The Atlas point of view 4

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Workshop on the Atlas of Lie Groups and Representations
University of Utah, Salt Lake City
July 10 - 21, 2017

Parameters

Recall from last time ...

An Atlas parameter is $p = (x, \lambda, \nu)$ where

x is an element of the KGB set for G

$\lambda \in X^* + \rho$

$\nu \in X^*_{\mathbb{Q}}$

λ represents a coset in $(X^* + \rho)/(1 - \theta_x)X^*$

ν represents a coset in $X^*_{\mathbb{Q}}/(1 + \theta_x)X^*_{\mathbb{Q}}$.
The software always replaces ν with a representative in $(X^*_{\mathbb{Q}})^{-\theta_x}$ (it isn't necessary for the user to do this).

The infinitesimal character is (the Weyl group orbit of)

$$\frac{1}{2}(1+\theta_x)\lambda + \frac{1}{2}(1-\theta_x)\nu.$$

p is standard if: $\langle\gamma,\alpha^\vee\rangle \ge 0$ for all $\alpha\in\Delta_i^+$.

p is final if: $\langle\nu,\alpha\rangle = 0 \Rightarrow \langle\gamma+\rho_r,\alpha^\vee\rangle\in 2\mathbb{Z}$ $\alpha\in\Delta_r^+$.

p is non-zero if $\langle\gamma,\alpha^\vee\rangle > 0$ for all compact simple roots of Δ_i.

There is a notion of equivalence of parameters (see the *Equivalence of parameters* notes on the conference web site).

Part of equivalence: α simple x-complex:

$$p(s_\alpha x, s_\alpha\lambda, s_\alpha\nu) \sim p(x,\lambda,\nu)$$

Exercise: for any `x`, `p=(x,rho,rho)` is a (standard, final, nonzero) parameter with infinitesimal character ρ.

Example: limits of discrete series of $SL(2,\mathbb{R})$

The final condition essentially comes down to this example in $SL(2,\mathbb{R})$:

`p=(x_2,[1],[0])` is final (spherical principal series at $\nu=0$)

`p=(x_2,[1],[0])` is not final (non-spherical principal series at $\nu=0$)

In fact the non-spherical principal series at $\nu=0$ is the sum of two limits of discrete series)

Example: $SL(2,\mathbb{R})$

```
atlas> set G=SL(2,R)
Variable G: RealForm
atlas> set p=parameter(KGB(G,2),[1],[0])
Variable p: Param
atlas> p
Value: final parameter(x=2,lambda=[1]/1,nu=[0]/1)
atlas> infinitesimal_character (p)
Value: [ 0 ]/1
atlas> highest_weight(LKT(p))
Value: (KGB element #0,[ 0 ])
atlas>
atlas> set q=parameter(KGB(G,2),[0],[0])
Variable q: Param
atlas> q
Value: non-final parameter(x=2,lambda=[2]/1,nu=[0]/1)
atlas> for mu in LKTs(q) do highest_weight(mu,KGB(G,1))
Value: [(KGB element #1,[ 1 ]),(KGB element #1,[ -1 ])]
```

The Atlas version of the Langlands classification

Associated to a standard, final, non-zero parameter $p = (x, \lambda, \nu)$ is a standard module $I(p)$, constructed either by real parabolic or cohomological induction.

Theorem Suppose p is a standard, final, non-zero parameter. Then $I(p)$ has a unique irreducible quotient $J(p)$, and the map $p \mapsto J(p)$ is a bijection between the standard, final, non-zero parameters and the irreducible admissible representations of $G(\mathbb{R})$.

More precisely: given x_0, have a bijection to the set of irreducible $(\mathfrak{g}, K_{x_0})$-modules.

Digression: $\widehat{K}$

$\{$standard, final, non-zero parameters $p = (x, \lambda, \nu)\} \longleftrightarrow \widehat{G(\mathbb{R})}$

Also:

Theorem

$\{$standard, final, non-zero parameters $(x, \lambda, 0)\} \longleftrightarrow \widehat{K}$

via the map $p = (x, \lambda, 0) \rightarrow$ (unique) lowest K-type of $I(p)$

The Kazhdan-Lusztig-Vogan picture

Fix an infinitesimal character γ. Very often we take $\gamma = \rho$, the infinitesimal character of the trivial representation.

We work entirely in the Grothendieck group, in which two finite length modules are equivalent if they have the same irreducible composition factors.

Let $\mathcal{M}_\gamma$ be the Grothendieck group of representations with infinitesimal character γ. Thus: $\mathcal{M}_\gamma$ is the formal $\mathbb{Z}$-span of the (finite set of) irreducible representations with this infinitesimal character.

So an element of $\mathcal{M}_\gamma$ is a formal sum

$$\sum_{i=1}^{n} a_i J_i \quad a_i \in \mathbb{Z}, J_i \text{ irreducible.}$$

The Kazhdan-Lusztig-Vogan picture

Any standard module I has a finite composition series, i.e.

$$I = \sum_i a_i J_i$$

for some $a_i \in \mathbb{Z}, J_i$ irreducible.
(Equality in the Grothendieck group).

Fact: the set of standard modules of infinitesimal character γ also are a basis of $\mathcal{M}_\gamma$. With the appropriate ordering the change of basis matrices are upper triangular, with ones on the diagonal.

Basic idea: the standard modules I are easy to understand (like Verma modules in category $\mathcal{O}$). For example: the restriction of I to K is computable.

The irreducible modules J (like irreducible highest weight modules in category $\mathcal{O}$) are what we are interested in. For example, it is hard to compute the restriction of J to K.

The formula

$$I = \sum a_i J_i \quad (a_i \in \mathbb{Z} \geq 0)$$

expressing the standard module I as a sum of irreducible modules (in the Grothendieck group) is called the composition series of I.

An expression

$$J = \sum b_i I_i \quad (b_i \in \mathbb{Z})$$

(a linear combination of standard modules) is called the character formula of J.

It is possible to compute the distribution character of the I_i, so the second expression give a formula for the character of J.

Calculating the restriction of J to K:
write $J = \sum_i a_i I_i$, then

$$J|_K = \sum_i a_i I_i|_K$$

Vogan's lectures: generalize this to

$$(J, \langle\,\rangle_J) = \sum_i z_i (I_i, \langle\,\rangle_i)$$

Example: $SL(2, \mathbb{R})$

$\gamma = \rho$

`p_0=(x_0,[1],[0]` (holomorphic discrete series)

`p_1=(x_1,[1],[0]` (anti-holomorphic discrete series)

`p_2=(x_2,[1],[1]` (spherical principal series)

`p_3=(x_2,[2],[1]` (non-spherical principal series)

$$
\begin{aligned}
I(p_0) &= J(p_0)\\
I(p_1) &= J(p_1)\\
I(p_2) &= J(p_2) + J(p_0) + J(p_1)\\
I(p_3) &= J(p_3)
\end{aligned}
$$

$$
\begin{aligned}
J(p_0) &= I(p_0)\\
J(p_1) &= I(p_1)\\
J(p_2) &= I(p_2) - I(p_0) - I(p_1)\\
J(p_3) &= I(p_3)
\end{aligned}
$$

Example: $SL(2, \mathbb{R})$

Change of basis matrices:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

This demonstrates that $\mathcal{M}_\lambda$ is the direct sum of blocks. Each block has two bases (standard and irreducible modules).

The Kazhdan-Lusztig-Vogan polynomials compute these matrices.

Example: the trivial representation

Let $\mathbb{C}$ be the trivial representation of G. Zuckerman computed its character formula.

In Atlas terms:

(For simplicity assume $\rho \in X^*$).

Theorem Suppose $p = (x, \lambda, \nu)$ is a parameter with infinitesimal character ρ. Associated to p is a character χ_p of a real form $H(\mathbb{R})$ of the fixed Cartan subgroup (with Cartan involution θ_x). Then χ_p occurs in the character formula for the trivial representation if and only if χ_p is the restriction of an algebraic character of $H(\mathbb{R})$ (with differential ρ). This term has coeffient $(-1)^{\text{length}}$.

Example: the trivial representation

These are precisely the parameters for which the corresponding local system (in the $\mathcal{D}$-module picture) is trivial.

In `atlas` terms:

$$\mathbb{C} = \sum_x (-1)^{\text{length}(x)} \mathtt{I}(\mathtt{x}, \mathtt{rho}, \mathtt{rho})$$

Example: $Sp(4, \mathbb{R})$

```
atlas> print_block(trivial(Sp(4,R)))
Parameter defines element 10 of the following block:
 0: 0 [i1,i1] (x= 0,lambda=[2,1], nu= [0,0]/1) e
 1: 0 [i1,i1] (x= 1,lambda=[2,1], nu= [0,0]/1) e
 2: 0 [ic,i1] (x= 2,lambda=[2,1], nu= [0,0]/1) e
 3: 0 [ic,i1] (x= 3,lambda=[2,1], nu= [0,0]/1) e
 4: 1 [r1,C+] (x= 4,lambda=[2,1], nu= [1,-1]/2) 1^e
 5: 1 [C+,r1] (x= 5,lambda=[2,1], nu= [0,1]/1) 2^e
 6: 1 [C+,r1] (x= 6,lambda=[2,1], nu= [0,1]/1) 2^e
 7: 2 [C-,i1] (x= 7,lambda=[2,1], nu= [2,0]/1) 1x2^e
 8: 2 [C-,i1] (x= 8,lambda=[2,1], nu= [2,0]/1) 1x2^e
 9: 2 [i2,C-] (x= 9,lambda=[2,1], nu= [3,3]/2) 2x1^e
10: 3 [r2,r1] (x=10,lambda=[2,1], nu= [2,1]/1) 1^2x1^e
11: 3 [r2,rn] (x=10,lambda=[3,2], nu= [2,1]/1) 1^2x1^e
```

Example: $Sp(4, \mathbb{R})$

```
print_character_formula(trivial(G))
1*final parameter(x=10,lambda=[2,1]/1,nu=[2,1]/1)
-1*final parameter(x=9,lambda=[2,1]/1,nu=[3,3]/2)
-1*final parameter(x=8,lambda=[2,1]/1,nu=[2,0]/1)
-1*final parameter(x=7,lambda=[2,1]/1,nu=[2,0]/1)
1*final parameter(x=6,lambda=[2,1]/1,nu=[0,1]/1)
1*final parameter(x=5,lambda=[2,1]/1,nu=[0,1]/1)
1*final parameter(x=4,lambda=[2,1]/1,nu=[1,-1]/2)
-1*final parameter(x=3,lambda=[2,1]/1,nu=[0,0]/1)
-1*final parameter(x=2,lambda=[2,1]/1,nu=[0,0]/1)
-1*final parameter(x=1,lambda=[2,1]/1,nu=[0,0]/1)
-1*final parameter(x=0,lambda=[2,1]/1,nu=[0,0]/1)
```

Example: F_4

Suppose G is the split group F_4. Here is a subset of the block of the trivial representation:

```
316: 13 (x=226,lam_rho= [0,0,0,0], nu= [2,3,0,3]/2)
317: 13 (x=226,lam_rho= [0,0,0,-1], nu= [2,3,0,3]/2)
318: 13 (x=226,lam_rho= [1,0,0,0], nu= [2,3,0,3]/2)
319: 13 (x=226,lam_rho= [1,0,0,-1], nu= [2,3,0,3]/2)
```

These parameters all have the same x, which means that they correspond to different local systems on the same K-orbit, or equivalently, characters of $H(\mathbb{R})$ with the same differential.

Here is the beginning of the character formula for $\mathbb{C}$ (note: `rho=[1,1,1,1]`):

```
(x=228,lambda=[1,1,1,1]/1,nu=[1,1,1,1]/1)
-(x=227,lambda=[1,1,1,1]/1,nu=[2,2,3,0]/2)
-(x=226,lambda=[1,1,1,1]/1,nu=[2,3,0,3]/2)
-(x=225,lambda=[1,1,1,1]/1,nu=[3,0,4,2]/2)
-(x=224,lambda=[1,1,1,1]/1,nu=[0,3,2,2]/2)
```

Composition series and character formulas in atlas

`print_composition_series(p)` and
`print_character_formula(p)`

```
atlas> print_composition_series (trivial(G))
1*final parameter(x=2,lambda=[1]/1,nu=[1]/1)
1*final parameter(x=1,lambda=[1]/1,nu=[0]/1)
1*final parameter(x=0,lambda=[1]/1,nu=[0]/1)
```

```
atlas> print_character_formula (trivial(G))
1*final parameter(x=2,lambda=[1]/1,nu=[1]/1)
-1*final parameter(x=1,lambda=[1]/1,nu=[0]/1)
-1*final parameter(x=0,lambda=[1]/1,nu=[0]/1)
```

Modules notation

```
atlas> set G=SU(2,1)
Variable G: RealForm
atlas> set p=trivial(G)
Variable p: Param
atlas> set I=I(p)
Variable I: (Param,string)
atlas> set J=J(p)
Variable J: (Param,string)
atlas> show(composition_series (I))
1*J(x=5,lambda=[2/1,1/1],nu=[2/1,1/1])
1*J(x=4,lambda=[2/1,1/1],nu=[1/2,-1/2])
1*J(x=3,lambda=[2/1,1/1],nu=[1/2,1/1])
1*J(x=0,lambda=[2/1,1/1],nu=[0/1,0/1])
atlas> show(character_formula (J))
1*I(x=5,lambda=[2/1,1/1],nu=[2/1,1/1])
-1*I(x=4,lambda=[2/1,1/1],nu=[1/2,-1/2])
-1*I(x=3,lambda=[2/1,1/1],nu=[1/2,1/1])
1*I(x=2,lambda=[2/1,1/1],nu=[0/1,0/1])
1*I(x=1,lambda=[2/1,1/1],nu=[0/1,0/1])
1*I(x=0,lambda=[2/1,1/1],nu=[0/1,0/1])
```