Unitary representations of reductive groups

David Vogan

University of Utah
July 10–14, 2017

Outline

What's a representation?

Definition. Representation of group G on vec space V_ρ is group homomorphism

$$\rho\colon G \to GL(V_\rho).$$

Equivalently: action of G on V_ρ by linear maps.

Main example. $G \curvearrowright X$, $V_\rho =$ functions on X.

Definition. Hilbert space is cplx vec space $\mathcal{H}$ with form $\langle, \rangle$:

1. $\langle v, w\rangle = \overline{\langle w, v\rangle} \quad \langle av_1 + bv_2, w\rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle$;
2. $\langle v, v\rangle > 0 \quad (0 \neq v \in \mathcal{H})$;
3. $\mathcal{H}$ complete in metric $d(v, w) =_{\text{def}} \langle v - w, v - w\rangle^{1/2}$.

Definition. Unitary representation of G on Hilbert space $\mathcal{H}_\pi$ is a group homomorphism

$$\pi\colon G \to U(\mathcal{H}_\pi).$$

Equiv: action of G on $\mathcal{H}_\pi$ by unitary linear maps.

Main example. $G \curvearrowright (X, dx)$, $\mathcal{H}_\pi = L^2(X)$.

Abstract harmonic analysis for dummies

Group G acts on X, have questions about X.

Step 1. Attach to X vector space V of functions on X. Questions about $X \rightsquigarrow$ questions about V.

Step 2. Find finest G-eqvt decomp $V = \oplus_\rho V_\rho$. Questions about $V \rightsquigarrow$ questions about each V_ρ.

Each V_ρ is irreducible representation of G.

Step 3. Understand $\widehat{G} =$ all irreducible representations of G.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$). Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\pi \mathcal{H}_\pi$. Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\pi$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Topic for these lectures: **Step 3** for reductive G.

Why are unitary representations better?

Why Gelfand's unitary rep $\rightsquigarrow \oplus$ irr unitary $>>$ dummies' any rep $\rightsquigarrow \oplus$ irr reps?

Programs seek $V = \oplus_\rho V_\rho, \qquad \mathcal{H} = \oplus_\pi \mathcal{H}_\pi$.

$\leftrightsquigarrow$ eigenspace decomp of lin op $=$ spectral theory.

Spectral theory of unitary ops on Hilb spaces $>>$ spectral theory of linear ops on top vec spaces.

Easy: $\mathcal{H}_1 \subset \mathcal{H}_\pi$ G-invt closed $\implies \mathcal{H}_\pi = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$.

$\rightsquigarrow$ 1ST: get direct integral decomposition

$$\mathcal{H} = \oplus_{\pi \in \widehat{G}_u} M_\pi \otimes \mathcal{H}_\pi$$

arb unitary $\rightsquigarrow$ irr unitary under weak hyps.

2ND: $\exists$ plenty of unitary (e.g. G-invt msres).

3RD: non-Hilb space questions (e.g. Schwartz space for $\mathbb{R}$) can be studied using unitary reps:

$$FT(\text{Schwartz space}) = \text{Schwartz space}.$$

Why study nonunitary representations?

$\widehat{G}$ = all irr reps = cplx alg variety.

Reason: $\mathrm{Hom}_{\mathrm{groups}}(G, GL(V)) \approx$ alg variety.

Reason for the reason: if G has N generators, then

$\mathrm{Hom}(G, GL(n, \mathbb{C}))$ = N-tuples of matrices (alg variety)
satisfying relations of G (alg subvariety).

Alg varieties can admit beautiful descriptions.

Langlands classif is beautiful description of $\widehat{G(\mathbb{R})}$

$\widehat{G(\mathbb{R})}_h$ = reps with invt Herm form
= real form of $\widehat{G(\mathbb{R})}$ (Knapp-Zuckerman).

$\widehat{G(\mathbb{R})}_u$ = reps with pos invt form
defined by inequalities: less beautiful.

Our plan this week: $G(\mathbb{R})$ real reductive Lie...

1. Langlands classification of $\widehat{G(\mathbb{R})}$;
2. Knapp-Zuckerman identification $\widehat{G(\mathbb{R})}_h \subset \widehat{G(\mathbb{R})}$;
3. signature of invt Herm form on each $\rho \in \widehat{G(\mathbb{R})}_h$.

Moral of the story

Aiming at `atlas` classification of $\widehat{G(\mathbb{R})}_u$, equiv classes of irr unitary reps of real reductive $G(\mathbb{R})$.

First: Langlands classification of $\widehat{G(\mathbb{R})}$, "all" irr reps of $G(\mathbb{R})$, as cplx alg variety.

Second: Knapp-Zuckerman classification of $\widehat{G(\mathbb{R})}_h$, irr Hermitian reps of $G(\mathbb{R})$, as real points of $\widehat{G(\mathbb{R})}$.

Third: `atlas` computation of signature of any invt Herm form.

Fourth: inspect answers to signature computations: unitary reps $\leftrightsquigarrow$ definite signatures.

What can we ask about representations?

Start with a reasonable category of representations…

Example: cplx reductive $\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$; BGG category $\mathcal{O}$ consists of $U(\mathfrak{g})$-modules V subject to

1. fin gen: $\exists V_0 \subset V,\ \dim V_0 < \infty,\ U(\mathfrak{g}) V_0 = V$.
2. $\mathfrak{b}$-locally finite: $\forall v \in V,\ \dim U(\mathfrak{b}) v < \infty$.
3. $\mathfrak{h}$-semisimple: $V = \sum_{\gamma \in \mathfrak{h}^*} V_\gamma$.

Want precise information about reps in the category.

Example: V in category $\mathcal{O}$

1. $\dim V_\gamma$ is *almost polynomial* as function of γ.
2. V has a *formal character* $\left[\sum_{\lambda \in \mathfrak{h}^*} a_V(\lambda) e^\lambda\right] / \Delta$.

Want construction/classification of reps in the category.

Example: $\lambda \in \mathfrak{h}^* \rightsquigarrow I(\lambda) =_{\text{def}} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda =$ *Verma module*.

1. (STRUCTURE THM): $I(\lambda)$ has highest weight $\mathbb{C}_\lambda \hookrightarrow I(\lambda)^{\mathfrak{n}}$.
2. (QUOTIENT THM): $I(\lambda)$ has unique irr quo $J(\lambda)$.
3. (CLASSIF THM): Each irr in $\mathcal{O}$ is $J(\lambda)$, unique $\lambda \in \mathfrak{h}^*$.

How do you do that?

$\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ roots, Δ^+ roots in $\mathfrak{n}$.

$\rightsquigarrow$ partial order on $\mathfrak{h}^*$:

$$\mu' \le \mu \iff \mu' \in \mu - \mathbb{N}\Delta^+$$
$$\iff \mu' = \mu - \sum_{\alpha \in \Delta^+} n_\alpha \alpha, \quad (n_\alpha \in \mathbb{N})$$

Proposition. Suppose $V \in \mathcal{O}$.

1. If $V \neq 0, \exists$ *maximal* $\mu \in \mathfrak{h}^*$ subject to $V_\mu \neq 0$.
2. If $\mu \in \mathfrak{h}^*$ is maxl subj to $V_\mu \neq 0$, then $V_\mu \subset V^{\mathfrak{n}}$.
3. If $V \neq 0, \exists \mu$ with $0 \neq V_\mu \subset V^{\mathfrak{n}}$.
4. $\forall \lambda \in \mathfrak{h}^*$, $\mathrm{Hom}_{\mathfrak{g}}(I(\lambda), V) \simeq \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda, V^{\mathfrak{n}})$.

Parts (1)–(3) guarantee existence of "highest weights;" based on formal calculations with lattices in vector spaces, and $\mathfrak{n} \cdot V_{\mu'} \subset \sum_{\alpha \in \Delta^+} V_{\mu'+\alpha}$.

Sketch of proof of (4):

$$\mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, V) \simeq \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, V) = \mathrm{Hom}_{U(\mathfrak{h})}(\mathbb{C}_\lambda, V^{\mathfrak{n}}).$$

First isom: "change of rings." Second: $\mathfrak{n} \cdot \mathbb{C}_\lambda =_{\text{def}} 0$.

Moral of the story

For category $\mathcal{O}$, three key ingredients:

1. Change of rings $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \cdot \rightsquigarrow$ Verma mods $I(\lambda)$.
2. Universality: $\mathrm{Hom}_{\mathfrak{g}}(I(\lambda), V) \simeq \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda, V^{\mathfrak{n}})$.
3. Highest weight exists: J irr $\implies J^{\mathfrak{n}} \neq 0$.

#2 is homological alg, #3 is comb/geom in $\mathfrak{h}^*$.

Irrs J in $\mathcal{O} \leftrightsquigarrow \lambda \in \mathfrak{h}^*$; characteristic is $\mathbb{C}_\lambda \subset J(\lambda)^{\mathfrak{n}}$.

Same three ideas apply to $(\mathfrak{g}, K)$-modules.

Technical problem: change of rings needed is not projective, so $\otimes$ has to be supplemented by Tor.

Parallel problem: replace $J^{\mathfrak{n}} = H^0(\mathfrak{n}, J)$ by some derived functors $H^p(\mathfrak{n}, J)$.

Irr $G(\mathbb{R})$-reps $J \leftrightsquigarrow \gamma \in \widehat{H(\mathbb{R})}$, some θ-stable Cartan $H(\mathbb{R}) \subset G(\mathbb{R})$; characteristic is $\mathbb{C}_\gamma \subset H^s(\mathfrak{n}, J)$.

END OF LECTURE ONE

BEGINNING OF LECTURE TWO

Principal series for $SL(2,\mathbb{R})$

To understand Harish-Chandra's category of group representations, need a serious example.

Use principal series repns for $SL(2,\mathbb{R}) =_{\mathrm{def}} G(\mathbb{R})$.

$G(\mathbb{R}) \curvearrowright \mathbb{R}^2$, so get rep of $G(\mathbb{R})$ on functions on $\mathbb{R}^2$:

$$[\rho(g)f](v) = f(g^{-1}\cdot v).$$

Lie algs easier than Lie gps $\rightsquigarrow$ write $\mathfrak{sl}(2,\mathbb{R})$ action, basis

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Action on functions on $\mathbb{R}^2$ is by vector fields:

$$\rho(D)f = -x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2}, \quad \rho(E) = -x_2\frac{\partial f}{\partial x_1}, \quad \rho(F) = -x_1\frac{\partial f}{\partial x_2}.$$

General principle: representations on function spaces are reducible $\leftrightsquigarrow$ exist $G(\mathbb{R})$-invt differential operators.

Euler deg operator $E = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ commutes with $G(\mathbb{R})$.

Conclusion: interesting reps of $G(\mathbb{R})$ on eigenspaces of E.

Principal series for $SL(2,\mathbb{R})$ (continued)

Previous slide: expect interesting reps of $G(\mathbb{R}) = SL(2,\mathbb{R})$ on homogeneous functions on $\mathbb{R}^2$.

For $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, define

$$W^{\nu,\epsilon} = \{f\colon (\mathbb{R}^2 - 0) \to \mathbb{C} \mid f(tx) = |t|^{-\nu-1}\operatorname{sgn}(t)^\epsilon f(x)\},$$

functions on the plane homog of degree $-(\nu+1,\epsilon)$.

$\nu \rightsquigarrow \nu+1$ simplifies MANY things later…

Study $W^{\nu,\epsilon}$ by restriction to circle $\{(\cos\theta,\sin\theta)\}$:

$$W^{\nu,\epsilon} \simeq \{w\colon S^1 \to \mathbb{C} \mid w(-s) = (-1)^\epsilon w(s)\},\ f(r,\theta) = r^{-\nu-1}w(\theta).$$

Compute Lie algebra action in polar coords using

$$\frac{\partial}{\partial x_1} = -x_2\frac{\partial}{\partial\theta} + x_1\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial x_2} = x_1\frac{\partial}{\partial\theta} + x_2\frac{\partial}{\partial r},$$
$$\frac{\partial}{\partial r} = -\nu-1, \qquad x_1 = \cos\theta, \qquad x_2 = \sin\theta.$$

Plug into formulas on preceding slide: get

$$\rho^{\nu,\epsilon}(D) = 2\sin\theta\cos\theta\frac{\partial}{\partial\theta} + (-\cos^2\theta + \sin^2\theta)(\nu+1),$$

$$\rho^{\nu,\epsilon}(E) = \sin^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)(\nu+1),$$

$$\rho^{\nu,\epsilon}(F) = -\cos^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)(\nu+1).$$

A more suitable basis

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2,\mathbb{R})$ defined on functions on S^1 of homogeneity (or parity) ϵ:

$$\rho^{\nu,\epsilon}(D) = 2\sin\theta\cos\theta\frac{\partial}{\partial\theta} + (-\cos^2\theta + \sin^2\theta)(\nu+1),$$
$$\rho^{\nu,\epsilon}(E) = \sin^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)(\nu+1),$$
$$\rho^{\nu,\epsilon}(F) = -\cos^2\theta\frac{\partial}{\partial\theta} + (-\cos\theta\sin\theta)(\nu+1).$$

Hard to make sense of. Clear: family of reps analytic (actually linear) in complex parameter ν.

Big idea: see how properties change as function of ν.

Problem: $\{D, E, F\}$ adapted to wt vectors for diagonal Cartan subalgebra; rep $\rho^{\nu,\epsilon}$ has no such wt vectors.

But rotation matrix $E - F$ acts simply by $\partial/\partial\theta$.

Suggests new basis of the complexified Lie algebra:

$$H = -i(E-F), \quad X = \frac{1}{2}(D + iE + iF), \quad Y = \frac{1}{2}(D - iE - iF).$$

$$\rho^{\nu,\epsilon}(H) = \frac{1}{i}\frac{\partial}{\partial\theta}, \; \rho^{\nu,\epsilon}(X) = \frac{e^{2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right), \; \rho^{\nu,\epsilon}(Y) = \frac{-e^{-2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right).$$

Matrices for principal series, bad news

Have family $\rho^{\nu,\epsilon}$ of reps of $SL(2, \mathbb{R})$ defined on functions on S^1 of homogeneity (or parity) ϵ:

$$\rho^{\nu,\epsilon}(H) = \frac{1}{i}\frac{\partial}{\partial\theta}, \quad \rho^{\nu,\epsilon}(X) = \frac{e^{2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right), \quad \rho^{\nu,\epsilon}(Y) = \frac{-e^{-2i\theta}}{2i}\left(\frac{\partial}{\partial\theta} + i(\nu+1)\right).$$

These ops act simply on basis $w_m(\cos\theta, \sin\theta) = e^{im\theta}$:

$$\rho^{\nu,\epsilon}(H)w_m = mw_m,$$

$$\rho^{\nu,\epsilon}(X)w_m = \frac{1}{2}(m+\nu+1)w_{m+2},$$

$$\rho^{\nu,\epsilon}(Y)w_m = \frac{1}{2}(-m+\nu+1)w_{m-2}.$$

Suggests reasonable function space to consider:

$$W^{\nu,\epsilon,K(\mathbb{R})} = \text{fns homog of deg } (\nu,\epsilon), \text{ finite under rotation}$$
$$= \mathrm{span}(\{w_m \mid m \equiv \epsilon \pmod 2\}).$$

$W^{\nu,\epsilon,K(\mathbb{R})}$ has beautiful rep of $\mathfrak{g}$: irr for most ν, easy submods otherwise. Not preserved by $G(\mathbb{R}) = SL(2,\mathbb{R})$: $\exp(A) \in G(\mathbb{R}) \rightsquigarrow \sum A^k/k!$: $A^k \curvearrowright W^{\nu,\epsilon,K(\mathbb{R})}$, sum not.

Structure of principal series: good news

Original question was action of $G(\mathbb{R}) = SL(2,\mathbb{R})$ on

$$W^{\nu,\epsilon,\infty} = \{f \in C^\infty(\mathbb{R}^2 - 0) \mid f \text{ homog of deg } -(\nu+1,\epsilon)\}:$$

what are the closed $G(\mathbb{R})$-invt subspaces. . . ?

Found nice subspace $W^{\nu,\epsilon,K(\mathbb{R})}$, explicit basis, explicit action of Lie algebra $\rightsquigarrow$ easy to describe $\mathfrak{g}$–invt subspaces.

Theorem (Harish-Chandra) There is one-to-one corr

$$\text{closed } G(\mathbb{R})\text{-invt } S \subset W^{\nu,\epsilon,\infty} \leftrightsquigarrow \mathfrak{g}(\mathbb{R})\text{-invt } S^K \subset W^{\nu,\epsilon,K}$$

$$S \rightsquigarrow K\text{-finite vectors in } S, \quad S^K \rightsquigarrow \overline{S^K}.$$

Content of thm: closure carries $\mathfrak{g}$-invt to G-invt.

Why this isn't obvious: $SO(2)$ acting by translation on $C^\infty(S^1)$. Lie alg acts by $\frac{d}{d\theta}$, so closed subspace

$$E = \{f \in C^\infty(S^1) \mid f(\cos\theta, \sin\theta) = 0, \theta \in (-\pi/2, \pi/2) + 2\pi\mathbb{Z}\}$$

is preserved by $\mathfrak{so}(2)$; *not* preserved by rotation.

Reason: Taylor series for in $f \in E$ doesn't converge to f.

Same formalism, general $G(\mathbb{R})$

Lesson of $SL(2,\mathbb{R})$ princ series: vecs finite under $SO(2)$ have nice/comprehensible/meaningful Lie algebra action.

Back to general setting: $G(\mathbb{R})$ real pts of conn reductive complex algebraic group $\rightsquigarrow$ can embed

$$G(\mathbb{R}) \hookrightarrow GL(n,\mathbb{R}), \quad \text{stable by transpose}, \quad G(\mathbb{R})/G(\mathbb{R})_0 \text{ finite.}$$

Recall *polar decomposition*:

$$\begin{aligned} GL(n,\mathbb{R}) &= O(n) \times (\text{pos def symmetric matrices}) \\ &= O(n) \times \exp(\text{symmetric matrices}). \end{aligned}$$

Inherited by $G(\mathbb{R})$ as Cartan decomposition for $G(\mathbb{R})$:

$$K(\mathbb{R}) = O(n) \cap G, \quad \mathfrak{s}_0 = \mathfrak{g}_0 \cap (\text{symm mats}), \quad S = \exp(\mathfrak{s}_0)$$

$$G(\mathbb{R}) = K(\mathbb{R}) \times S = K(\mathbb{R}) \times \exp(\mathfrak{s}_0).$$

(ρ, W) rep of G on complete loc cvx top vec W;

$$W^{K(\mathbb{R})} = \{w \in W \mid \dim \operatorname{span}(\rho(K(\mathbb{R}))w) < \infty\},$$

$$W^{\infty} = \{w \in W \mid G(\mathbb{R}) \to W, \;\; g \mapsto \rho(g)w \text{ smooth}\}.$$

Definition. The Harish-Chandra-module of W is $W^{K(\mathbb{R}),\infty}$: representation of Lie algebra $\mathfrak{g}(\mathbb{R})$ and of group $K(\mathbb{R})$.

Category of $(\mathfrak{h}(\mathbb{R}), L(\mathbb{R}))$-modules

Setting: $\mathfrak{h}(\mathbb{R}) \supset \mathfrak{l}(\mathbb{R})$ real Lie algebras, $L(\mathbb{R})$ compact Lie group acting on $\mathfrak{h}(\mathbb{R})$ by Lie algebra automorphisms Ad.

Definition. An $(\mathfrak{h}(\mathbb{R}), L(\mathbb{R}))$-module is complex vector space W, with reps of $\mathfrak{h}(\mathbb{R})$ and of $L(\mathbb{R})$, subject to

1. each $w \in W$ belongs to fin-diml $L(\mathbb{R})$-invt W_0, so that action of $L(\mathbb{R})$ on W_0 continuous (hence smooth);
2. differential of $L(\mathbb{R})$ action is $\mathfrak{l}(\mathbb{R})$ action;
3. For $k \in L(\mathbb{R})$, $Z \in \mathfrak{h}(\mathbb{R})$, $w \in W$,
   $k \cdot (Z \cdot (k^{-1} \cdot w)) = [\mathrm{Ad}(k)(Z)] \cdot w$.

Proposition. Passage to smooth $K(\mathbb{R})$-finite vectors defines a functor

$$(\text{reps of } G(\mathbb{R}) \text{ on complete locally convex } W)$$
$$\longrightarrow (\mathfrak{g}(\mathbb{R}), K(\mathbb{R}))\text{-modules } W^{K(\mathbb{R}),\infty}$$

.

Complexified is better

Complex vector spaces $>>$ real vector spaces.

Reason: linear maps are (nearly) diagonalizable.

Example: Motion of pendulum $\leftrightsquigarrow$ real-valued

$$\phi\colon \mathbb{R}_{\text{time}} \to \mathbb{R}_{\text{displacement}}, \qquad \frac{d^2\phi}{dt^2} = -\lambda^2\phi.$$

Solutions $\phi(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t) \quad (c_1, c_2 \in \mathbb{R})$.

Easier to study complex-valued

$$\phi\colon \mathbb{R}_{\text{time}} \to \mathbb{C}_{\text{displacement}}, \qquad \frac{d^2\phi}{dt^2} = -\lambda^2\phi.$$

Solutions $\phi(t) = a_1 e^{i\lambda t} + a_2 e^{-i\lambda t} \quad (a_1, a_2 \in \mathbb{C})$.

If you need to build a clock, real-valued $\iff$ $a_2 = \overline{a_1}$.

Complexified Lie algebras

real Lie algebra $\mathfrak{h}(\mathbb{R}) \rightsquigarrow$ complex Lie algebra

$$\mathfrak{h} = \mathfrak{h}(\mathbb{C}) =_{\text{def}} \mathfrak{h}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$
$$= \{X + iY \mid X, Y \in \mathfrak{h}(\mathbb{R}).\}$$

complexification of $\mathfrak{h}(\mathbb{R})$.

Proposition. Representation (π_0, V) of $\mathfrak{h}(\mathbb{R}) \leftrightsquigarrow$ representation (π_1, V) of $\mathfrak{h}(\mathbb{C})$:

$$\pi_1(X + iY) = \pi_0(X) + i\pi_0(Y), \qquad \pi_0(X) = \pi_1(X).$$

Identification $\pi_0 \leftrightsquigarrow \pi_1$ is perfect; write π for both.

Complexified compact Lie groups

Same thing works for compact groups...

real compact $L(\mathbb{R}) \subset U(n) \rightsquigarrow$ complex reductive alg

$$L = L(\mathbb{C}) =_{\text{def}} L(\mathbb{R}) \exp(i\mathfrak{l}(\mathbb{R}) \subset GL(n, \mathbb{C})$$

complexification of $L(\mathbb{R})$.

Coordinate-free definition:

reg fns on $L(\mathbb{C}) = L(\mathbb{R})$-finite $\mathbb{C}$-valued fns on $L(\mathbb{R})$

Proposition. Fin-diml continuous (π_0, V) of $L(\mathbb{R}) \leftrightsquigarrow$ fin-diml algebraic representation (π_1, V) of $L(\mathbb{C})$:

$$\pi_1(l \exp(iY)) = \pi_0(l) \exp(id\pi_0(Y)), \qquad \pi_0(l) = \pi_1(l).$$

Identification $\pi_0 \leftrightsquigarrow \pi_1$ is perfect; write π for both.

$L(\mathbb{R})$-finite cont reps of $L(\mathbb{R})$ = algebraic reps of $L(\mathbb{C})$.

Category of $(\mathfrak{h}, L)$-modules

Now we can complexify Harish-Chandra's category...

Setting: $\mathfrak{h} \supset \mathfrak{l}$ complex Lie algebras, L complex reductive algebraic acting on $\mathfrak{h}$ by Lie algebra automorphisms Ad.

Definition. An $(\mathfrak{h}, L)$-module is complex vector space W, with reps of $\mathfrak{h}$ and of L, subject to

1. L action is algebraic (hence smooth);
2. differential of L action is $\mathfrak{l}$ action;
3. For $k \in L$, $Z \in \mathfrak{h}$, $w \in W$,
   $k \cdot (Z \cdot (k^{-1} \cdot w)) = [\mathrm{Ad}(k)(Z)] \cdot w$.

Write $\mathcal{M}(\mathfrak{h}, L)$ for category of $(\mathfrak{h}, L)$-modules.

Proposition. Smooth K-finite vectors define functor

$$W \in (\text{reps of } G(\mathbb{R}) \text{ on complete locally convex space}) \longrightarrow W^{K,\infty} \in \mathcal{M}(\mathfrak{g}, K)$$

Definition of $\mathcal{M}(\mathfrak{h}, L)$ makes sense for L algebraic (not necessarily reductive).

Representations and R-modules

Rings and modules familiar and powerful $\rightsquigarrow$ try to make representation categories into module categories.

Category of reps of $\mathfrak{h}(\mathbb{R})$ = category of $U(\mathfrak{h}(\mathbb{R}))$-modules.

Seek parallel for locally finite reps of compact $L(\mathbb{R})$:

$$R(L(\mathbb{R})) = \text{conv alg of } \mathbb{R}\text{-valued } L(\mathbb{R})\text{-finite msres on } L(\mathbb{R})$$

$$\simeq_{\text{(Peter-Weyl)}} \left[\sum_{(\mu,E_\mu)\in\widehat{L(\mathbb{R})}} \mathrm{End}(E_\mu)\right](\mathbb{R})$$

$1 \notin R(L(\mathbb{R}))$ if $L(\mathbb{R})$ is infinite: convolution identity is delta function at $e \in L(\mathbb{R})$, not $L(\mathbb{R})$-finite.

$\alpha \subset \widehat{L(\mathbb{R})}$ finite, self-dual $\rightsquigarrow 1_\alpha =_{\text{def}} \sum_{\mu\in\alpha} \mathrm{Id}_\mu \in R(L(\mathbb{R}))$.

Elements 1_α are approximate identity:
$\forall r \in R(L(\mathbb{R}))\ \ \exists\alpha(r)$ finite so $1_\beta \cdot r = r \cdot 1_\beta = r$ if $\beta \supset \alpha(r)$.

$R(L(\mathbb{R}))$-module M is approximately unital if
$\forall m \in M\ \ \exists\alpha(m)$ finite so $1_\beta \cdot m = m$ if $\beta \supset \alpha(m)$.

Loc fin reps of $L(\mathbb{R})$ = approx unital $R(L(\mathbb{R}))$-modules.

R-mod $=_{\text{def}}$ category of approximately unital R-modules.

Representations and R-modules complexified

Category of reps of cplx $\mathfrak{h}$ = category of $U(\mathfrak{h})$-modules.

Parallel for locally finite reps of reductive algebraic L, $\mathcal{O}(L)$ = algebra of regular functions on L:

$$R(L) = L\text{-finite linear functionals} \subset \mathcal{O}(L)^*$$
$$\simeq \sum_{(\mu, E_\mu) \in \widehat{L}} \mathrm{End}(E_\mu)$$

Algebra structure $R(L) \otimes R(L) \to R(L)$ is dual to coproduct $\mathcal{O}(L) \to \mathcal{O}(L) \otimes \mathcal{O}(L)$ ($\leftrightsquigarrow$ group multiplication $L \times L \to L$).

$\alpha \subset \widehat{L}$ finite $\rightsquigarrow$ $1_\alpha =_{\mathrm{def}} \sum_{\mu \in \alpha} \mathrm{Id}_\mu$.

Alg reps of L = approx unital $R(L)$-modules.

Exercise: define $R(L)$ for any complex algebraic group.

Hecke algebras

Setting: $\mathfrak{h} \supset \mathfrak{l}$ cplx Lie algs, L reductive alg $\curvearrowright \mathfrak{h}$ by Lie alg automorphisms Ad.

Definition. The Hecke algebra $R(\mathfrak{h}, L)$ is

$$\begin{aligned} R(\mathfrak{h}, L) &= U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} R(L) \\ &\simeq [\text{conv alg of } L(\mathbb{R})\text{-finite } U(\mathfrak{h})\text{-valued msres on } L]/U(\mathfrak{l}) \end{aligned}$$

$R(\mathfrak{h}, L)$ inherits approx identity from subalgebra $R(L)$.

Proposition. $\mathcal{M}(\mathfrak{h}, L) = R(\mathfrak{h}, L)$-mod: $(\mathfrak{h}, L)$ modules are approximately unital modules for Hecke algebra $R(\mathfrak{h}, L)$.

Immediate corollary: $\mathcal{M}(\mathfrak{h}, L)$ has projective resolutions, so derived functors. . .

Group reps and Lie algebra reps

$G(\mathbb{R})$ reductive $\supset K(\mathbb{R})$ max cpt, $\mathfrak{Z}(\mathfrak{g})$ = center of $U(\mathfrak{g})$.

Definition. Representation (π, V) of $G(\mathbb{R})$ on complete locally convex V is *quasisimple* if $\pi^\infty(z) =$ scalar, all $z \in \mathfrak{Z}(\mathfrak{g})$. Algebra homomorphism $\chi_\pi\colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ is the *infinitesimal character of π*.

Theorem (Segal, Harish-Chandra)

1. Any irreducible $(\mathfrak{g}, K)$-module is quasisimple.
2. Any irreducible unitary rep of $G(\mathbb{R})$ is quasisimple.
3. Suppose V quasisimple rep of $G(\mathbb{R})$. Then $W \mapsto W^{K,\infty}$ is bijection between subrepresentations
$$(\text{closed } W \subset V) \leftrightarrow (W^{K,\infty} \subset V^{K,\infty}).$$
4. (irr quasisimple reps of $G(\mathbb{R})$) $\rightsquigarrow$ (irr $(\mathfrak{g}, K)$-modules), $W_\pi \rightsquigarrow W_\pi^{K,\infty}$ is surjective.

Idea of proof: $G(\mathbb{R})/K(\mathbb{R}) \simeq \mathfrak{s}_0$, vector space. Describe anything analytic on $G(\mathbb{R})$ by Taylor expansion along $K(\mathbb{R})$.

END OF LECTURE TWO

BEGINNING OF LECTURE THREE

So where are we now?

Harish-Chandra's notion of all irreducible representations π of $G(\mathbb{R})$: continuous irreducible on complete loc cvx top vec space W_π, quasisimple: center of $U(\mathfrak{g})$ acts by scalars on W_π^∞.

$\rightsquigarrow$ $W_\pi^{K,\infty}$ irr $(\mathfrak{g}, K)$-module of $K(\mathbb{R})$-finite smooth vecs.

π and π' infinitesimally equivalent if $W_\pi^{K,\infty} \simeq W_{\pi'}^{K,\infty}$.

$\widehat{G(\mathbb{R})} =_{\text{def}}$ infinitesimal equiv classes of irr quasisimple, so

$$\widehat{G(\mathbb{R})} \simeq_{\text{def}} \text{simple } R(\mathfrak{g}, K)\text{-modules.}$$

Notice right side depends only on `atlas` data: complex G, involutive automorphism θ, $K = G^\theta$.

atlas point of view for Cartans

Complex torus H is naturally $H \simeq \mathbb{C}^\times \otimes_{\mathbb{Z}} X_*(H)$,

Consequently H has unique compact real form

$$\sigma_c(z \otimes \xi) =_{\text{def}} \overline{z}^{-1} \otimes \xi \qquad (z \in \mathbb{C}^\times, \xi \in X_*(H)),$$

$$H(\mathbb{R}, \sigma_c) = S^1 \otimes_{\mathbb{Z}} X_*(H) \simeq (S^1)^{\text{rk}(H)}.$$

Reason is that $S^1 = \{z \in \mathbb{C}^\times \mid z = \overline{z}^{-1}\}$.

Proposition.

1. Real form σ of $H \leftrightsquigarrow$ inv aut $\theta \in \text{Aut}(H)$:
$$\theta(h) =_{\text{def}} \sigma(\sigma_c(h)).$$
2. Unique maximal compact subgroup of $H(\mathbb{R}, \sigma)$ is
$$T(\mathbb{R}) =_{\text{def}} H(\mathbb{R}, \sigma)^\theta = H(\mathbb{R}, \sigma) \cap H(\mathbb{R}, \sigma_c).$$
3. Complexification of $T(\mathbb{R})$ is $T(\mathbb{C}) = H^\theta$.

atlas point of view for structure

$H(\mathbb{R})$ real torus with Cartan involution $\theta \in \mathrm{Aut}(H)$.

Cartan decomp of $\mathfrak{h}(\mathbb{R})$ is into ± 1 eigenspaces of θ

$$\mathfrak{h}(\mathbb{R}) = \mathfrak{t}(\mathbb{R}) + \mathfrak{a}(\mathbb{R}), \quad \mathfrak{a}(\mathbb{R}) = \{X \in \mathfrak{h}(\mathbb{R}) \mid \theta X = -X\}$$

$$H(\mathbb{R}) \simeq T(\mathbb{R}) \exp(\mathfrak{a}(\mathbb{R})).$$

Because $H(\mathbb{R}, \sigma)$ is abelian, this is isomorphism of groups (but not of algebraic groups).

$\exp(\mathfrak{a}(\mathbb{R}))$ is just identity component of real algebraic group

$$A(\mathbb{R}) = \{h \in H(\mathbb{R}) \mid \theta(z) = z^{-1}\}.$$

Examples

θ	$H(\mathbb{R})$	$T(\mathbb{R})$	$\exp \mathfrak{a}(\mathbb{R})$
$\theta(z) = z^{-1}$	$\mathbb{R}^\times$	$\{\pm 1\}$	$\{e^t \mid t \in \mathbb{R}\}$
$\theta(z) = z$	S^1	$\{e^{is} \mid s \in \mathbb{R}\}$	$\{1\}$
$\theta(z, w) = (w, z)$	$\mathbb{C}^\times$	$\{(e^{is}, e^{is})\}$	$\{(e^t, e^{-t})\}$

All real algebraic tori are products of these three.

atlas point of view for characters

$$H(\mathbb{R}) \simeq T(\mathbb{R})\exp(\mathfrak{a}(\mathbb{R})).$$

Unitary characters of $T(\mathbb{R})$ are restrictions of algebraic characters of $T = H^\theta$, namely $X^*/(1-\theta)X^*$.

Lie algebra chars of $\mathfrak{h}(\mathbb{R})$ are complexified differentials of algebraic characters of H, namely $X^* \otimes_{\mathbb{Z}} \mathbb{C}$.

$$\begin{aligned}\widehat{H(\mathbb{R})} &= \text{one-dimensional } (\mathfrak{h}, T)\text{-modules}\\ &= \{(\gamma,\phi) \mid \gamma \in X^*(T),\ \phi \in \mathfrak{h}^*,\ \phi|_{\mathfrak{t}} = d\gamma\}\\ &= \{(\overline{\lambda},\phi) \mid \overline{\lambda} \in X^*/(1-\theta)X^*,\ \phi \in X^* \otimes_{\mathbb{Z}} \mathbb{C},\\ &\qquad (1+\theta)\lambda = (1+\theta)\phi\}\\ &= \{(\overline{\lambda},\overline{\nu}) \mid \overline{\nu} \in [X^*/(1+\theta)X^*] \otimes_{\mathbb{Z}} \mathbb{C}\}\end{aligned}$$

Last identification is $\phi = \frac{1+\theta}{2}\lambda + \frac{1-\theta}{2}\nu \in X^* \otimes_{\mathbb{Z}} \mathbb{C}$.

atlas considers only

$$\widehat{H(\mathbb{R})}_{\mathbb{Q}} =_{\text{def}} \{(\gamma,\phi) \mid \phi \in \mathfrak{h}^*_{\mathbb{Q}} =_{\text{def}} X^* \otimes_{\mathbb{Z}} \mathbb{Q}\}$$

Reason: all interesting rep theory happens in $\widehat{H(\mathbb{R})}_{\mathbb{Q}}$.

Lie algebra cohomology

$\mathfrak{n}$ Lie alg (e.g. nil radical of a parabolic in reductive $\mathfrak{g}$.)
Study *functor of $\mathfrak{n}$-invts* $V \mapsto V^{\mathfrak{n}}$ on reps of $\mathfrak{n}$.

Extra: $\mathfrak{n} \lhd \mathfrak{b}$, V rep of $\mathfrak{b} \implies V^{\mathfrak{n}}$ is rep of $\mathfrak{b}/\mathfrak{n}$.

Functor left exact; not right exact unless $\mathfrak{n} = 0$.

Definition 1. $H^p(\mathfrak{n}, \cdot)$ is the pth right derived functor of $\cdot^{\mathfrak{n}}$.

Definition 2. Suppose

$$0 \to V \to I_0 \to \cdots \to I_{p-1} \to I_p \to I_{p+1} \to \cdots$$

is an injective resolution of V as a $U(\mathfrak{n})$-module. Then

$$H^p(\mathfrak{n}, V) = \ker[I_p^{\mathfrak{n}} \to I_{p+1}^{\mathfrak{n}}] / \operatorname{im}[I_{p-1}^{\mathfrak{n}} \to I_p^{\mathfrak{n}}].$$

Definition 3. $H^p(\mathfrak{n}, V) =$ pth coh of cplx $\operatorname{Hom}(\bigwedge^p \mathfrak{n}, V)$.

Extra structure: $\mathfrak{n} \lhd \mathfrak{b} \implies H^p(\mathfrak{n}, V)$ is $\mathfrak{b}/\mathfrak{n}$-module.

$0 \to V_1 \to V_2 \to V_3 \to 0$ exact seq of $\mathfrak{n}$-modules $\implies$

$$\begin{array}{lll} 0 \longrightarrow H^0(\mathfrak{n}, V_1) \longrightarrow & H^0(\mathfrak{n}, V_2) \longrightarrow & H^0(\mathfrak{n}, V_3) \\ \longrightarrow H^1(\mathfrak{n}, V_1) \longrightarrow & H^1(\mathfrak{n}, V_2) \longrightarrow & H^1(\mathfrak{n}, V_3) \\ \vdots & \vdots & \vdots \\ \longrightarrow H^d(\mathfrak{n}, V_1) \longrightarrow & H^d(\mathfrak{n}, V_2) \longrightarrow & H^d(\mathfrak{n}, V_3) \longrightarrow 0 \end{array}$$

Casselman-Osborne theorem

$K(\mathbb{R}) \subset G(\mathbb{R})$ max compact in real reductive, θ Cartan involution $\rightsquigarrow$ pair $(\mathfrak{g}, K)$.

$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ Levi decomp of parabolic subalg; assume $\mathfrak{l} = \theta\mathfrak{l} = \bar{\mathfrak{l}}$. Get $L(\mathbb{R})$, Levi pair $(\mathfrak{l}, L \cap K)$.

Theorem Lie algebra cohomology is a cohomological family of functors $H^p(\mathfrak{u}, \cdot) \colon \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{l}, L \cap K)$. Each carries modules of finite length to modules of finite length.

"Finite length" close to "quasisimple." Proof of thm depends on analyzing $\mathfrak{Z}(\mathfrak{g})$...

$U(\mathfrak{g}) = U(\mathfrak{u}) \otimes U(\mathfrak{l}) \otimes U(\mathfrak{u}^-)$ gives linear projection

$$\tilde{\xi} \colon U(\mathfrak{g}) \to U(\mathfrak{l}); \quad \tilde{\xi} \colon U(\mathfrak{g})^{\mathfrak{z}(\mathfrak{l})} \to U(\mathfrak{l})^{\mathfrak{z}(\mathfrak{l})} \text{ alg hom.}$$

Theorem (Casselman-Osborne) If V is a $\mathfrak{g}$-module, then $\mathfrak{Z}(\mathfrak{g})$ acts on $H^p(\mathfrak{u}, V)$. This action is related to the $\mathfrak{l}$ action by $z \cdot \omega = \tilde{\xi}(z) \cdot \omega$.

Interlude: Chevalley isomorphism

Cplx reductive $\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$; $W = W(\mathfrak{g}, \mathfrak{h}) \curvearrowright \mathfrak{h},\ \mathfrak{h}^*$.

$\rho =$ half sum of pos roots $\in \mathfrak{h}^*$. Twisted action $*$ of W is

$$w * \lambda =_{\text{def}} w(\lambda + \rho) - \rho, \quad (w * p)(\lambda) =_{\text{def}} p(w^{-1} * \lambda)$$

$(\lambda \in \mathfrak{h}^*, p \in S(\mathfrak{h}))$.

Theorem (Chevalley). Algebra hom $\tilde{\xi}\colon \mathfrak{Z}(\mathfrak{g}) \to S(\mathfrak{h})$ from previous slide is injection with image equal to $S(\mathfrak{h})^{W,*}$, the invts of the twisted W action. Consequently maxl ideals in $\mathfrak{Z}(\mathfrak{g})$ are in one-to-one corr with twisted W orbits on $\mathfrak{h}^*$.

Here should introduce ρ-twisted version ξ of $\widetilde{\xi}$,

$$\xi\colon \mathfrak{Z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^W.$$

Corollary of Thm and Casselman-Osborne: if $\mathfrak{g}$-module V has infl char $\lambda \in \mathfrak{h}^*$, then $H^p(\mathfrak{u}, V)$ has finite filtration with each level of infl char $w * \lambda$, some $w \in W(\mathfrak{l}, \mathfrak{h}) \backslash W(\mathfrak{g}, \mathfrak{h})$.

END OF LECTURE THREE

BEGINNING OF LECTURE FOUR

Strategy for Langlands classification

Still aiming at Langlands classification:

$\widehat{G(\mathbb{R})} =$ irr rep of $G(\mathbb{R})$ / infinitesimal equivalence

$\leftrightsquigarrow$ irr $(\mathfrak{g}, K)$-module

$\leftrightsquigarrow$ character of real Cartan $\gamma \in \widehat{H(\mathbb{R})}$ $/G(\mathbb{R})$

$\leftrightsquigarrow$ one-diml $(\mathfrak{h}, T)$-module $(\overline{\lambda}, \overline{\nu})/(K(\mathbb{C}))$

IDEA: start with M irr $(\mathfrak{g}, K)$-module.

Find nice-for-M Borel subalg $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, $\theta\mathfrak{h} = \mathfrak{h}$; $\rightsquigarrow$ $T = H^\theta$.

Find nice cohomology class in $H^*(\mathfrak{n}, M)$; action of $(\mathfrak{h}, T)$ defines Langlands parameter.

How this works for $SL(2,\mathbb{R})$

$G(\mathbb{R}) = SL(2,\mathbb{R})$, $K \simeq \mathbb{C}^\times$, M irr $(\mathfrak{g}, K)$-module.

$\widehat{K} \simeq \mathbb{Z}$, so $M = \sum_{\mu\in\mathbb{Z}} M_\mu$. Recall basis (H, X, Y) for $\mathfrak{g}$:

$$X \cdot M_\mu \subset M_{\mu+2}, \quad Y \cdot M_\mu \subset M_{\mu-2}, \quad H \cdot v = \mu v \quad (v \in M_\mu).$$

Lowest K-type of M is smallest μ_0 such that $M_{\mu_0} \neq 0$.

Case DS+: $\mu_0 \geq 2$. In this case

1. $M_{\mu_0-2} = 0$, so $Y \cdot M_{\mu_0} = 0$, so $M_{\mu_0} \subset M^{\mathbb{C}Y}$
2. Define $\mathfrak{n}_Y = \mathbb{C}Y$; get T weight μ_0 in $H^0(\mathfrak{n}_Y, M)$.
3. M is (irreducible) $\mathfrak{b}_Y$-Verma of highest weight μ_0.
4. Langlands parameter is $(T, \overline{\lambda} = \mu_0 - 1, \overline{\nu} = 0)$.

Case DS-: $\mu_0 \leq -2$. In this case

1. $M_{\mu_0+2} = 0$, so $X \cdot M_{\mu_0} = 0$, so $M_{\mu_0} \subset M^{\mathbb{C}X}$
2. Define $\mathfrak{n}_X = \mathbb{C}X$; get T weight μ_0 in $H^0(\mathfrak{n}_X, M)$.
3. M is (irreducible) $\mathfrak{b}_X$-Verma of highest weight μ_0.
4. Langlands parameter is $(T, \overline{\lambda} = \mu_0 + 1, \overline{\nu} = 0)$.

Case PS: $\mu_0 = 0$ or ± 1 is something completely different...

$SL(2,\mathbb{R})$: lowest K-type 0 or ± 1

Suppose M irr $(\mathfrak{g}, K)$-module containing K-type 0 or ± 1.

(Almost always) $M \rightsquigarrow$ diagonal Cartan H_s: $\theta =$ inverse, $T_s = \{\pm I\}$, $\mathfrak{a}_s = \mathfrak{h}_s$.

Define $\mathfrak{b}_s = \mathfrak{n}_s + \mathfrak{a}_s =$ upper triangular Borel.

Define $\nu_s(M) =$ infinitesimal character of M in $\mathfrak{a}_s^*$, $\epsilon =$ parity of μ_0.

Langlands parameter: $(T_s, \overline{\lambda} = \epsilon + 1, \overline{\nu}_s = \nu_s)$,

Proposition.

1. M is a composition factor of the principal series representation $\rho^{\nu(M)+1,\epsilon(M)}$ defined in Lecture 2.
2. if $\nu_s < 0$, M is unique irr sub of $\rho^{\nu_s+1,\epsilon(M)}$, and $H_0(\mathfrak{n}_s, M)$ has T_s-weight $(\epsilon, \nu_s + 1)$.
3. If $\nu_s > 0$, M is unique irr quo of $\rho^{\nu_s+1,\epsilon(M)}$.

Case $\nu_s = 0$ causes some complications; ignore for now.

How this works for $G(\mathbb{R})$

$G \supset K = G^\theta$, M irr $(\mathfrak{g}, K)$-module.

Fix maximal torus $T_0 \subset K$; $T_f = G^{T_0} = \theta$-stable Cartan in G, fundamental Cartan.

$X^*(T_f) \twoheadrightarrow X^*(T_f^\theta)$

Fix Borel $\mathfrak{b}_K = \mathfrak{t} + \mathfrak{n}_K$, pos roots $\Delta^+(\mathfrak{k}, \mathfrak{t}) \subset X^*(T_f^\theta)$.

Write $\mathfrak{b}_K^{\mathrm{op}} = \mathfrak{t} + \mathfrak{n}_K^{\mathrm{op}}$, $2\rho_c = \sum_{\alpha \in \Delta^+(\mathfrak{k},\mathfrak{t})} \alpha$, $s = \dim(\mathfrak{n}_K)$.

Any irr $(\tau, E_\tau) \in \widehat{K} \rightsquigarrow$ highest weights $\{\mu_j \in X^*(T_f^\theta)\}$:

$$\mu_j \text{ appears in } H^0(\mathfrak{n}_K, E_\tau), \quad \mu_j + 2\rho_c \text{ appears in } H^s(\mathfrak{n}_K^{\mathrm{op}}, E_\tau)$$

Given highest weight μ of (τ, E_τ), choose θ-stable $\Delta^+(\mathfrak{g}, \mathfrak{t}_f)$ so that $\langle \mu + 2\rho_c, (1+\theta)\alpha^\vee \rangle \geq 0$ (all $\alpha \in \Delta^+$).

Define rough height of τ $\tilde{h}(\tau) = \sum_{\alpha \in \Delta^+} \langle \mu + 2\rho_c, \alpha^\vee \rangle$.

Lowest K-type of M is τ_0 minimizing $\tilde{h}(\tau_0)$ with $M_{\tau_0} \neq 0$.

Constructing cohomology

$T_f(\mathbb{R}) = T_f(\mathbb{R})^\theta \exp(\mathfrak{a}_f(\mathbb{R}))$ fundamental Cartan subgroup.

M irr $(\mathfrak{g}, K)$-module, τ_0 lowest K-type, $\mu_0 \in X^*(T_f^\theta)$ highest weight, $\mathfrak{b}_f^{\mathrm{op}} = \mathfrak{t}_f + \mathfrak{n}_K^{\mathrm{op}} + \mathfrak{n}_{\mathfrak{p}}^{\mathrm{op}}$ making $\mu_0 + 2\rho_c$ antidominant.

Theorem. Assume $\mu_0 + 2\rho_c - \rho$ regular antidominant.

1. $H^*(\mathfrak{n}, M_f^{\mathrm{op}})_{\mu_0+2\rho_c} \neq 0$ only in degree $s = \dim \mathfrak{n}_K^{\mathrm{op}}$.
2. $\dim H^s(\mathfrak{n}^{\mathrm{op}}, M)_{\mu_0+2\rho_c} = \dim M_{\tau_0}$.
3. $H^s(\mathfrak{n}_f^{\mathrm{op}}, M)_{\mu_0+2\rho_c}$ has at least one $\mathfrak{a}_f$-weight $\nu_f \in \mathfrak{a}_f^*$.

If $\mu_0 + 2\rho_c - \rho$ regular antidominant,

$$M \rightsquigarrow \gamma(M) = (T_f, \overline{\lambda} = \mu_0 + 2\rho_c - \rho, \overline{\nu} = \nu_f(M)).$$

In this case M is unique irreducible quotient of cohomologically induced module

$$I(\gamma) =_{\mathrm{def}} \mathcal{L}_s^{\mathfrak{b}_f^{\mathrm{op}}}(\mathbb{C}_{\mu+2\rho_c,\nu_f}).$$

Consequence: $I(\mu + 2\rho_c, \nu_f')$ algebraic in ν_f'.

Study by deformation in continuous parameters.

Moral of these stories

M irr $(\mathfrak{g}, K)$-module $\rightsquigarrow$ Langlands parameter $\gamma = (H, \overline{\lambda}, \overline{\nu})$.

$T =_{\text{def}} H^\theta \subset K$, $\mathfrak{a} = \mathfrak{h}^{-\theta}$.

1. Infl char of $M = \frac{(1+\theta)}{2}\overline{\lambda} + \frac{(1-\theta)}{2}\overline{\nu}$.
2. Lowest K-type(s) of $M \leftrightsquigarrow \overline{\lambda} \in \widehat{H^\theta}$
3. (Langlands original idea) growth of matrix coefficients of $M \leftrightsquigarrow \overline{\nu} \in \mathfrak{a}^*$

END OF LECTURE FOUR

BEGINNING OF LECTURE FIVE

Invariant Hermitian forms

(π, W_π) continuous rep of $G(\mathbb{R})$ on complete locally convex topological vector space W_π.

Definition. $G(\mathbb{R})$-invariant Hermitian form is continuous Hermitian pairing $\langle, \rangle_\pi$ on W_π satisfying

$$\langle \pi(g)w_1, \pi(g)w_2\rangle_\pi = \langle w_1, w_2\rangle_\pi \quad (g \in G(\mathbb{R})).$$

Definition. Invariant Hermitian form on $(\mathfrak{g}, K)$-module M is Hermitian pairing $\langle, \rangle_M$ on M satisfying

$$\langle Z \cdot m_1, m_2\rangle_M + \langle m_1, \sigma(Z)\cdot m_2\rangle_\pi = 0 \quad Z \in \mathfrak{g}(\mathbb{C})),$$

$$\langle k \cdot m_1, \sigma(k)\cdot m_2\rangle_M = \langle m_1, m_2\rangle_M \quad k \in K(\mathbb{C})).$$

Prop. $G(\mathbb{R})$-invt $\langle, \rangle_\pi \xrightarrow{\text{restrict}} (\mathfrak{g}, K)$-invt $\langle, \rangle_\pi^{K,\infty}$ on $W_\pi^{K,\infty}$. Conversely, $(\mathfrak{g}, K)$-invt $\langle, \rangle_M$ on finite length $M \xrightarrow{\text{extend}}$ $G(\mathbb{R})$-invt $\langle, \rangle_{M_{\min}}$ on Schmid minimal globalization $M_{\min}$.

Forms and dual spaces

V cplx vec space (or alg rep of K, or $(\mathfrak{g}, K)$-module...)

Hermitian dual of V

$$V^h = \{\xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v)\}$$

(V alg K-rep $\rightsquigarrow$ require ξ K-finite; V topolog. $\rightsquigarrow$ require ξ cont.)

Sesquilinear pairings between V and W are

$$\mathrm{Sesq}(V, W) = \{\langle , \rangle \colon V \times W \to \mathbb{C}, \text{lin in } V, \text{conj-lin in } W\}$$

$$\mathrm{Sesq}(V, W) \simeq \mathrm{Hom}(V, W^h), \quad \langle v, w\rangle_T = (Tv)(w).$$

Complex conjugation of forms is (conj linear) isom

$$\mathrm{Sesq}(V, W) \simeq \mathrm{Sesq}(W, V).$$

Corresponding (conj lin) isom is Hermitian transpose:

$$\mathrm{Hom}(V, W^h) \simeq \mathrm{Hom}(W, V^h), \quad (T^h w)(v) = \overline{(Tv)(w)}.$$

$$(TS)^h = S^h T^h, \qquad (zT)^h = \overline{z}(T^h).$$

Sesq form $\langle , \rangle_T$ on V ($\leftrightsquigarrow$ $T \in \mathrm{Hom}(V, V^h)$) Hermitian if

$$\langle v, v'\rangle_T = \overline{\langle v', v\rangle}_T \iff T^h = T.$$

David Vogan

Defining Herm dual repn(s)

(π, V) $(\mathfrak{g}, K)$-module; Recall Herm dual V^h of V.

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators.

Def REQUIRES twist by conj lin antiaut of $\mathfrak{g}$, conjugate linear group antiaut of K. We have one!

Define $(\mathfrak{g}, K)$-module π^h on V^h,

$$\pi^h(Z) \cdot \xi =_{\text{def}} [\pi(-\sigma_0(Z))]^h \cdot \xi \quad (Z \in \mathfrak{g},\ \xi \in V^h),$$

$$\pi^h(k) \cdot \xi =_{\text{def}} [\pi(\sigma_0(k^{-1}))]^h \cdot \xi \quad (k \in K,\ \xi \in V^h).$$

Variant: compact real form $\sigma_c = \theta\sigma_0 \rightsquigarrow \pi^{h,c}$,

$$\pi^{h,c}(Z) \cdot \xi =_{\text{def}} [\pi(-\sigma_c(Z))]^h \cdot \xi \quad (Z \in \mathfrak{g},\ \xi \in V^h),$$

$$\pi^{h,c}(k) \cdot \xi =_{\text{def}} [\pi(\sigma_c(k)^{-1})]^h \cdot \xi \quad (k \in K,\ \xi \in V^h).$$

c-Invariant Hermitian forms

$V = (\mathfrak{g}, K)$-module.

A c-invt sesq form on V is sesq pairing $\langle , \rangle^c$ such that

$$\langle Z \cdot v, w \rangle = \langle v, -\sigma_c(Z) \cdot w \rangle, \quad \langle k \cdot v, w \rangle = \langle v, \sigma_c(k^{-1}) \cdot w \rangle$$

$$(Z \in \mathfrak{g};\ k \in K;\ v, w \in V).$$

Proposition.

1. c-invt sesq form on V $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-map $T\colon V \to V^{h,c}$:
$$\langle v, w \rangle_T = (Tv)(w).$$
2. Form is Hermitian $\iff$ $T^h = T$.

Assume from now on V is irreducible.

3. $V \simeq V^{h,c} \iff \exists$ c-invt sesq $\iff \exists$ c-invt Herm
4. c-invt Herm form on V unique up to real scalar mult.

$T \to T^h \leftrightsquigarrow$ real form of cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^{h,c})$.

Deciding existence of c-invt Hermitian form amounts to computing the involution $\pi \mapsto \pi^{h,c}$ on $\widehat{G}$.

Same conclusion for invariant Hermitian forms.

Hermitian forms and unitary reps

π rep of G on complete loc cvx V_π, $(\pi^h V_\pi^h)$ (continuous) Hermitian dual representation.

Because infl equiv easier than topol equiv, $V_\pi \simeq V_\pi^{h,\tau} \not\Rightarrow$ continuous map $V_\pi \to V_\pi^h$. So invt forms may not exist on topological reps even if they exist on $(\mathfrak{g}, K)$-modules.

Theorem (Harish-Chandra). Passage to K-finite vectors defines bijection from the unitary dual $\widehat{G}_u$ onto equivalence classes of irreducible $(\mathfrak{g}, K)$ modules admitting a pos def invt Hermitian form.

Despite warning, get perfect alg param of $\widehat{G}_u$.

Knapp-Zuckerman computed involution $\pi \mapsto \pi^{h,c}$ on $\widehat{G}$.

So we know which irr $(\mathfrak{g}, K)$ modules admit invt forms.

Remaining task: compute signatures of these forms.

END OF LECTURE FIVE

BEGINNING OF LECTURE SIX

Big idea for computing signatures

Here's what theory gives for Hermitian reps...

Comprehensible/computable parameters $\gamma = (x, \overline{\lambda}, \nu)$.

Hermitian parameter $\gamma \rightsquigarrow$

1. Standard rep $I^{\text{quo}}(\gamma) \twoheadrightarrow J(\gamma)$ unique irr quotient.
2. deformation fam $\gamma_t = (x, \overline{\lambda}, t\nu)$ $(0 \le t \le 1)$
3. deformation fam of invt Herm forms $\langle , \rangle_t$ on $I^{\text{quo}}(\gamma_t)$.

Key properties:

1. $\text{Rad}(\langle , \rangle_t) = \ker(I^{\text{quo}}(\gamma_t) \twoheadrightarrow J(\gamma_t))$. Consequently
2. $\langle , \rangle_t$ descends to nondeg form on $J(\gamma_t)$.
3. $J(\gamma_0) = I(\gamma_0)$ tempered irr, so $\langle , \rangle_0$ pos definite.

Plan for computing signature of $\langle , \rangle_t$:

1. start with known pos def signature at $t = 0$
2. compute changes in signature at reducibility points t_i
3. $\rightsquigarrow$ formula for signature at $t = 1$.

Hermitian duals for $SL(2, \mathbb{R})$

Recall ρ^ν ($\nu \in \mathbb{C}$) family of reps of $SL(2, \mathbb{R})$ defined on

$W =$ even trig polys on $S^1 = \text{span}(w_m(\theta) = e^{im\theta}, m \in 2\mathbb{Z})$

Rotation by θ in $SO(2)$ acts on w_m by $e^{im\theta}$, Lie alg acts by

$$\rho^\nu(H)w_m = mw_m, \qquad \rho^{\nu,h}(H)w_m = mw_m,$$

$$\rho^\nu(X)w_m = \frac{1}{2}(m + \nu + 1)w_{m+2}, \qquad \rho^{\nu,h}(X)w_m = \frac{1}{2}(m - \overline{\nu} + 1)w_{m+2},$$

$$\rho^\nu(Y)w_m = \frac{1}{2}(-m + \nu + 1)w_{m-2} \qquad \rho^{\nu,h}(Y)w_m = \frac{1}{2}(-m - \overline{\nu} + 1)w_{m-2}.$$

If we identify $W \simeq W^h$ by pos def inner product

$$\left\langle \sum_r a_r w_r, \sum_s b_s w_s \right\rangle = \sum_p a_p \overline{b_p},$$

then Hermitian transpose of $T = (t_{ij})$ is $T^h = {}^t\overline{T} = (\overline{t_{ji}})$.

See that $(\rho^\nu)^h = \rho^{-\overline{\nu}}$. So ν imag $\implies \rho^\nu$ Herm; invt form $=$ pos def standard form, so $\nu \in i\mathbb{R} \implies \rho^\nu$ unitary. These are (tempered) unitary principal series.

There is more to say!

$SL(2,\mathbb{R})$-invariant forms $\langle , \rangle_\nu$

Calculated $(\rho^\nu)^h = \rho^{-\overline{\nu}}$. Know $\rho^\nu \approx \rho^{-\nu}$; so expect invariant forms $\langle , \rangle_\nu, \quad \nu \in \mathbb{R}$.

Recall basis (H, X, Y), $\qquad \sigma_0(H) = -H$, $\sigma_0(X) = Y$.

Condition for invariant form is

$$\langle \rho^\nu(Z)w, w'\rangle_\nu + \langle w, \rho^\nu(\sigma_0(Z))w'\rangle_\nu = 0. \qquad (INVT)$$

$\rho^\nu(H)w_m = mw_m$; plugging in (INVT) gives

$$\langle w_m, w_n\rangle_\nu = a_m(\nu)\delta_{mn} \qquad (a_m \in \mathbb{R}).$$

$$\rho^\nu(X)w_m = \frac{1}{2}(m+\nu+1)w_{m+2}, \quad \rho^\nu(Y)w_{m+2} = \frac{1}{2}(-m+\nu-1)w_m.$$

Plugging in (INVT) gives

$$(m+1+\nu)a_{m+2}(\nu) = (m+1-\nu)a_m(\nu) \qquad (a_m \in \mathbb{R}).$$

Proposition. For $\nu \geq 0$, ρ^ν has unique invt $\langle , \rangle_\nu$ char by $\langle w_0, w_0\rangle_\nu = 1$. Each $\langle w_{2m}, w_{2m}\rangle_\nu$ is rational in ν:

$$a_{\pm 2}(\nu) = \frac{1-\nu}{1+\nu}, \quad a_{\pm 4} = \frac{(1-\nu)(3-\nu)}{(1+\nu)(3+\nu)}, \ldots .$$

Signatures for $SL(2,\mathbb{R})$

Recall $W^\nu =$ even fns homog of deg $-\nu - 1$ on the plane.

Need "signature" of Herm form on this inf-diml space.

Harish-Chandra (or Fourier) idea:
use $K = SO(2)$ break into fin-diml subspaces

$$W^\nu_{2m} = \{f \in W^\nu \mid \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot f = e^{2im\theta} f\}.$$

$$W^\nu \supset \sum_m W^\nu_m, \qquad \text{(dense subspace)}$$

Decomp is orthogonal for any invariant Herm form.

Signature is $+$ or $-$ for each m. For $3 < |\nu| < 5$

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$
$\cdots$	$+$	$+$	$-$	$+$	$-$	$+$	$+$	$\cdots$

Deforming signatures for $SL(2, \mathbb{R})$

Here's how signatures of the reps $I(\nu)$ change with ν.

$\nu = 0$, $I(0)$ "$\subset$" $L^2(G)$: unitary, signature positive.

$0 < \nu < 1$, $I(\nu)$ irr: signature remains positive.

$\nu = 1$, form finite pos on $J(1)$ $\leftrightsquigarrow$ $SO(2)$ rep 0.

$\nu = 1$, form has zero, pos derivative on $I(1)/J(1)$.

$1 < \nu < 3$, across zero at $\nu = 1$, signature changes.

$\nu = 3$, form finite $- + -$ on $J(3)$.

$\nu = 3$, form has zero, neg derivative on $I(3)/J(3)$.

$3 < \nu < 5$, across zero at $\nu = 3$, signature changes. ETC.

Conclude: $J(\nu)$ unitary, $\nu \in [0, 1]$; nonunitary, $\nu \in [1, \infty)$.

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$	$SO(2)$ reps
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 0$
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$0 < \nu < 1$
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 1$
$\cdots$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$\cdots$	$1 < \nu < 3$
$\cdots$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$\cdots$	$\nu = 3$
$\cdots$	$+$	$+$	$-$	$+$	$-$	$+$	$+$	$\cdots$	$3 < \nu < 5$

Formulas for signatures

M $(\mathfrak{g}, K)$-module $\rightsquigarrow M = \sum_{\tau \in \widehat{K}} M_\tau \otimes E_\tau$, $m_M(\tau) = \dim M_\tau$.

FIX positive K-invariant form $\langle , \rangle_\tau$ on each K-irr (τ, E_τ) . . .

invariant Hermitian form $\langle , \rangle_M \rightsquigarrow$ Hermitian forms $\langle , \rangle_{M_\tau}$.

Define signature functions from $\widehat{K}$ to $\mathbb{N}$,

$$(p_M(\tau), q_M(\tau), z_M(\tau)) = \text{signature of } \langle , \rangle_{M_\tau}.$$

M is unitary $\iff$ function $q_M = 0$.

Recall from Annegret: lowest K-type is bijection

$$(\text{ parameters with } \nu = 0) =_{\text{def}} \mathcal{T}_{\mathbb{R}} \longrightarrow \widehat{K}$$

Notation refers to $\mathcal{T}$empered reps of $\mathbb{R}$eal infinitesimal character.

Theorem. Each function m_M, p_M, q_M, and z_M can be written uniquely as finite integer combination

$$p_M = \sum_{\gamma \in \mathcal{T}_{\mathbb{R}}} a(\gamma) m_{I(\gamma)}$$

of K-mult fns for tempered reps of real infl char.

Signature formulas for $SL(2,\mathbb{R})$

Here's what deformation says about signatures on $I(\nu)$.

$\nu = 0$, $I(0)$ unitary, signature positive:

$$p_{J(0)} = m_{I(0)}, \quad q_{J(0)} = 0, \quad z_{J(0)} = 0.$$

$0 < \nu < 1$, $I(\nu)$ irr: signature remains positive:

$$p_{J(\nu)} = m_{I(0)}, \quad q_{J(\nu)} = 0, \quad z_{J(\nu)} = 0.$$

$\nu = 1$: form has simple zero on radical = $DS_+(1) + DS_-(1)$:

$$p_{J(1)} = m_{I(0)} - m_{DS_+(1)} - m_{DS_-(1)}, \quad q_{J(1)} = 0,$$

$$z_{J(1)} = m_{DS_+(1)} + m_{DS_-(1)}.$$

$1 < \nu < 3$, across zero at $\nu = 1$, signature changes:

$$p_{J(\nu)} = m_{I(0)} - m_{DS_+(1)} - m_{DS_-(1)}$$

$$q_{J(\nu)} = m_{DS_+(1)} + m_{DS_-(1)}, \quad z_{J(\nu)} = 0.$$

$\nu = 3$: form form has simple zero on radical = $DS_+(3) + DS_-(3)$:

$$p_{J(\nu)} = m_{J(0)} - m_{DS_+(1)} - m_{DS_-(1)},$$

$$q_{J(3)} = m_{DS_+(1)} + m_{DS_-(1)} - m_{DS_+(3)} - m_{DS_-(3)},$$

$$z_{J(3)} = m_{DS_+(3)} + m_{DS_-(3)}.$$

Invariant forms on standard reps

Recall multiplicity formula

$$I(x) = \sum_{y \le x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

for standard $(\mathfrak{g}, K)$-mod $I(x)$.

Want parallel formulas for invt Hermitian forms. Need forms on standard modules.

Form on irr $J(x) \xrightarrow{\text{deformation}}$ Jantzen filt $I^k(x)$ on std, nondeg forms $\langle , \rangle^k$ on I^k/I^{k+1}.

Details (proved by Beilinson-Bernstein):

$$I(x) = I^0 \supset I^1 \supset I^2 \supset \cdots, \qquad I^0/I^1 = J(x)$$

$$I^k/I^{k+1} \text{ completely reducible}$$

$$[J(y) \colon I^k/I^{k+1}] = \text{coeff of } q^{(\ell(x)-\ell(y)-k)/2} \text{ in KL poly } Q_{y,x}$$

Hence $\langle , \rangle_{I(x)} \stackrel{\text{def}}{=} \sum_k \langle , \rangle^k$, nondeg form on gr $I(x)$.

Restricts to original form on irr $J(x)$.

Virtual Hermitian forms

$$\mathbb{Z} = \text{Groth group of vec spaces.}$$

These are mults of irr reps in virtual reps.

$$\mathbb{Z}[\widehat{G(\mathbb{R})}] = \text{Groth grp of finite length reps.}$$

For invariant forms...

$$\mathbb{W} = \mathbb{Z} \oplus \mathbb{Z} = \begin{array}{c}\text{Grothendieck group of}\\ \text{finite-dimensional forms.}\end{array}$$

Ring structure

$$(p, q)(p', q') = (pp' + qq', pq' + q'p).$$

Mult of irr-with-forms in virtual-with-forms is in $\mathbb{W}$:

$\mathbb{W}[\widehat{G(\mathbb{R})}_h] \approx$ Groth grp of fin lgth reps with invt forms.

Problem: invt form $\langle , \rangle_J$ may not be preferable to

What's a Jantzen filtration?

V cplx, $\langle,\rangle_t$ Herm forms analytic in t, generically nondeg.

$$V = V^0(t) \supset V^1(t) = \mathrm{Rad}(\langle,\rangle_t), \quad J(t) = V^0(t)/V^1(t)$$

$$(p^0(t), q^0(t)) = \text{signature of } \langle,\rangle_t \text{ on } J(t).$$

Question: how does $(p^0(t), q^0(t))$ change with t?

First answer: locally constant on open set $V^1(t) = 0$.

Refine answer. . . define form on $V^1(t)$

$$\langle v, w\rangle^1(t) = \lim_{s\to t} \frac{1}{t-s} < v, w >_s, \qquad V_2(t) = \mathrm{Rad}(\langle,\rangle^1(t))$$

$$(p^1(t), q^1(t)) = \text{signature of } \langle,\rangle^1(t).$$

Continuing gives Jantzen filtration

$$V = V^0(t) \supset V^1(t) \supset V^2(t) \cdots \supset V^{m+1}(t) = 0$$

From $t-\epsilon$ to $t+\epsilon$, signature changes on odd levels:

$$p(t+\epsilon) = p(t-\epsilon) + \sum[-p^{2k+1}(t) + q^{2k+1}(t)].$$

Hermitian KL polynomials: multiplicities

Fix invt Hermitian form $\langle , \rangle_{J(x)}$ on each irr having one; recall Jantzen form $\langle , \rangle^n$ on $I(x)^n/I(x)^{n+1}$.

MODULO problem of irrs with no invt form, write

$$(I^n/I^{n+1}, \langle , \rangle^n) = \sum_{y \leq x} w_{y,x}(n)(J(y), \langle , \rangle_{J(y)}),$$

coeffs $w(n) = (p(n), q(n)) \in \mathbb{W}$; summand means

$$p(n)(J(y), \langle , \rangle_{J(y)}) \oplus q(n)(J(y), -\langle , \rangle_{J(y)})$$

Define Hermitian KL polynomials

$$Q^h_{y,x} = \sum_n w_{y,x}(n) q^{(l(x)-l(y)-n)/2} \in \mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q = 1 \leftrightarrow$ form $\langle , \rangle_{I(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W} \to \mathbb{Z} \leftrightarrow$ KL poly $Q_{y,x}$.

Hermitian KL polynomials: characters

Matrix $Q^h_{y,x}$ is upper tri, 1s on diag: INVERTIBLE.

$$P^h_{x,y} \overset{\text{def}}{=} (-1)^{l(x)-l(y)}((x,y) \text{ entry of inverse}) \in \mathbb{W}[q].$$

Definition of $Q^h_{x,y}$ says

$$(\text{gr } I(x), \langle , \rangle_{I(x)}) = \sum_{y \le x} Q^h_{x,y}(1)(J(y), \langle , \rangle_{J(y)});$$

inverting this gives

$$(J(x), \langle , \rangle) = \sum_{y \le x} (-1)^{l(x)-l(y)} P^h_{x,y}(1)(\text{gr } I(y), \langle , \rangle).$$

Next question: how do you compute $P^h_{x,y}$?

Herm KL polys for σ_c

σ_c = cplx conj for cpt form of G, $\sigma_c(K) = K$.

Plan: study σ_c-invt forms, relate to σ_0-invt forms.

Proposition

Suppose $J(x)$ irr $(\mathfrak{g}, K)$-module, real infl char. Then $J(x)$ has σ_c-invt Herm form $\langle , \rangle^c_{J(x)}$, characterized by

$\langle , \rangle^c_{J(x)}$ is pos def on the lowest K-types of $J(x)$.

Proposition $\implies$ Herm KL polys $Q^c_{x,y}$, $P^c_{x,y}$ well-def.

Coeffs in $\mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}$; $s = (0,1) \leftrightsquigarrow$ one-diml neg def form.

ALMOST: $Q^c_{x,y}(q) = s^{\frac{\ell_o(x)-\ell_o(y)}{2}} Q_{x,y}(qs), \quad P^c_{x,y}(q) = s^{\frac{\ell_o(x)-\ell_o(y)}{2}} P_{x,y}(qs).$

Equiv: if $J(y)$ occurs at level k of Jantzen filt of $I(x)$, then Jantzen form is $(-1)^{(l(x)-l(y)-k)/2}$ times $\langle , \rangle_{J(y)}$.

ALMOST is false...but not seriously so. Need an extra power of s on the right side.

Orientation number

ALMOST $\leftrightarrow$ KL polys $\leftrightarrow$ *integral* roots.

Simple form of ALMOST *implies* Jantzen-Zuckerman translation across non-integral root walls preserves signatures of (σ_c-invariant) Hermitian forms.

It ain't necessarily so.

$SL(2, \mathbb{R})$: translating spherical principal series from (real non-integral positive) ν to (negative) $\nu - 2m$ changes sign of form iff $\nu \in (0, 1) + 2\mathbb{Z}$.

Orientation number $\ell_o(x)$ is

1. # pairs $(\alpha, -\theta(\alpha))$ cplx nonint, pos on x; PLUS
2. # real β s.t. $\langle x, \beta^\vee \rangle \in (0, 1) + \epsilon(\beta, x) + 2\mathbb{N}$.

$\epsilon(\beta, x) = 0$ spherical, 1 non-spherical.

Deforming to $\nu = 0$

Have computable ALMOST formula (omitting ℓ_o)

$$(J(x), \langle , \rangle^c_{J(x)}) = \sum_{y \le x} (-1)^{l(x)-l(y)} P^c_{x,y}(s)(\mathrm{gr}\, I(y), \langle , \rangle^c_{I(y)})$$

for σ^c-invt forms in terms of forms on stds, same inf char.

Polys $P^c_{x,y}$ are KL polys, computed by `atlas` software.

Std rep $I = I(\nu)$ deps on cont param ν. Put $I(t) = I(t\nu), \; t \ge 0$.

Apply Jantzen formalism to deform t to 0...

$$\langle , \rangle^c_J = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle^c_{I'(0)} \qquad (v_{J,I'} \in \mathbb{W}).$$

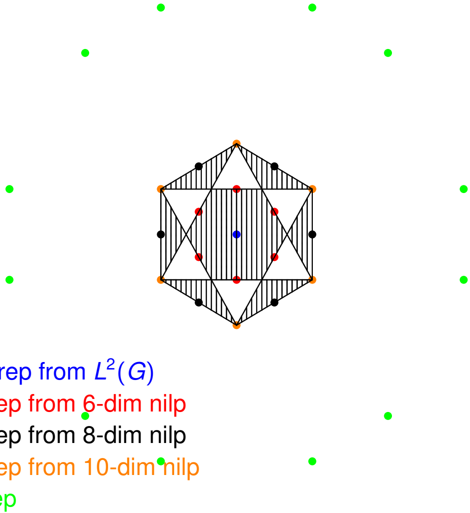
More rep theory gives formula for $G(\mathbb{R})$-invt forms:

$$\langle , \rangle^0_J = \sum_{I'(0) \text{ std at } \nu' = 0} s^{\epsilon(I')} v_{J,I'} \langle , \rangle^0_{I'(0)}.$$

$I'(0)$ unitary, so J unitary $\iff$ all coeffs are $(p, 0) \in \mathbb{W}$.

Example of $G_2(\mathbb{R})$

Real parameters for spherical unitary reps of $G_2(\mathbb{R})$



From σ_c to σ_0

Cplx conjs σ_c (compact form) and σ_0 (our real form) differ by Cartan involution θ: $\sigma_0 = \theta \circ \sigma_c$.

Irr $(\mathfrak{g}, K)$-mod $J \rightsquigarrow J^\theta$ (same space, rep twisted by θ).

Proposition

J admits σ_0-invt Herm form if and only if $J^\theta \simeq J$. If $T_0 \colon J \xrightarrow{\sim} J^\theta$, and $T_0^2 = \mathrm{Id}$, then

$$\langle v, w\rangle_J^0 = \langle v, T_0 w\rangle_J^c.$$

$T \colon J \xrightarrow{\sim} J^\theta \Rightarrow T^2 = z \in \mathbb{C} \Rightarrow T_0 = z^{-1/2} T \rightsquigarrow \sigma$-invt Herm form.

To convert formulas for σ_c invt forms $\rightsquigarrow$ formulas for σ_0-invt forms need intertwining ops $T_J \colon J \xrightarrow{\sim} J^\theta$, consistent with decomp of std reps.

Equal rank case

rk K = rk $G \Rightarrow$ Cartan inv inner: $\exists \tau \in K$, $\mathrm{Ad}(\tau) = \theta$.

$\theta^2 = 1 \Rightarrow \tau^2 = \zeta \in Z(G) \cap K$.

Study reps π with $\pi(\zeta) = z$. Fix square root $z^{1/2}$.

If ζ acts by z on V, and $\langle,\rangle_V^c$ is σ_c-invt form, then $\langle v, w\rangle_V^0 \overset{\text{def}}{=} \langle v, z^{-1/2}\tau \cdot w\rangle_V^c$ is σ_0-invt form.

$$\langle,\rangle_J^c = \sum_{I'(0) \text{ std at } \nu'=0} v_{J,I'} \langle,\rangle_{I'(0)}^c \qquad (v_{J,I'} \in \mathbb{W}).$$

translates to

$$\langle,\rangle_J^0 = \sum_{I'(0) \text{ std at } \nu'=0} v_{J,I'} \langle,\rangle_{I'(0)}^0 \qquad (v_{J,I'} \in \mathbb{W}).$$

I' has LKT $\mu' \Rightarrow \langle,\rangle_{I'(0)}^0$ definite, sign $z^{-1/2}\mu'(\tau)$.

J unitary $\iff$ each summand on right pos def.

Computability of $v_{J,I'}$ needs conjecture about $P_{x,y}^{\sigma_c}$.

General case

Fix "distinguished involution" δ_0 of G inner to θ

Define extended group $G^\Gamma = G \rtimes \{1, \delta_0\}$.

Can arrange $\theta = \mathrm{Ad}(\tau\delta_0)$, some $\tau \in K$.

Define $K^\Gamma = \mathrm{Cent}_{G^\Gamma}(\tau\delta_0) = K \rtimes \{1, \delta_0\}$.

Study $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-mods V with $D_0 \colon V \xrightarrow{\sim} V^{\delta_0}$, $D_0^2 = \mathrm{Id}$.

Beilinson-Bernstein localization: $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ action of δ_0 on K-eqvt perverse sheaves on G/B.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate σ_c-invt forms to σ_0 invt forms

$$\langle v, w \rangle_V^0 \overset{\text{def}}{=} \langle v, z^{-1/2}\tau\delta_0 \cdot w \rangle_V^c$$

on $(\mathfrak{g}, K^\Gamma)$-mods as in equal rank case.