The Atlas point of view 3

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Inner classes

$\mathrm{Aut}(G)$, $\mathrm{Int}(G)$ (inner automorphisms)$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$

Note: If G is semisimple, $\mathrm{Out}(G)$ is a subgroup of the automorphism group of the Dynkin diagram.

Fact: the exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

canonically splits (up to inner automorphism).

So: an inner class is given by $\delta \in \mathrm{Aut}(G)_2$.

Without loss of generality: $\delta(H) = H, \delta(B) = B$.

The compact inner class is $\delta = 1$.

Fact: θ is in the compact inner class if and only if $G(\mathbb{R})$ has a compact Cartan subgroup.

Extended Group

Definition: ${}^{\delta}G = G \rtimes \langle\delta\rangle$

${}^{\delta}G = G \cup G\delta, \delta g \delta^{-1} = \delta(g), \delta^2 = 1$

If $\delta = 1$ then ${}^{\delta}G = G \times \mathbb{Z}/2\mathbb{Z}$ and we can ignore the extension.

$K\backslash G/B$ in general:

$x_0 \in {}^{\delta}G - G$, $x_0^2 \in Z(G) \to \theta_{x_0}, K$

$\mathcal{X}[x_0] = \{g \in \mathrm{Norm}_{G\delta}(H)\}/H$

Theorem

$$\boxed{\mathcal{X}[x_0] \leftrightarrow K\backslash G/B}$$

Inner classes

Example: $G = SL(n)$, $\mathrm{Out}(G) = \mathbb{Z}/2Z$ ($g \to {}^tg^{-1}$)

Inner class of 1: $SU(p,q)$, $\theta = \mathrm{int}(\mathrm{diag}(I_p, -I_q))$.

Inner class of $g \to {}^tg^{-1}$: $SL(n,\mathbb{R})$, $\theta(g) = {}^tg^{-1}$.

Any others?...
$SL(n/2,\mathbb{H})$ ($K = Sp(n)$)

Exercise: which inner class is this in?

Example: $SO(2n,\mathbb{C})$

Inner class of 1: $SO(p,q)$ (p,q even), also $SO^*(n) = Sp(n,\mathbb{H})$

Other inner class: $SO(p,q)$ (p,q odd)

Parameters

Last time: $G, \sigma, \theta, K = G^\theta \to \mathcal{X} \leftrightarrow K\backslash G/B$.

This is *most* of the hard work in computing the Langlands classification.

Recall from Vogan's lecture: the irreducible representations of $G(\mathbb{R})$ are (approximately) parametrized by

$$\{(H(\mathbb{R}), \chi)\}/G(\mathbb{R})$$

where $H(\mathbb{R})$ is a real Cartan subgroup, and χ is a character of (the ρ-cover of) $H(\mathbb{R})$.

Also recall: replace the RHS with $\{H, \mu)\}$ where H is θ-stable, and μ is a $(\mathfrak{h}, H^\theta)$-module.

Digression: covers of tori

G, H

$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$

Definition: H_ρ:
$H_\rho = \{(h,z) \mid h \in H, z \in \mathbb{C}^\times, 2\rho(h) = z^2\}$

$$\begin{array}{ccc} H_\rho & \xrightarrow{\rho=\mathrm{proj}_2} & \mathbb{C}^\times \\ \Big\downarrow{\scriptstyle \mathrm{proj}_1} & & \Big\downarrow{\scriptstyle z^2} \\ H & \xrightarrow[2\rho]{} & \mathbb{C}^\times \end{array}$$

Easy Lemma: ρ is a character of H if and only if $H_\rho \simeq H \times \mathbb{Z}/2\mathbb{Z}$

Ignore H_ρ in this case (for example if G is simply connected).

Example: $SL(2,\mathbb{R})$

$H(\mathbb{R}) = S^1$, $\widehat{H(\mathbb{R})} \simeq \mathbb{Z}$

$H(\mathbb{R}) = \mathbb{R}^\times$, $\widehat{H(\mathbb{R})} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{C}$

$(\epsilon, \nu) : x \to |x|^\nu$ or $\mathrm{sgn}(x)|x|^\nu$ $(\nu \in \mathbb{C}, \epsilon = \pm 1)$.

Need to choose the coordinates wisely, and (critically) *uniformly* as $H(\mathbb{R})$ varies.

Characters of real tori

A real algebraic torus is the real points of a complex algebraic torus $H(\mathbb{C}) \simeq \mathbb{C}^{\times n}$.

Not entirely trivial exercise:

$$H(\mathbb{R}) \simeq \mathbb{R}^{\times a} \times S^{1b} \times \mathbb{C}^{\times c}$$

Theorem (see Vogan's lecture)
H, σ, θ

$$\widehat{H(\mathbb{R})} \leftrightarrow \{(\overline{\lambda}, \overline{\nu})\}$$

$\overline{\lambda} \in X^*/(1-\theta)X^*, \overline{\nu} \in X^*_{\mathbb{Q}}/(1+\theta)X^*_{\mathbb{Q}}$

WLOG $\nu \in (X^*_{\mathbb{Q}})^{-\theta}$

$\lambda \in X^*$ is a representative of $\overline{\lambda} \in X^*/(1-\theta)X^*$

Basic Examples

H	S^1	$\mathbb{R}^\times$	$\mathbb{C}^\times$
X^*	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$
θ	I	$-I$	$\theta(a,b)=(b,a)$
$X^*/(1-\theta)X^*$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\{(a,b)\}/\{(x,-x)\}\simeq\mathbb{Z}$
$(X^*_{\mathbb{C}})^{-\theta}$	0	$\mathbb{C}$	$\{(\nu,-\nu)\}\simeq\mathbb{C}$
character	k $e^{i\theta}\to e^{ik\theta}$	(ϵ,ν) $x\to \mathrm{sgn}(x)^{\frac{1-\epsilon}{2}}\|x\|^\nu$	(k,ν) $re^{i\theta}\to e^{ik\theta}r^\nu$

Parameters in Atlas

$(G, B, H), \theta, X^*, X_* \dots$

Recall: Atlas chooses coordinates making $X^* \simeq \mathbb{Z}^n, X_* \simeq \mathbb{Z}^n$, with the pairing $\langle w, v\rangle = w \cdot v$.

$\Delta^+ = \Delta(B, H)$

$\rho = \frac{1}{2}\sum_{\alpha\boldsymbol{\Delta}^+} \alpha$

$\langle \rho, R^\vee\rangle \in \mathbb{Z}$

Parameters in Atlas

Definition: A parameter is a triple: (x, λ, ν) where

- $x \in K\backslash G/B$
- $\overline{\lambda} \in X^* + \rho/(1-\theta_x)X^* \quad (\lambda \in X^* + \rho)$
- $\nu \in (X^*_{\mathbb{Q}})^{-\theta_x}$

$K\backslash G/B$ is numbered $0, 1, \dots, n$

$\lambda \in X^* + \rho/(1-\theta_x)X^* \mapsto \lambda \in X^* + \rho \simeq \mathbb{Z}^n + \rho \quad (\texttt{ratvec})$

$\nu \in X^* \otimes \mathbb{Q} \simeq \mathbb{Q}^n \quad (\texttt{ratvec})$

So, concretely:
$(x, \lambda, \nu) = (k \in \mathbb{Z}, [a_1, \dots, a_n] \in \frac{1}{2}\mathbb{Z}^n, [b_1, \dots, b_n] \in \mathbb{Q}^n)$

Digression: Cartan subgroups

Fix $x_0 \in K\backslash G/B$, use it do define θ_{x_0} (an automorphism of all of G) and $K_{x_0} = G^{\theta_{x_0}}$.

Recall

$$K_{x_0}\backslash G/B \longleftrightarrow \{\theta_{x_0}\text{-stable Cartan subgroups}\}/K_{x_0}$$

In particular

$$K_{x_0}\backslash G/B \twoheadrightarrow \{\theta_{x_0}\text{-stable Cartan subgroups}\}/K_0$$

The map is

$$x = gx_0g^{-1} \to g^{-1}Bg$$

The fiber of the map is $W/W(K_0, H)$: K_0-conjugacy classes of Borel subgroups containing a given θ-stable Cartan subgroup.

Parameters in Atlas

$p = (x, \lambda, \nu)$

Roughly: this gives a Cartan subgroup $H(\mathbb{R})$ of $G(\mathbb{R})$, and $\chi \in \widehat{H(\mathbb{R})}$.

More precisely, once we've fixed x_0, x gives a conjugacy class of θ_{x_0}-stable Cartan subgroups, and given one H' we get a $(\mathfrak{h}', H' \cap K_{x_0})$-module for it.

It is frequently possible, in fact necessary, to gloss over the technical details.In particular the software never actually constructs H' (it only knows about the fixed Cartan subgroup H).What the software directly produces is an involution θ_x of H, and a $(\mathfrak{h}, H^{\theta_x})$-module. It is up to the user to interpret this in the traditional context.

Digression: Types of roots

Fix $x \in \mathcal{X}$, $\theta_x = \mathrm{int}(x)$ (automorphism of H).

θ_x acts on the roots

α is x-imaginary if $\theta_x(\alpha) = \alpha$)

α is x-real if $\theta_x(\alpha) = -\alpha$)

α is x-complex if $\theta_x(\alpha) \neq \pm\alpha$)

Note:
The imaginary roots define a root system Δ_i of Δ;
$\Delta_i^+ = \Delta_, \cap \Delta^+, \;\; \rho_i = \sum_{\alpha\in\Delta_i} \alpha$

Similarly the real roots define Δ_r, Δ_r^+ and ρ_r.

(The complex roots are usually not a root system).

Parameters in Atlas

$p = (x, \lambda, \nu) \to \gamma(p) = \frac{1}{2}(-\theta_x)\lambda + \frac{1}{2}(1 + \theta)\nu \in \mathfrak{h}^*_{\mathbb{Q}}$

$\gamma(p)$ is the infinitesimal character

Definition: $p = (x, lambda, nu)$ is standard if

$$\langle \gamma, \alpha^\vee \rangle \quad \text{for all } \alpha \in \Delta_i^+$$

(γ is imaginary-dominant)

Defintion: p is final if

$$\langle \nu, \alpha \rangle = 0 \Rightarrow \langle \lambda + \rho_r, \alpha^\vee \rangle \in 2\mathbb{Z} \quad \alpha \text{ real}$$

(a “parity condition”)

Parameters in Atlas

Suppose $p = (x, \lambda, \nu)$ is a parameter. Associated to p is:

- the involution θ_x of H
- the corresponding real form $H(\mathbb{R})$ of H
- a character χ of (the ρ-cover of) $H(\mathbb{R})$
- χ satisfies:

$$\begin{aligned} d\chi &= \frac{1}{2}(1+\theta_x)\lambda + \frac{1}{2}(1-\theta_x)\nu \in \mathfrak{h}^* \\ &= \frac{1}{2}(1+\theta_x)\lambda + \nu \\ &= \gamma(p) \end{aligned}$$

The
hard part is to identify these with the classical picture.

Example: $SL(2,\mathbb{R})$

$$G = SL(2),\, x = t = \mathrm{diag}(i,-i), -t, w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$K\backslash G/B$=`{0,1,2}`

$\theta_0 = \theta_1 = I, \theta_w = -I.$

`p=(0,[k],[0])` $\rightarrow$ $H(\mathbb{R})_{x=0} = S^1, \chi = e^{ik\theta}$

`p=(1,[k],[0])` $\rightarrow$ $H(\mathbb{R})_{x=1} = S^1, \chi = e^{ik\theta}$

`p=(1,[2k],r)` $\rightarrow$ $H(\mathbb{R}) = \mathbb{R}^\times$, $\chi(x) = |x|^r$

`p=(1,[2k+1],r)` $\rightarrow$ $H(\mathbb{R}) = \mathbb{R}^\times$, $\chi(x) = \mathrm{sgn}(x)|x|^r$

Example: $SL(2, \mathbb{R})$

Summary:

$$(\mathtt{0}, [\mathtt{k}], \mathtt{0}) \to H(\mathbb{R})_{x=0} = S^1, \chi = e^{ik\theta} \quad (k > 0)$$
$$(\mathtt{1}, [\mathtt{k}], \mathtt{0}) \to H(\mathbb{R})_{x=1} = S^1, \chi = e^{ik\theta} \quad (k > 0)$$
$$(\mathtt{1}, [\mathtt{0}], \mathtt{r}) \to H(\mathbb{R}) = \mathbb{R}^\times, \chi(x) = |x|^r \quad (r > 0)$$
$$(\mathtt{1}, [\mathtt{1}], \mathtt{r}) \to H(\mathbb{R}) = \mathbb{R}^\times, \chi(x) = \mathrm{sgn}(x)|x|^r \quad (r > 0)$$

The last two cases unambiguosly give representations $\mathrm{Ind}_B^G(\chi)$, where B is a Borel subgroup; the technical details don't matter here.

The first two cases require some care. Suppose we fix x_0 to be KGB element #0. Then `p=(0,[k],0)` gives the character $e^{ik\theta}$ of S^1, and (say by cohomological induction) the holomorphic discrete series representation (lowest weight module), with Harish-Chandra parameter k.

What about `p=(1,[k],0)`? In order to understand this is a parameter for our real group, we need to conjugate KGB element #1 to KGB element #0. This is achieved by an element of the Weyl group, which takes $e^{ik\theta}$ to $e^{-ik\theta}$. So this gives the antiholomorphic discrete series (highest weight module) with Harish-Chandra parameter $-k$.

Note: Suppose we choose x_0 to be KGB-element 1 instead? Then

$$\mathtt{p} = (\mathtt{0}, [\mathtt{k}], \mathtt{0}) \rightarrow \text{ holomorphic discrete series}$$
$$\mathtt{p} = (\mathtt{1}, [\mathtt{k}], \mathtt{0}) \rightarrow \text{ anti-holomorphic discrete series}$$

(the reverse of the earlier case).

Remark There is an outer automorphism of $SL(2, \mathbb{R})$ which exchanges the holomorphic and anti-holomorphc discrete series. Consequently it is not possible to intrinsically resolve this ambiguity.