Cohomological Induction

Peter Trapa

University of Utah

Workshop on the Atlas of Lie Groups and Representations
University of Utah, Salt Lake City
July 10 - 21, 2017

Goals

Goals

- Provide theoretical background for Annegret's next talk (loosely "Borel-Weil-Bott as change of rings"), and some groundwork for the ABV theory of special unipotent packets.

- Set up some interesting examples (possibly to be explored in evening sessions).

Throughout, (G, K) will be a reductive pair, $K = G^\theta$, and $G(\mathbb{R})$ will be the corresponding real form.

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a θ-stable Cartan, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ a Borel subalgebra.

Cohomological Induction: heuristic construction

starting point

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic of $\mathfrak{g}$ containing $\mathfrak{b}$; set

$$Q = LU = \mathrm{Stab}_G(\mathfrak{q}).$$

Fix a character $\mathbb{C}_\lambda$ of L with unique weight $\lambda \in \mathfrak{h}^*$.

Form the G-equivariant holomorphic line bundle

$$\begin{array}{c} \mathcal{L}_\lambda = G \times_Q \mathbb{C}_\lambda \\ \downarrow \\ G/Q. \end{array}$$

This is the Borel-Weil-Bott setting for constructing representations of G.

Cohomological Induction: heuristic construction

We want to construct irreducible representations of $G(\mathbb{R})$.

So restrict $\mathcal{L}_\lambda$ to a $G(\mathbb{R})$ orbit $\mathcal{O} \subset G/Q$, and obtain a $G(\mathbb{R})$-equivariant line bundle

$$\begin{array}{c} \mathcal{L}_\lambda|_{\mathcal{O}} \\ \Big\downarrow \\ \mathcal{O}. \end{array}$$

Imitate Borel-Weil-Bott and consider $H^{0,k}(\mathcal{O}, \mathcal{L}_\lambda|_{\mathcal{O}})$.

Senseless unless $\mathcal{L}_\lambda|_{\mathcal{O}}$ has a holomorphic structure.

Cohomological Induction: heuristic construction

If $\mathfrak{q}$ is θ-stable, then

$$\mathcal{O}_{\mathfrak{q}} := G(\mathbb{R}).eQ \subset G/Q \text{ is open.}$$

So

$$\begin{array}{c} \mathcal{L}_\lambda|_{\mathcal{O}} \\ \big\downarrow \\ \mathcal{O} \end{array}$$

inherits a holomorphic structure.

This leads us to a provisional definition:

$$\boxed{A_{\mathfrak{q}}(\lambda) := H^{0,S}(\mathcal{O}_{\mathfrak{q}}, \mathcal{L}_\lambda \otimes \wedge^{top} \mathfrak{u})}$$

with $S = \dim(\mathfrak{u} \cap \mathfrak{k})$; here $\mathcal{L}_\lambda \otimes \wedge^{top}\mathfrak{u} = \mathcal{L}_{\lambda+2\rho(\mathfrak{u})}$.

Cohomological Induction: reality check

What does

$$\boxed{A_{\mathfrak{q}}(\lambda) := H^{0,S}(\mathcal{O}_{\mathfrak{q}}, \mathcal{L}_{\lambda} \otimes \wedge^{top}\mathfrak{u})}$$

with $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ look like when $G(\mathbb{R})$ is compact (i.e. $G = K$ and $\theta \equiv 1$)?

Take $\mathfrak{q} = \mathfrak{b}$, so $\mathcal{O}_{\mathfrak{q}} = G/B$ and $\dim(\mathfrak{u} \cap \mathfrak{k}) = \dim(G/B)$. Then Serre + Dolbeault says

$$A_{\mathfrak{b}}(\lambda) := H^{0,\dim(G/B)}(G/B, \mathcal{L}_{\lambda} \otimes \wedge^{top}\mathfrak{u}) \simeq H^{0}(G/B, \mathcal{L}_{-\lambda}).$$

Borel-Weil tells us that this is either an irreducible representation of G or zero.

Cohomological Induction: reality check

More precisely: if $G(\mathbb{R})$ is compact (i.e. $G = K$) and connected,

$$A_{\mathfrak{b}}(\lambda) \simeq H^0(G/B, \mathcal{L}_{-\lambda}) = \begin{cases} V_\lambda & \text{if } \langle \lambda, \alpha \rangle \geq 0 \ \forall\, \alpha \in \Delta(\mathfrak{h}, \mathfrak{u}); \\ 0 & \text{else.} \end{cases}$$

where V_λ is irreducible, highest weight λ.

For later use, rewrite nonvanishing condition:

$$\langle \lambda + \rho, \alpha \rangle > 0 \ \forall\, \alpha \in \Delta(\mathfrak{h}, \mathfrak{u}).$$

Cohomological Induction: reality check

So

$$\boxed{A_{\mathfrak{q}}(\lambda) := H^{0,S}(\mathcal{O}_{\mathfrak{q}}, \mathcal{L}_\lambda \otimes \wedge^{top}\mathfrak{u})}$$

is the simplest generalization of the Borel-Weil construction to the case of noncompact groups (proposed by Kostant and Langlands in the early 60's).

But there are serious analytic complications in the noncompact case. Some were surmounted in Schmid's thesis (and the work of many others), but others still remain. More on that in a moment.

On the `atlas` perspective...

In the above, $\mathcal{O}_{\mathfrak{q}} = G(\mathbb{R}).eQ \subset G/Q$ are living in different places as $\mathfrak{q}$ varies.

Better: fix (once and for all) a base point in the partial flag variety $\mathcal{Q}$ of parabolics of a fixed type. (Special case: fixing a base point in the flag variety, as discussed by Jeff.)

Vary the choice of open $G(\mathbb{R})$ orbit $\mathcal{O}_{\mathfrak{q}}$ in $\mathcal{Q}$.

Getting warmer, but $G(\mathbb{R})$ orbits on $\mathcal{Q}$ are not something `atlas` is designed to handle.

Interlude: Matsuki Duality

What's the connection with Annegret's parametrization of θ-stable parabolics in terms of closed K orbits on G/Q (thinking now as Q fixed)?

Matsuki Duality

There is an order-reversing bijection between the $G(\mathbb{R})$ orbits on G/Q and the K-orbits on G/Q.

Two corresponding orbits intersect in a $G(\mathbb{R})^\theta$ orbit. In particular, an open $G(\mathbb{R})$ orbit consists of θ-stable parabolic subalgebras iff the corresponding closed K orbits does.

Interlude: Matsuki Duality

Matsuki Duality

There is an order-reversing bijection between the $G(\mathbb{R})$ orbits on G/Q and the K-orbits on G/Q. (Two corresponding orbits intersect in a $G(\mathbb{R})^\theta$ orbit.)

Example: $SL(2, \mathbb{R})$

- $K \simeq \mathbb{C}^\times$ orbits on $G/B \simeq \mathbb{P}^1$ consists of two closed orbits (0 and ∞) and their open complement.
- $SL(2, \mathbb{R})$ orbits on $\mathbb{P}^1$ consist of the two open hemispheres and the closed equator.

Example: $G(\mathbb{R})$ split F4, S^c middle short simple root.

Example (Split F4)

```
atlas> set G=F4_s
Variable G: RealForm
atlas> simple_roots (G)
Value:
|  2, -1,  0,  0 |
| -1,  2, -1,  0 |
|  0, -2,  2, -1 |
|  0,  0, -1,  2 |
```

So the long middle simple root is 1; the complement is $\{0,2,3\}$.

Preview of coming attractions: S consists of the zero labels for the weighted Dynkin diagram for the even nilpotent orbit F4(a3).

Example: $G(\mathbb{R})$ split F4, S^c middle short simple root.

Example (Split F4)

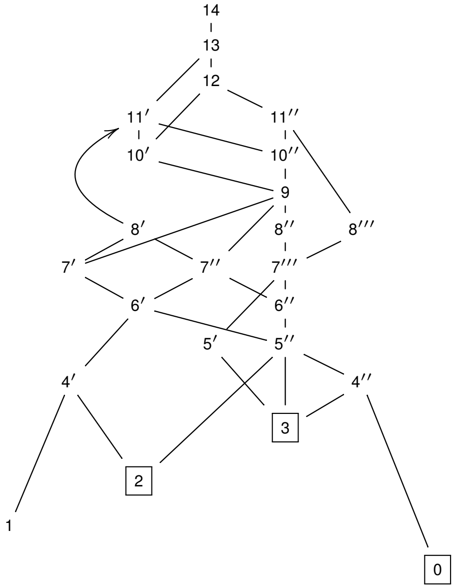
The long middle simple root is 1; the complement is $\{0,2,3\}$.

```
atlas> set G=F4_s
Variable G: RealForm
atlas> #theta_stable_parabolics(G)
Value: 92
atlas> theta_stable_parabolics(G)
Value:...([0,2,3],KGB element #8),([0,2,3],KGB element #25),
([0,2,3],KGB element #31),([0,2,3],KGB element #47),...
atlas> #KGP(G,[0,2,3])
Value: 24
```

So there are four closed orbits among 24 K orbits on G/P.

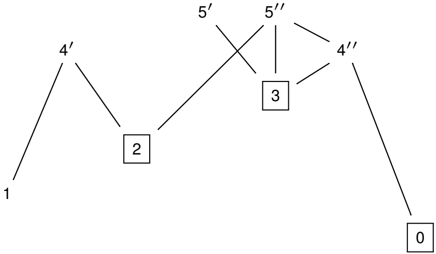
Example continued: closure order on KGP

Labels are dimensions:



Example continued: closed orbits on KGP

There are four closed orbits of course. But there is some extra structure: the three boxed ones canonically match the three real forms of the nilpotent orbit F4(a3).



How? I hope to return to this next week. (Jonathan Fernandes has written some nice code for this.)

Back to Cohomological Induction

In the general case,

$$\boxed{A_{\mathfrak{q}}(\lambda) := H^{0,S}(\mathcal{O}_{\mathfrak{q}}, \mathcal{L}_{\lambda} \otimes \wedge^{top}\mathfrak{u}).}$$

is only a heuristic construction of a $G(\mathbb{R})$ representation.

Zuckerman discovered an algebraic way to construct (what should be) its underlying $(\mathfrak{g}, K)$ module.

Back to Cohomological Induction

Start with something like formal Taylor series at eQ of sections:

$$\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), \mathbb{C}_\lambda \otimes \wedge^{top}\mathfrak{u}),$$

a $U(\mathfrak{g})$ module. Extract the formal Taylor series that "extend to sections" by looking at the largest "K-finite subspace" to obtain a $(\mathfrak{g}, K)$ module:

$$\Gamma(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), \mathbb{C}_\lambda \otimes \wedge^{top}\mathfrak{u})).$$

Problem: this might be zero (just like sections of $\mathcal{L}$ need not exist). So look for analogs of cohomology.

Back to Cohomological Induction

Seek functors

$$\mathcal{R}^j_{\mathfrak{q}} \ : \ \mathcal{M}(\mathfrak{l}, L \cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, K).$$

Zeroeth step: Twist Z by $\wedge^{\mathrm{top}}\mathfrak{u}$ and extend it trivially to a $(\overline{\mathfrak{q}}, L \cap K)$ module $Z^{\#}_{\mathfrak{q}}$.

First step: form

$$\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), Z^{\#})_{L\cap K}.$$

Second step: The "maximal K-finite vector subspace" functor

$$\Gamma : \mathcal{M}(\mathfrak{g}, L \cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$$

is covariant left exact, and has derived functors Γ^j.

$$\boxed{\mathcal{R}^j_{\mathfrak{q}}(Z) := \Gamma^j(\mathrm{Hom}_{\overline{\mathfrak{q}}}(U(\mathfrak{g}), Z^{\#})_{L\cap K})}$$

Better: change of rings

After the zeroth step $Z \mapsto Z_{\mathfrak{q}}^{\#}$, we are seeking a functor

$$\mathcal{M}(\mathfrak{q}, L \cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, K),$$

i.e.

$$R(\mathfrak{q}, L \cap K)\text{-mod} \longrightarrow R(\mathfrak{g}, K)\text{-mod}.$$

There is an obvious change of rings functor:

$$I(X) := \mathrm{Hom}_{R(\mathfrak{q}, L \cap K)}(R(\mathfrak{g}, K), X)_K.$$

And another one too:

$$P(X) := R(\mathfrak{g}, K) \otimes_{R(\mathfrak{q}, L \cap K)} X.$$

Better: change of rings

Focus on I for now:

$$R(\mathfrak{q}, L\cap K)\text{-mod} \xrightarrow{I} R(\mathfrak{g}, K)\text{-mod}.$$

$$I(X) := \mathrm{Hom}_{R(\mathfrak{l}, L\cap K)}(R(\mathfrak{g}, K), X)_K$$

can be factored as a composite:

$$R(\mathfrak{q}, L\cap K)\text{-mod} \xrightarrow{I_1} R(\mathfrak{g}, L\cap K)\text{-mod} \xrightarrow{I_2} R(\mathfrak{g}, K)\text{-mod}.$$

with

$$I_1(X) := \mathrm{Hom}_{R(\mathfrak{q}, L\cap K)}(R(\mathfrak{g}, L\cap K), Z)_{L\cap K} \simeq \mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), X)_{L\cap K}$$

exact, and

$$I_2(X) := \mathrm{Hom}_{R(\mathfrak{g}, L\cap K)}(R(\mathfrak{g}, L\cap K), X)_K \simeq \Gamma(X).$$

left exact. This explains our old friend

$$\boxed{\mathcal{R}^j_{\mathfrak{q}}(Z) = \Gamma^j(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), Z^{\#}_{\mathfrak{q}})_{L\cap K}).}$$

Better: change of rings

So cohomological induction is the change of rings functors

$$R(\mathfrak{q}, L\cap K)\text{-mod} \longrightarrow R(\mathfrak{g}, K)\text{-mod}.$$

with the mild complication that the functor is not exact.

What if you do the analogous change of rings when $\mathfrak{p}$ is a *real* with corresponding real parabolic subgroup $P(\mathbb{R})$? In this case

$$Z \mapsto \mathrm{Hom}_{R(\mathfrak{q}, M\cap K)}(R(\mathfrak{g}, K), Z_{\mathfrak{p}}^{\#})_K$$

is exact, and is the underlying $(\mathfrak{g}, K)$ module of a real parabolically induced representation, roughly

$$\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(Z_{P(\mathbb{R})}^{\#}).$$

So standard modules are all just change of rings!

Cohomological Induction Recap

Have functors

$$\mathcal{R}^j_{\mathfrak{q}} \;:\; \mathcal{M}(\mathfrak{l}, L \cap K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$$

$$\mathcal{R}^j_{\mathfrak{q}}(Z) := \Gamma^j(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), Z^{\#})_{L\cap K})$$

Special case:

$$A_{\mathfrak{q}}(\lambda) := \mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}_\lambda).$$

Very special case (Borel-Weil-Bott): $G(\mathbb{R})$ compact and $\mathfrak{q} = \mathfrak{b}$; then

$$A_{\mathfrak{b}}(\lambda) = \begin{cases} V_\lambda & \text{if } \langle \lambda + \rho, \alpha \rangle \geq 0 \; \forall \, \alpha \in \Delta(\mathfrak{h}, \mathfrak{u}); \\ 0 & \text{else.} \end{cases}$$

Cohomological Induction: Properties

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable, and let Z be an $(\mathfrak{l}, L \cap K)$-module with infinitesimal character γ_L. Then $\mathcal{R}^j_{\mathfrak{q}}(Z)$ has infinitesimal character $\gamma_L + \rho(\mathfrak{u})$.

- Z is in the **good range** for $\mathfrak{q}$ if for all $\alpha \in \Delta(\mathfrak{u})$,

$$Re\langle \gamma_L + \rho(\mathfrak{u}), \alpha^\vee \rangle > 0.$$

- Z is in the **weakly good range** if for all $\alpha \in \Delta(\mathfrak{u})$,

$$Re\langle \gamma_L + \rho(\mathfrak{u}), \alpha^\vee \rangle \geq 0.$$

- If Z is in the good range $\mathcal{R}^S_{\mathfrak{q}}(Z)$ is irreducible (resp. unitary) iff Z is. For $j \neq S, \mathcal{R}^j_{\mathfrak{q}}(Z) \neq 0$.
  In the weakly good range, $\mathcal{R}^S_{\mathfrak{q}}(Z)$ could vanish.

Cohomological Induction: Properties

Specialize to $Z = \mathbb{C}_\lambda$, a unitary character. So

$$\gamma_L + \rho(\mathfrak{u}) = \lambda + \rho(\mathfrak{l}) + \rho(\mathfrak{u}) = \lambda + \rho.$$

- λ is in the **good range** for $\mathfrak{q}$ if for all $\alpha \in \Delta(\mathfrak{u})$,

$$Re\langle \lambda + \rho, \alpha^\vee \rangle > 0.$$

- λ is in the **weakly good range** if for all $\alpha \in \Delta(\mathfrak{u})$,

$$Re\langle \lambda + \rho, \alpha^\vee \rangle \geq 0.$$

- If λ is in the good range $A_\mathfrak{q}(\lambda)$ is irreducible and unitary.
- In the weakly good range, easy to tell if $A_\mathfrak{q}(\lambda) \neq 0$.

Example: Discrete Series

Suppose $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\mathfrak{k})$, and $\mathfrak{b}$ is a θ-stable Borel subalgebra. Suppose λ is in the good range. Then

$A_{\mathfrak{b}}(\lambda)$ is a discrete series with infinitesimal character $\lambda + \rho$;

all arise this way, and there are no coincidences.

Discrete series with a fixed infinitesimal character are parametrized by closed K orbits on the flag variety.

(Weakly good range: get all limits of discrete series.)

Example: Discrete Series

Discrete Series of $SL(2,\mathbb{R})$ at ρ.

```
atlas> set G=SL(2,R)
atlas> print_KGB(G)
0:  0  [n]   1    2  (0)#0 e
1:  0  [n]   0    2  (1)#0 e
2:  1  [r]   2    *  (0)#1 1^e
```

So there are of course two closed orbits.

```
atlas> all_discrete_series (G,rho(G))
final parameter(x=0,lambda=[1]/1,nu=[0]/1)
final parameter(x=1,lambda=[1]/1,nu=[0]/1)
```

The two discrete series at ρ.

Example: Discrete Series

Discrete Series of $Sp(4,\mathbb{R})$ at ρ.

```
atlas> set G=Sp(4,R)
atlas> print_KGB(G)
kgbsize: 11
 0:  0  [n,n]    1   2     4   5  (0,0)#0 e
 1:  0  [n,n]    0   3     4   6  (1,1)#0 e
 2:  0  [c,n]    2   0     *   5  (0,1)#0 e
 3:  0  [c,n]    3   1     *   6  (1,0)#0 e
....

atlas> all_discrete_series (G,rho(G))
final parameter(x=0,lambda=[2,1]/1,nu=[0,0]/1)
final parameter(x=1,lambda=[2,1]/1,nu=[0,0]/1)
final parameter(x=2,lambda=[2,1]/1,nu=[0,0]/1)
final parameter(x=3,lambda=[2,1]/1,nu=[0,0]/1)]
```

The four discrete series at ρ.

Role of $A_{\mathfrak{q}}(\lambda)$

Theorem (Salamanca-Riba)

If X has strongly regular infinitesimal character then X is unitary if and only if it is a good $A_{\mathfrak{q}}(\lambda)$ module.

Vogan showed that there a larger range of unitarity:

- $A_{\mathfrak{q}}(\lambda) = \mathcal{R}_{\mathfrak{q}}^{S}(\mathbb{C}_\lambda)$ is in the **weakly fair** range if for all $\alpha \in \Delta(\mathfrak{u})$,

  $$Re\langle \lambda + \rho(\mathfrak{u}), \alpha^\vee \rangle \geq 0.$$

  A weakly fair $A_{\mathfrak{q}}(\lambda)$ module is either 0 or unitary (but could reduce).

Lots of interesting questions here: Determine nonvanishing, reducibility, and coincidences among weakly fair $A_{\mathfrak{q}}(\lambda)$. Extend to larger ranges?

Preview of coming attractions

Annegret's next lecture will explain the `atlas` functions to address these interesting questions.

For example, Vogan/Borho-Brylinski showed that if the moment map $\mu : T^*\mathcal{Q} \to \mathcal{N}$ is birational onto normal image, then $A_{\mathfrak{q}}(\lambda)$ is irreducible or zero.

Always true in type A. In that case, Vogan conjectured that *any* irreducible unitary representation of $U(p, q)$ with integral infinitesimal is a weakly fair $A_{\mathfrak{q}}(\lambda)$ module. (Can check any particular infinitesimal character in `atlas` now.)

First place to look for reducibility: $G = Sp(4, \mathbb{R})$ and $\mathcal{Q}$ corresponding to Levi $GL(1) \times Sp(2)$. Then μ is not birational (but the image is normal). Annegret will look closely at this.