The Atlas point of view 2

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Complex Reductive Groups and Root Data

$G = G(\mathbb{C})$: connected, complex reductive group

$GL(n, \mathbb{C}), SL(n, \mathbb{C}), Spin(n, \mathbb{C}), G_2(\mathbb{C}), E_8(\mathbb{C}), \dots$

Such a group is classified by its Root Datuma basic concept we assume you are familiar with (see Bourbaki, Lie Groups and Lie Algebras, Chapters 4-6).

Complex Reductive Groups and Root Data

Given G, *choose* a Cartan subgroup H, then

$$X^* = X^*(H) = \mathrm{Hom}_{\mathrm{alg}}(H, \mathbb{C}^\times) \simeq \mathbb{Z}^{\times n}$$
$$X_* = X_*(H) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}^\times, H) \simeq \mathbb{Z}^{\times n}$$

Perfect pairing $X_* \times X^* \to \mathbb{Z}$:

$$\langle \mu^\vee, \gamma \rangle = n : \quad \gamma(\mu^\vee(z)) = z^n \quad (\mathbb{C}^\times \xrightarrow{\mu^\vee} H \xrightarrow{\gamma} \mathbb{C}^\times)$$

$$R \subset X^* \quad \text{the roots}$$
$$R^\vee \subset X_* \quad \text{the coroots}$$

(Finite sets, in bijection via:)

$R \ni \alpha \to \alpha^\vee \in R^\vee$:

$$\langle \alpha, \alpha^\vee \rangle = 2$$

Complex Groups and Root Data

Although X^* and X_* are both isomorphic to $\mathbb{Z}^n$, never identify them. They are naturally dual to each other.

$X_* \times X^* \to \mathbb{Z}$

In `atlas`:

`v=[a_1,a_2,...,a_n]`$\in X_*$
`w=[b_1,b_2,...,b_n]`$\in X^*$

Then the pairing is the dot product `v*w=a_1b_1+...+a_nb_n`.

However the software does not prevent you from computing `v*v` or `w*w`, although these values are probaly not what you intended.

Root Datum in Atlas

A root datum in `atlas` is a pair of $n \times m$ integral matrices (A, B).

n : rank ($X^* \simeq \mathbb{Z}^n$)
m: semisimple rank (rank of the span of the roots)

Satisfying: ${}^tB * A$ is a Cartan matrix (size $m \times m$)

Equivalence: $(A, B) \sim (gA, {}^tg^{-1}B)$ ($g \in GL(2, \mathbb{Z})$)

Example: rank 2, semisimple rank 1

Exercise: Suppose $\mathrm{rank}(G) = 2$, $\text{semisimple-rank}(G) = 1$.
Then up to equivalence (A, B)=

$$(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}) \leftrightarrow SL(2) \times GL(1)$$

$$(\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}) \leftrightarrow PSL(2) \times GL(1)$$

$$(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}) \leftrightarrow GL(2)$$

Exercise: What group has root datum $(\begin{pmatrix}2005\\2017\end{pmatrix}, \begin{pmatrix}336\\334\end{pmatrix})$?

Complex Groups and Root Data

G: connected complex reductive group

Also fix a Cartan subgroup H $\rightarrow \Delta(G,H)=(X^*,R,X_*,R^\vee)$

Also fix a Borel subgroup $H\subset B$, equivalently a set of positive roots, equivalently a Π of simple roots

$$(G,B,H)\rightarrow\Delta_b=(X^*,\Pi,X_*,\Pi^\vee)$$

(Based rood datum)
Recall:

The Atlas point of view: The Cartan subgroup H and Borel subgroup B are fixed, fixed, fixed forever. Everything else, including θ and K, can vary.

$K\backslash G/B$

G,σ,$G(\mathbb{R}) = G^\sigma$, θ, $K = G^\theta$

(Remember: $G = G(\mathbb{C}), B = B(\mathbb{C}), K = K(\mathbb{C})$)

Recall: G/B is the set of Borel subgroups

$B\backslash G/B \simeq W$ (a finite set): geometry behind Category $\mathcal{O}$/Verma modules

In the setting of $(\mathfrak{g}, K)$-modules: K is playing an important role

$$
\begin{aligned}
K\backslash G/B &= \{K\text{-orbits of } K \text{ acting on } \mathcal{B}\} \\
&= \{K - \text{conjugacy classes of Borel subgroups of } G\}
\end{aligned}
$$

$K\backslash G/B$

Theorem K acts on G/B with finitely many orbits.

This goes back to (Wolf 1969); this version is (Matsuki 1979).

Problem: Parametrize the orbit space $K\backslash G/B$.

Remember: strictly speaking this depends on the choice of K, and it is *not* possible in general to canonically identify these spaces (even for K's which are conjugate).

Example: $SL(2,\mathbb{R})$

$G(\mathbb{R}) = SU(1,1)$, $K = H = \{\mathrm{diag}(z, 1/z)\} \simeq \mathbb{C}^\times$.

$G/B = \mathbb{C}P^1 = \mathbb{C} \cup \infty$, $\mathrm{diag}(z, 1/z) : w \to z^2 w$

$K \backslash G/B = \{0, \infty, \mathbb{C}^\times\}$

In terms of Borel subgroups:

$$0 \to \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$$\infty \to \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

$$\mathbb{C}^\times \to \text{ all other Borel subgroups}$$

$K\backslash G/B$

One of the first things the software does is: compute a parameter set for $K\backslash G/B$

Simplifying assumption: θ is an inner automorphism. This is the compact inner class (more on this later)

$K\backslash G/B$

Theorem Suppose $G(\mathbb{R})$ is in the compact inner class. Suppose θ is a Cartan involution for $G(\mathbb{R})$. Then $\theta = \mathrm{int}(x_0)$ for $x_0 \in G, x_0^2 \in Z(G)$. After conjugating by G we may assume $x_0 \in H$; let $K = \mathrm{Cent}_G(x_0)$.

Then

$$\boxed{K\backslash G/B \leftrightarrow \{x \in \mathrm{Norm}_G(H) \mid x \sim x_0\}/H}$$

This set is very amenable to computation by computer.

Note: There is a map from this set to the set of involutions in the Weyl group. The latter is comprehensible combinatorially (and the fibers of the map are essentially finite).

Example: $SL(2,\mathbb{R})$

$G = SL(2), G(\mathbb{R}) = SU(1,1)$, $x_0 = \mathrm{diag}(i,-i)$, $x_0^2 = -I$

Compute $\{x \in \mathrm{Norm}_G(H) \mid x^2 = x_0^2\}$/conjugation by H

Case 1: $x \in H$: $x = \pm x_0 = \pm(\mathrm{diag}(i,-i)$
(the H-action by conjugation is trivial)

Case 2: $x \notin H$, $\to x = \begin{pmatrix} 0 & z \\ -\frac{1}{z} & 0 \end{pmatrix}$

Easy exercise: these elements are all conjugate, and have the same square: $x^2 = x_0^2 = -I$.

Example: $SL(2,\mathbb{R})$

Conclusion:

$$K\backslash G/B \leftrightarrow \{\mathrm{diag}(i,-i), \mathrm{diag}(-i,i), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$$

Obviously the last element goes to the open orbit. But how do we match $\pm\mathrm{diag}(i,-i)$ with $\{0,\infty\}$?

This depends on the choice of x_0 defining $K = \mathrm{Cent}_G(x_0)$

Write $K = K_{x_0}$, and think of G/B as the set of Borel subgroups $\mathcal{B}$. Unwinding the bijection

$$K_{x_0}\backslash G/B \leftrightarrow \{x \in \mathrm{Norm}_G(H) \mid x \sim x_0\}/H$$

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we see: suppose x is in the RHS. Write $x = gx_0g^{-1}$. Then $x \to g^{-1}Bg$ (B is the fixed, fixed, fixed Borel subgroup).

In our example: $x_0 \to B, -x_0 \to B^{\mathrm{op}}$

The Space $\mathcal{X}$

Consider the compact inner class of real forms of G

$\theta = \mathrm{int}(x)$ ($x^2 \in Z$)

Definition

$$\mathcal{X} = \{x \in \mathrm{Norm}_G(H) \mid x^2 \in Z\}/H$$

(the quotient is by conjugation by H)

If $x_0 \in \mathcal{X}$ then

$$\mathcal{X}[x_0] = \{x \in \mathcal{X} \mid x \sim x_0\}/H$$

$\mathcal{X}$ is a finite set if G is semisimple, and $X[x_0]$ is always finite.

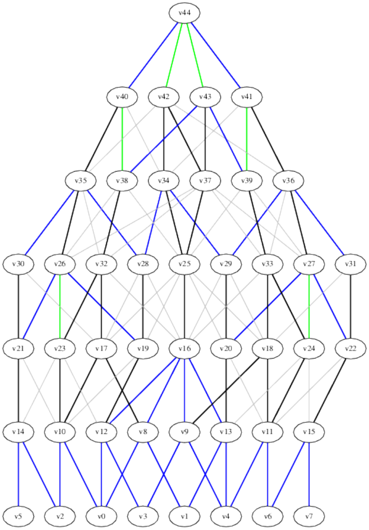
The Space $\mathcal{X}$

Here is a slight restatement of the preceding Theorem.

Theorem There is a surjective map from $\mathcal{X}$ to real forms of G, in the compact inner class, given by $x \to \theta_x = \mathrm{int}(x)$.
$x, x' \in \mathcal{X}$ map to the same real form if $x \simeq x'$.
Fix $x_0 \in \mathcal{X}$. Let $\theta = \theta_{x_0} = \mathrm{int}(x_0)$, $K_0 = G^\theta$. Then

$$K_0 \backslash G/B \leftrightarrow \mathcal{X}[x_0] = \{x \in \mathrm{Norm}_H | x \sim x_0\}/H$$

Stricly speaking θ_x is only a well-defined involution of H (since x is only an H-conjugacy class of elements). Thus: the involution θ_x of H is well defined, but to make an involution of G requires a further choice (of an element in G mapping to the H-conjugacy class x). We will usually gloss over this distinction.


v44
v40 v42 v43 v41
v35 v38 v34 v37 v39 v36
v30 v26 v32 v28 v25 v29 v33 v27 v31
v21 v23 v17 v19 v16 v20 v18 v24 v22
v14 v10 v12 v8 v9 v13 v11 v15
v5 v2 v0 v3 v1 v4 v6 v7

Further properties of $\mathcal{X}$

Fix $x_0 \in \mathcal{X}$, let $\theta = \theta_{x_0}$ (an involution of H, and also G), $K = G^\theta$.

Theorem

$$\mathcal{X}[x_0] \leftrightarrow K\backslash G/B \tag{1}$$

$$\begin{aligned} \mathcal{X}[x_0]/W &\leftrightarrow \{\theta\text{-stable Cartan subgroups}\}/K \\ &\leftrightarrow \{\text{real Cartan subgroups}\}/G(\mathbb{R}) \end{aligned} \tag{2}$$

$$W(K,H) \simeq W(G(\mathbb{R}),H_{\mathbb{R}}) \simeq \mathrm{Stab}_W(x)k \tag{3}$$

Example: $SL(2,\mathbb{R})$

$G = SL(2)$, $x_0 = t = \mathrm{diag}(i,-i)$

$$\mathcal{X}[t] = \{t, -t, w\} = \{\mathrm{diag}(i,-i), \mathrm{diag}(-i,i)\} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1}$$

$$\mathcal{X}[t]/W = \{t, w\} \leftrightarrow \{H_f(\mathbb{R}) = S^1, H_s(\mathbb{R}) = \mathbb{R}^\times\} \tag{2}$$

$$\begin{aligned} \mathrm{Stab}_W(t) = 1 &\to W_{G(\mathbb{R})}(S^1) = 1 \\ \mathrm{Stab}_W(w) = W &\to W_{G(\mathbb{R})}(\mathbb{R}^\times) = W = \mathbb{Z}/2\mathbb{Z} \end{aligned} \tag{3}$$

Example: $Sp(4, \mathbb{R})$

In Atlas $K\backslash G/B$ is a numbered list `1,2,...,n`

```
atlas> print_KGB(Sp(4,R))
kgbsize: 11
Base grading: [11].
 0:  0  [n,n]    1   2     4   5  (0,0)#0 e
 1:  0  [n,n]    0   3     4   6  (1,1)#0 e
 2:  0  [c,n]    2   0     *   5  (0,1)#0 e
 3:  0  [c,n]    3   1     *   6  (1,0)#0 e
 4:  1  [r,C]    4   9     *   *  (0,0) 1 1^e
 5:  1  [C,r]    7   5     *   *  (0,0) 2 2^e
 6:  1  [C,r]    8   6     *   *  (1,0) 2 2^e
 7:  2  [C,n]    5   8     *  10  (0,0)#2 1x2^e
 8:  2  [C,n]    6   7     *  10  (0,1)#2 1x2^e
 9:  2  [n,C]    9   4    10   *  (0,0)#1 2x1^e
10:  3  [r,r]   10  10     *   *  (0,0)#3 1^2x1^e
```

Example: $Sp(4, \mathbb{R})$

KGB#	dimension	Cartan	$\theta_x \in W$
0	0	$S^1 \times S^1$	e
1	0	$S^1 \times S^1$	e
2	0	$S^1 \times S^1$	e
3	0	$S^1 \times S^1$	e
4	1	$\mathbb{C}^\times$	s_1
5	1	$S^1 \times \mathbb{R}^\times$	s_2
6	1	$S^1 \times \mathbb{R}^\times$	s_2
7	2	$S^1 \times \mathbb{R}^\times$	$s_1 s_2 s_1$
8	2	$S^1 \times \mathbb{R}^\times$	$s_1 s_2 s_1$
9	2	$\mathbb{C}^\times$	$s_2 s_1 s_2$
10	3	$\mathbb{R}^\times \times \mathbb{R}^\times$	$-id$

Inner classes

$\mathrm{Aut}(G)$, $\mathrm{Int}(G)$ (inner automorphisms)$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$

Note: If G is semisimple, $\mathrm{Out}(G)$ is a subgroup of the automorphism group of the Dynkin diagram.

Fact: the exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

canonically splits (up to inner automorphism).

So: an inner class is given by $\delta \in \mathrm{Aut}(G)_2$.

Without loss of generality: $\delta(H) = H, \delta(B) = B$.

The compact inner class is $\delta = 1$.

Fact: θ is in the compact inner class if and only if $G(\mathbb{R})$ has a compact Cartan subgroup.

Extended Group

Definition: ${}^{\delta}G = G \rtimes \langle\delta\rangle$

${}^{\delta}G = G \cup G\delta, \delta g \delta^{-1} = \delta(g), \delta^2 = 1$

If $\delta = 1$ then ${}^{\delta}G = G \times \mathbb{Z}/2\mathbb{Z}$ and we can ignore the extension.

$K\backslash G/B$ in general:

$x_0 \in {}^{\delta}G - G$, $x_0^2 \in Z(G) \to \theta_{x_0}, K$

$\mathcal{X}[x_0] = \{g \in \mathrm{Norm}_{G\delta}(H)\}/H$

Theorem

$$\boxed{\mathcal{X}[x_0] \leftrightarrow K\backslash G/B}$$