The Atlas point of view 1

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Overview

The Atlas Software is a comprehensive set of tools for studying representations of Lie groups. In writing the software we rewrote the entire theory from the ground up, with implementation by computer in mind.

These talks will proceed along two parallel tracks. David Vogan will lecture from a more conventional mathematical perspective. My talks will be about the Atlas point of view: our reformulation of the theory.

Peter Trapa and Annegret Paul's talks will be somewhere in the middle.

Marc van Leeuwen will be talking about the Atlas programming language itself.

Example: $SL(2,\mathbb{R})$

Classic example of a Lie group: $SL(2,\mathbb{R})$

Representations of $SL(2,\mathbb{R})$ are associated to characters of Cartan subgroups.

(1) $H_s(\mathbb{R}) = \mathrm{diag}(x, 1/x) \simeq \mathbb{R}^\times \subset B$ (Borel subgroup of upper triangular matrices)
$\chi \in \widehat{\mathbb{R}^\times} \to \mathrm{Ind}_B^G(\chi)$ (principal series)

(2) $H_c(\mathbb{R}) = \{\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\} \simeq S^1$

$\chi \in \widehat{S^1} \to \mathrm{DS}(\chi)$ (discrete series representation)

Example: $SL(2,\mathbb{R})$

All Cartan subgroups of a complex group are conjugate.

In our example $H(\mathbb{C}) \simeq \mathbb{C}^\times \subset SL(2,\mathbb{C})$

(1) $H_s(\mathbb{R}) \simeq \mathbb{R}^\times,\ H_s(\mathbb{C}) = \{\mathrm{diag}(z, 1/z) \simeq \mathbb{C}^\times$

(2) $H_c(\mathbb{R}) \simeq S^1$

$$H_c(\mathbb{C}) = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{C},\ a^2+b^2 = 1\} \simeq \mathbb{C}^\times \quad (\to a+ib \in \mathbb{C}^\times)$$

Better way to handle H_c:
$G(\mathbb{R}) = SU(1,1) \simeq SL(2,\mathbb{R})$

$$SU(1,1) = \{g \mid gJ\,{}^t\overline{g} = J\} \quad (J = \mathrm{diag}(1,-1))$$

Example: $SL(2, \mathbb{R})$

Diagonal matrices in $SL(2, \mathbb{R})$:

$$H_s(\mathbb{R}) = diag(x, 1/x) \simeq \mathbb{R}^\times$$
$$H_s(\mathbb{C}) = diag(z, 1/z) \simeq \mathbb{C}^\times$$

Diagonal matrices in $SU(1, 1)$:

$$H_c(\mathbb{R}) = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\} \simeq S^1$$
$$H_c(\mathbb{C}) = \{\mathrm{diag}(z, 1/z) \mid z \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times$$

Moral of the story

Rather than using one $SL(2,\mathbb{R})$, and two different Cartan subgroups:

$$H_s(\mathbb{R}) = \mathrm{diag}(x, 1/x), \quad H_c(\mathbb{R}) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\} \simeq S^1]$$

choose $G(\mathbb{R})$ depending on the Cartan subgroup of interest:

$$H_s(\mathbb{R}): \quad G(\mathbb{R}) = SL(2,\mathbb{R}), \quad H_s(\mathbb{R}) = \mathrm{diag}(x, 1/x)],$$

$$H_c(\mathbb{R}): \quad G(\mathbb{R}) = SU(1,1), \quad H_c(\mathbb{R}) = \mathrm{diag}(e^{i\theta}, e^{-i\theta})]$$

Example: $SL(2,\mathbb{R})$ and $SU(1,1)$

(1) $SL(2,\mathbb{R}) = SL(2,\mathbb{C})^{\sigma_1}$ where $\sigma_1(g) = \overline{g}$

(1) $SU(1,1) = SL(2,\mathbb{C})^{\sigma_2}$ where $\sigma_2(g) = J\,{}^t\overline{g}J$

Key point: σ_1, σ_2 are conjugate by an inner automorphism:

$$\boxed{\sigma_2 = \mathrm{int}(g)\circ\sigma_1\circ\mathrm{int}(g^{-1})} \quad g = \frac{1}{2i}\begin{pmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4}\end{pmatrix}$$

$(\mathrm{int}(g)(h) = ghg^{-1})$.

$$\boxed{SU(1,1) = G^{\sigma_2} = gG^{\sigma_1}g^{-1} = gSL(2,\mathbb{R})g^{-1}}$$

Real forms of complex groups

How this works in general...

$G = G(\mathbb{C})$: connected, complex, reductive group ($SL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$, $SO(n, \mathbb{C})$, etc.)

What is a "real form" of $G(\mathbb{C})$?

Definition: A real form of G is an antiholomorphic involution σ of $G = G(\mathbb{C})$.

Then $G(\mathbb{R}) = G(\mathbb{C})^\sigma$.

Galois and Cartan involutions

The antiholomorphic involution σ is called a Galois involution.

Special feature of real (as opposed to *p*-adic) groups: real forms can also be classified by their **Cartan** involutions.

Definition If σ is a real form of G, then a Cartan involution for σ is a holomorphic involution of G, commuting with σ, such that $G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$.

Note: Some references view θ as an involution only of $G(\mathbb{R})$, but it can be shown that it extends to a (holomorphic) involution of $G(\mathbb{C})$.

Galois and Cartan involutions

Theorem (Cartan) The map $\sigma \to \theta$ is a bijection

$\{$anti-holom. involutions $\sigma\}/G(\mathbb{C}) \leftrightarrow \{$holom. involutions $\theta\}/G(\mathbb{C})$.

Example: The "obvious" θ is: $\theta = 1 \leftrightarrow G(\mathbb{R})$ is compact

Example: What is the "obvious" σ?: $G(\mathbb{R})$ is split (the Chevalley group, defined over $\mathbb{Z}$)

The Atlas approach is to almost exclusively use θ instead of σ.

The σ/θ Picture

$$
\begin{array}{ccccc}
 & & G(\mathbb{C}) & & \\
 & \swarrow^{\sigma} & & \searrow^{\theta} & \\
G(\mathbb{R}) = G(\mathbb{C})^{\sigma} & & & & G(\mathbb{C})^{\theta} = K(\mathbb{C}) \\
 & \searrow^{\theta} & & \swarrow^{\sigma} & \\
 & & G(\mathbb{R})^{\theta} = G(\mathbb{C})^{\sigma,\theta} = K(\mathbb{C})^{\sigma} & &
\end{array}
$$

σ picture: $G(\mathbb{R})$ is the basic object

θ picture: $K(\mathbb{C})$ is the basic object.

The σ/θ Picture

	σ		θ
Real forms	$G(\mathbb{R})$		$K(\mathbb{C})$
θ-basepoint	$G(\mathbb{R})$ compact		$\theta = 1, K = G$
σ-basepoint	$G(\mathbb{R})$ split		
"base" Cartan	H_s (most split)		H_f (most compact)
Cartan subgroups	$\{H(\mathbb{R})\}$ modulo $G(\mathbb{R})$	$\overset{1-1}{\leftrightarrow}$	$\{\theta$-stable $H(\mathbb{C})\}$ modulo $K(\mathbb{C})$
Weyl group	$\mathrm{Norm}_{G(\mathbb{R})}(Hd.e(\mathbb{R}))/H(\mathbb{R})$	$\overset{1-1}{\leftrightarrow}$	$\mathrm{Norm}_K(H)/H \cap K$
Nilpotent orbits	$\mathfrak{g}_0(\mathbb{R})_{nil}/G(\mathbb{R})$	$\overset{1-1}{\leftrightarrow}$	$\mathfrak{p}_{nil}/K$
Flag variety $\mathcal{B} = G/B$	$\mathcal{B}/G(\mathbb{R})$	$\overset{1-1}{\leftrightarrow}$	$\mathcal{B}/K$
component group	$G(\mathbb{R})/G(\mathbb{R})^0$	$\overset{1-1}{\leftrightarrow}$	$K(\mathbb{C})/K(\mathbb{C})^0$
Representations	$G(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ (analysis)		$(\mathfrak{g}, K)$-module (algebra)
group cohomology	$H^1(\sigma, G)$	$\overset{1-1}{\leftrightarrow}$	$H^1(\theta, G)$

The Atlas point of view

$G = G(\mathbb{C})$: connected complex reductive group

fix a Cartan subgroup H (unique up to conjugation by G)

Also fix a Borel subgroup $B \supset H$ (unique up to conjugation by $\mathrm{Norm}_G(H)$)

The Atlas point of view: The Cartan subgroup H and Borel subgroup B are fixed, fixed, fixed forever. Everything else, including θ and K, can vary.