  1[5]:  {2} --> 0,2
  2[7]:  {1} --> 1
```

From http://lie.math.okstate.edu/atlas/data/ (wcellreps)

$Sp(2, \mathbb{R})$

```
Sp(4,R) x SO(3,2) block:
⋮
cell #2
  cell size = 3
  cell W-rep = phi[[1],[1]]+phi[[1, 1],[]]
  special rep = phi[[1],[1]] ; dim = 2
  special orbit = [2, 2]
  tau-infinity partition completed in 1 step(s)
  2 parts
  partitioning = [[1, 1], [2, 1]]
  intersection with blocku = {2,5}
cell #3
⋮
```

More info:

- The special repn $\pi(\mathcal{O}^{\mathbb{C}}, 1)$ occurs in polys of homogeneous degree $d = \mid \Delta^+ \mid - \frac{1}{2}\dim(\mathcal{O}^{\mathbb{C}})$, and occurs in no lower degree polys.
- $V_{\mathcal{C}} = V_{\mathcal{C}}^{\mathrm{spec}} \oplus$ (other stuff), and the other stuff occurs in polys of degree higher than d.

Conclude: There exists a unique (up to scalar multiple)

$$\varphi : V_{\mathcal{C}} \to S^d(\mathfrak{h}), \; W\text{-h.m.}$$

WANT TO FIND IT.

This will tell us two things: multiplicity polys up to scalar multiple $(= \varphi(X))$ and annihilators.

Remark: David has an algorithm, likely to soon to be available in atlas, to find the AC of an irreducible HC module. This will give the multiplicity as an integer. If we are successful in finding φ, then the constant multiple of $\varphi(X)$ can be nailed down to get the mult poly exactly.

Somewhat weaker:

Let $(V_{\mathcal{C}}, \sigma)$ denote the cell repn for some arbitrary fixed cell. Let χ^{spec} be the character of the special repn corresponding to $\mathcal{C}$.

Consider the projection

$$P = \sigma\left(\frac{1}{\dim(\mathrm{spec})}\sum_{W}\chi^{\mathrm{spec}}(w)w\right) : V_{\mathcal{C}} \to Y_{\mathcal{C}}^{\mathrm{spec}}.$$

onto the special repn.

Then for X in $\mathcal{C}$, $P(X) = \sum b_i X_i$ and

The X_i are exactly the elements of $\mathcal{C}$ having the same annihilator as X.

The atlas can do this. (The character table for the special reps is needed.....)

Example: $Sp(4, R)$

There are 12 reps in the block of the trivial, three cells with just one element and three with 3 elements. Lets consider block #2: $\mathcal{C} = \{X_0, X_1, X_2\}$. The AV of annihilator is $[2, 2]$ (in partition notation), and the special repn has dimension 2 and occurs in $S^1(\mathfrak{h})$ (so is the reflection repn of W).

Coherent continuation:

$$\sigma(s_1) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \sigma(s_2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Get $P(X_0) = X_0 + X_2 = P(X_2)$ and $P(X_1) = 2X_1$, so $ann(X_0) = ann(X_2)$.

How do we find $\varphi : V_{\mathcal{C}} \to P^d(\mathfrak{h})$? ($d = 1$, special is reflection)??

X_0 generates the $V_{\mathcal{C}}$: $X_1 = (-1 + s_2)X_0$, etc.

Since $\tau(X_0) = \{1\} = \{\epsilon_1 - \epsilon_2\}$, $\varphi(X_0)$ transforms by the sign repn of $\{1, s_1\}$, so $\varphi(X_0) = \lambda_1 - \lambda_2$ (up to constant multiple). Now $\varphi(X_1) = (-1 + s_2) \cdot (\lambda_1 - \lambda_2) = (-\lambda_1 - \lambda_2) + (\lambda_1 + \lambda_2) = 2\lambda_2$.

Get

$$\varphi(X_0) = \varphi(X_2) = \lambda_1 - \lambda_2; , \varphi(X_1) = 2\lambda_2.$$

These are normalized because X_0 is h.d.s. which has multiplicity 1 at infinitesimal character ρ.

[We also have $V_{\mathcal{C}} = \mathbb{C} \cdot (X_0 - X_2) \oplus \mathbb{C} \cdot \{X_0 + X_2, X_1\}$.]

Thank You!