K-Types in `atlas`

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Overview

Tasks

- *What is K in* `atlas`*?*
- *K as a reductive real Lie group (in* `atlas`*: K versus K_0).*
- *How does* `atlas` *"think" of K-types and their highest weights?*
- *Given a parameter for $G(\mathbb{R})$, find the lowest K-type(s) of the representation.*
- *Find (part of) the K-spectrum: multiplicities of K-types up to height h.*
- *What about K-types for disconnected groups?*

The Group K

Let G be a complex group as allowed in `atlas`. Fix a `RealForm` $G(\mathbb{R})$, giving `KGB`.

- $G(\mathbb{R})$ determines the "maximal compact" complex subgroup $K \subset G$, up to conjugation by G.
- If `x` belongs to `KGB` then $K = G^{\theta_x}$.
- K is a complex reductive algebraic group, not necessarily connected.
- So it may not be an `atlas` group.
- K captures the disconnectedness of $G(\mathbb{R})$:

$$K/K_0 \cong G(\mathbb{R})/G(\mathbb{R})_0.$$

- If G is simply connected then K is connected.

The Group K

Example

- If $G(\mathbb{R}) = SL(2,\mathbb{R})$ then $K = SO(2,\mathbb{C})$.
- If $G(\mathbb{R}) = PGL(2,\mathbb{R}) \cong SO(2,1)$ then $K = O(2,\mathbb{C})$

- $K(\mathbb{R})$ (the intersection with $G(\mathbb{R})$) is a maximal compact subgroup of $G(\mathbb{R})$, and

$$K(\mathbb{R})/K(\mathbb{R})_0 \cong K/K_0 \cong G(\mathbb{R})/G(\mathbb{R})_0.$$

- K_0 is a connected reductive complex Lie group, with its own root datum.
- Choose `x` in the distinguished fiber. Then `atlas` can construct the group K_0 with `RealForm` $K_0(\mathbb{R})$, the maximal compact subgroup of $G(\mathbb{R})_0$.
- Choosing `x` carefully (as for constructing θ-stable parabolics) yields K_0 in more familiar coordinates.

Example ($G(\mathbb{R}) = Sp(4,\mathbb{R})$)

If $G(\mathbb{R}) = Sp(4,\mathbb{R})$ then $K_0(\mathbb{R}) = U(2) = K(\mathbb{R})$:

```
atlas> set G=Sp(4,R)
atlas> print_KGB(G)
 0:  0  [n,n]    1   2     4   5  (0,0)#0 e
 1:  0  [n,n]    0   3     4   6  (1,1)#0 e
 2:  0  [c,n]    2   0     *   5  (0,1)#0 e
 3:  0  [c,n]    3   1     *   6  (1,0)#0 e
...

atlas> set x=KGB(G,2)
atlas> rho_c(x)
Value: [  1, -1 ]/2

atlas> set K0=K_0(x)
Variable K0: RealForm
atlas> K0
Value: compact connected real group with Lie algebra 'su(2).u(1)'

atlas> simple_roots(K0)
Value:
|  1 |
| -1 |
```

The Group K

Example ($Sp(4,\mathbb{R})$ continued)

```
atlas> set y=KGB(G,0)
Variable y: KGBElt
atlas> rho_c(y)
Value: [ 1, 1 ]/2

atlas> set K0_alt=K_0(y)
Variable K0_alt: RealForm
atlas> K0_alt
Value: compact connected real group with Lie algebra 'su(2).u(1)'

atlas> simple_roots(K0_alt)
Value:
| 1 |
| 1 |

atlas> print_KGB(K0)
kgbsize: 1
0:  0  [c]   0    *  (1,0)#0 e
```

Example ($GL(3,\mathbb{R})$)

```
atlas> G:=GL(3,R)
atlas> print_KGB(G)
0:  0  [C,C]   2  1    *  *  (0,0,0)#0 e
1:  1  [n,C]   1  0    3  *  (0,0,0) 0 2xe
2:  1  [C,n]   0  2    *  3  (0,0,0) 0 1xe
3:  2  [r,r]   3  3    *  *  (0,0,0)#1 1^2xe
```

There is only one choice for `x`.

```
atlas> rho_c(KGB(G,1))
Runtime error:
  x is not in distinguished fiber
...

atlas> x:=KGB(G,0)
atlas> rho_c(x)
Value: [  1,  0, -1 ]/4
atlas> set K_GL=K_0(x)
atlas> K_GL
Value: compact connected real group with Lie algebra 'su(2)'

atlas> simple_roots (K_GL)
Value:
| -1 |
```

Lowest K-types

- We will parametrize K-types (irreducible representations of $K(\mathbb{R})$) using the notion of "lowest" or "minimal" K-types (LKTs) of standard or irreducible representations of $G(\mathbb{R})$.
- A LKT of such a representation is a K-type τ which is minimal with respect to the "Vogan norm": Choose a positive system of roots for K with associated $\rho(K)$, and an extremal weight μ of τ, dominant with respect to this system. Then

$$||\tau|| = \langle \mu + 2\rho(K), \mu + 2\rho(K) \rangle.$$

- Every standard or irreducible module of $G(\mathbb{R})$ has a finite number of LKTs, each with multiplicity 1.
- Certain tempered representations have **unique** LKTs, and we can use this fact to parametrize K-types.

K-Parameters

Let $G(\mathbb{R})$ be a real group (as defined in `atlas`) with maximal compact subgroup $K(\mathbb{R})$ and Cartan involution θ.

Theorem (Vogan)

There are natural bijections between the following three sets.

1. *Tempered irreducible representations π of $G(\mathbb{R})$ with real infinitesimal character.*
2. *Irreducible representations τ of $K(\mathbb{R})$ (K-types).*
3. *Discrete final limit parameters Φ attached to θ-stable Cartan subgroups of $G(\mathbb{R})$, modulo conjugation by $K(\mathbb{R})$.*

The bijection from (1) to (2) sends π to the unique lowest K-type of π. The bijection from (3) to (1) is the Knapp-Zuckerman parametrization of irreducible tempered representations.

`atlas` versions of these (K-conjugacy classes of) discrete final limit parameters are K-parameters, which parametrize K-types in `atlas`.

K-Parameters

In `atlas`, fix a `RealForm` $G(\mathbb{R})$ of the complex group G and the associated set `KGB`.

Definition

A K-parameter is an equivalence class of pairs (x, λ) with

1. $x \in \mathtt{KGB}$;
2. $\lambda \in X^*(H) + \rho(G)$.

The parameter must satisfy for each positive root α:

- If α is imaginary then $\langle \lambda, \alpha^\vee \rangle \geq 0$ (standard);
- If α is real then $\langle \lambda + \rho_r(x), \alpha^\vee \rangle$ is even (final);
- If α is imaginary-simple and $\langle \lambda, \alpha^\vee \rangle = 0$ then α is noncompact (nonzero).

Equivalence of parameters is generated by

1. $(x, \lambda) \equiv (x, \lambda')$ if $\lambda - \lambda' \in (1 - \theta_x) X^*(H)$;
2. $(x, \lambda) \equiv (s_\alpha \times x, s_\alpha \lambda)$ for α simple and complex.

K-Parameters

- If (x, λ) is a K-parameter then the standard module $I(x, \lambda, 0)$ corresponding to the parameter `(x,lambda,nu=0)` is a tempered irreducible representation of $G(\mathbb{R})$ with real infinitesimal character, as in the theorem.
- It has a unique lowest K-type τ.
- The corresponding `atlas` data type is `K_Type`.
- This gives a formal and clean parametrization of K-types; it works for all cases, including the disconnected case.
- This parametrization can be difficult to understand: What is (are) the highest weight(s) of a K-type parametrized in this way?

K-Parameters

Example ($SL(2,\mathbb{R})$)

Let $G(\mathbb{R}) = SL(2,\mathbb{R})$. Irreducible representations of $SO(2)$ may be given by integers.

- The tempered irreducible representations with real infinitesimal character are the discrete series, the two limits of discrete series, and the spherical principal series at infinitesimal character 0.
- The $SO(2)$-type with weight $k > 0$ may then be assigned to the K-parameter `(x,k-1)` with `x` the `KGB` element 0, since the (limit of) discrete series with LKT k has parameter `(x,k-1,0)`.
- The $SO(2)$-type with weight $k < 0$ may then be assigned to the K-parameter `(x,-k-1)` with `x` the `KGB` element 1.
- The trivial $SO(2)$-type is given by the K-parameter `(x,1)` with `x` the (open) `KGB` element 2.

K-Parameters

Example ($PGL(2,\mathbb{R}) \cong SO(2,1)$)

If $G(\mathbb{R}) = SO(2,1)$ then $K(\mathbb{R}) = O(2)$.

- The $O(2)$-types are the trivial one *triv*, the determinant *det* (both one-dimensional), and there is one for each positive integer k, $\tau(k)$ of dimension 2. The restriction of $\tau(k)$ to $SO(2)$ contains the $SO(2)$-types with weights $\pm k$.
- The irreduclble tempered representations at real inf. char. are discrete series with (positive) half-integral Harish-Chandra parameters, and two minimal principal series with $\nu = 0$.
- The $O(2)$-type $\tau(k)$ then corresponds to the K-parameter `(x,[2k-1]/2)` with `x` the unique closed orbit 0.
- *triv* corresponds to the K-parameter `(x,[1]/2)` with `x` the unique open orbit 1. Recall that $\lambda = \rho$ gives the spherical principal series.
- *det* corresponds to `(x,[3]/2)` with `x` the open orbit.

Lowest K-Types of Standard Modules

- With this parametrization, it is easy to compute the LKTs of a standard (or equivalently, irreducible) module $I(p)$ of $G(\mathbb{R})$.
- Since the K-spectrum of a standard module does not depend on the continuous parameter ν, setting $\nu = 0$ does not change the LKTs.
- However, setting $\nu = 0$ might change the reducibility, and whether the parameter is final or not.
- "Finalizing" the parameter turns it into a sum of final and, if $\nu = 0$, tempered, modules with real infinitesimal characters:

$$I(p_0) = \bigoplus_{i=1}^{r} I(p_i), \quad p_i = (x_i, \lambda_i, 0) \text{ final.}$$

- The list of K-parameters for the LKTs of the original module is obtained by removing the continuous parameter:

$$\{(x_i, \lambda_i) : 1 \leq i \leq r\}.$$

LKTs

Example ($SL(2,\mathbb{R})$)

Let's take the non-spherical principal series representation of $SL(2,\mathbb{R})$ with infinitesimal character $1/2$.

```
atlas> set G=SL(2,R)
Variable G: RealForm
atlas> set p=minimal_principal_series (G,[0],[1]/2)
Value: final parameter(x=2,lambda=[2]/1,nu=[1]/2)
atlas> set p_0=parameter(G,2,[2],[0])
Value: non-final parameter(x=2,lambda=[2]/1,nu=[0]/1)
atlas> set P=finalize(p_0)
Value:
1*parameter(x=1,lambda=[0]/1,nu=[0]/1) [0]
1*parameter(x=0,lambda=[0]/1,nu=[0]/1) [0]
```

Removing ν yields the $SO(2)$-types ± 1.

The `atlas` function is `LKTs`:

Example ($SL(2,\mathbb{R})$ continued)

```
atlas> LKTs(p)
Value: [(KGB element #1,[ 0 ]/1),(KGB element #0,[ 0 ]/1)]
```

Highest Weights

- Parametrizing K-types by standard final tempered limit parameters is clean and rigorous.
- However, it is often more helpful to parametrize K-types by highest weights. In the connected case, this can be done just as precisely.
- In the disconnected case, a K-type will not necessarily have a unique highest weight, and a weight may be a highest weight of more than one K-type.
- Focus on connected groups (much easier).
- `atlas` will compute highest weights of K-types; however, a suitable `KGB` element (associated to the fundamental Cartan subgroup) must be specified.
- As before, a good way to choose this element is by looking at which roots are made compact.

Highest Weights

Definition

A K-highest weight for $G(\mathbb{R})$ is a pair $\mu = (x, \kappa)$ (modulo equivalence as defined below), where

1. $x \in \mathrm{KGB}$, and the associated Cartan subgroup is the fundamental Cartan subgroup;
2. $\kappa \in X^*(H)$.

We define equivalence of K-highest weights to be generated by

1. $(x, \kappa) \equiv (x, \kappa')$ if $\kappa - \kappa' \in (1 - \theta_x)X^*$.
2. $(x, \kappa) \equiv (x, \kappa')$ if $\kappa' = w\kappa$ for some $w \in W(K_0, H_{K_0})$.

The corresponding `atlas` data type is `KHighestWeight`.

Highest Weights

- It is also often helpful to be able to move a highest weight to a different `KGB` element. In general, this may not be well defined; however, for K connected, it is: $(x, \kappa) \to (w \times x, w \cdot \kappa)$ for $w \in W^\delta = \{w \in W : w\delta = \delta w\}$, δ the distinguished involution.
- `atlas` does not consider two `KHighestWeights` related this way equal, but it can move them.
- In the case of a unique highest weight, the function `highest_weight` computes the highest weight of a K-type; one can also specify the desired `KGB` element.
- (In the disconnected case, use the function `highest_weights` instead.)
- If no `KGB` element is specified, the default element is #0.

HIghest Weights

Example ($SL(2,\mathbb{R})$)

For $G(\mathbb{R}) = SL(2,\mathbb{R})$, consider the K-type τ that is the LKT of the holomorphic discrete series at infinitesimal character ρ, with (highest) weight 2:

```
atlas> set G=SL(2,R)
Variable G: RealForm
atlas> set tau=K_Type:(KGB(G,0),[1])
Variable tau: (KGBElt,ratvec)
atlas> tau
Value: (KGB element #0,[ 1 ]/1)

atlas> set mu1=highest_weight(tau)
Variable mu1: (KGBElt,vec)
atlas> set mu2=highest_weight(tau,KGB(G,1))
Variable mu2: (KGBElt,vec)
atlas> mu1
Value: (KGB element #0,[ 2 ])
atlas> mu2
Value: (KGB element #1,[ -2 ])
atlas> mu1=mu2
Value: false
```

How does it work?

- Given a K-parameter `(x,lambda)`, the parameter `p=(x,lambda,nu=0)` corresponds to an irreducible (standard) representation with unique LKT.
- Associated to `p` is a set of θ-stable data (Q, q), where Q is a θ-stable parabolic with (relatively) split Levi $L(\mathbb{R})$, and q is a parameter for a minimal principal series representation X_L of $L(\mathbb{R})$ so that p is obtained from q by cohomological parabolic induction.
- X_L has a unique (one-dimensional, "G-spherical") LKT τ_L with (highest) weight $\mu_L \leftrightarrow (\mathrm{x_K}, \kappa_\mathrm{L})$; see D. Vogan: "Branching to a Maximal Compact Subgroup".
- The highest weight of τ is then obtained by adding $2\rho(\mathfrak{u} \cap \mathfrak{s})$, the sum of the non-compact roots in $\mathfrak{u}$.

Example ($Sp(4,\mathbb{R})$)

Look at one of the large discrete series of $Sp(4,\mathbb{R})$ at infinitesimal character ρ:

```
atlas> set G=Sp(4,R)
Variable G: RealForm
atlas> p:=large_discrete_series (G,rho(G))
Value: final parameter(x=0,lambda=[2,1]/1,nu=[0,0]/1)
atlas> set tau=LKT(p)
Value: (KGB element #0,[ 2, 1 ]/1)

atlas> highest_weight(p)
Value: (KGB element #0,[ 3, 1 ])

atlas> set x_K=KGB(G,2)
atlas> highest_weight(tau,x_K)
Value: (KGB element #2,[  3, -1 ])
```

What is the LKT of the homomorphic d.s.?

```
atlas> set q=parameter(KGB(G,2),rho(G),[0,0])
Variable q: Param
atlas> highest_weight(LKT(q),x_K)
Value: (KGB element #2,[ 3, 3 ])
```

K_0-Parameters

- The restriction of a K-type to the connected component $K_0(\mathbb{R})$ is a sum of irreducible representations of the `atlas` `RealForm` $K_0(\mathbb{R})$.
- The function `K0_params` takes a parameter p for $G(\mathbb{R})$ and returns the list of parameters for K_0 that represent the K_0-types occurring in the LKTs of the module given by p.

Example ($Sp(4, \mathbb{R})$)

The large discrete series on the previous slide had a unique LKT with highest weight $(3, -1)$. Compute the corresponding K_0-parameter (unique since the group is connected):

```
atlas> K0_params(p,x_K)
Value: [final parameter(x=0,lambda=[7,-3]/2,nu=[0,0]/1)]
```

K-Parameters from Highest Weights

- The calculation for finding the K-parameter (irreducible tempered final limit character) from a highest weight is the "Vogan Algorithm" (see the reference mentioned earlier, or Vogan's Green Book).
- This is implemented by the `atlas` function `K_type`.
- Some care is needed: the calculation may require a particular choice for `x_K`.
- Also, in the disconnected case, there may be more than one K-type with the same highest weight. In that case, use `K_types`.

K-Parameters from Highest Weights

Example ($Sp(4,\mathbb{R})$)

Consider the $U(2)$-type with highest weight $(5,1)$.

```
atlas> mu:=(x_K,[5,1])
Value: (KGB element #2,[ 5, 1 ])
atlas> K_type(mu)
Runtime error:
  x does not make 2rho_c dominant
    ....
```

Conjugate this to KGB element #0 by applying s_{α_1}:

```
atlas> mu:=(KGB(G,0),[5,-1])
Value: (KGB element #0,[  5, -1 ])
atlas> K_type(mu)
Value: (KGB element #5,[ 4, 1 ]/1)
```

This module is attached to KGB element #5, i.e., to the mixed Cartan. The corresponding parabolic has Levi factor $L(\mathbb{R}) = SL(2,\mathbb{R}) \times GL(1,\mathbb{R})$:

```
 5:  1   [C,r]    7   5     *   *  (0,0) 2 2^e
```

Branching to K

- There is a built-in function `branch_std` which returns a list of all K-types of a standard module up to a certain "height", in the form of a `ParamPol` with terms the corresponding tempered parameters.
- A corresponding function `branch_irr` does the same for the irreducible representation associated to a parameter.
- One can also print the information in more convenient form, in terms of highest weights, and with the dimension of each K-type included.

Branching to K

Example ($Sp(4,\mathbb{R})$)

Choose p to be parameter #7 in the block of the trivial representation of $Sp(4,\mathbb{R})$. It is attached to the mixed Cartan $U(1) \times \mathbb{R}^\times$.

```
atlas> p
Value: final parameter(x=7,lambda=[2,1]/1,nu=[2,0]/1)
atlas> branch_std(p,8)
Value:
1*parameter(x=2,lambda=[1,0]/1,nu=[0,0]/1) [3]
1*parameter(x=0,lambda=[1,0]/1,nu=[0,0]/1) [3]
1*parameter(x=5,lambda=[2,1]/1,nu=[0,0]/1) [6]
1*parameter(x=4,lambda=[2,1]/1,nu=[0,0]/1) [6]
1*parameter(x=0,lambda=[2,1]/1,nu=[0,0]/1) [7]

atlas> branch_irr(p,8)
Value:
1*parameter(x=2,lambda=[1,0]/1,nu=[0,0]/1) [3]
1*parameter(x=0,lambda=[1,0]/1,nu=[0,0]/1) [3]
atlas> LKTs(p)
Value: [(KGB element #2,[ 1, 0 ]/1),(KGB element #0,[ 1, 0 ]/1)]
```

Branching to K

Example ($Sp(4,\mathbb{R})$ continued)

To see the highest weights of these K-types, with respect to our chosen `x_K`:

```
atlas> x:=KGB(G,2)
Value: KGB element #2
atlas> print_branch_std(p,x,8)
(1+0s)*(KGB element #2,[ 2, 2 ])
(1+0s)*(KGB element #2,[ 2, 0 ])
(1+0s)*(KGB element #2,[ 3, 1 ])
(1+0s)*(KGB element #2,[  2, -2 ])
(1+0s)*(KGB element #2,[  3, -1 ])
```

The "..._long" function also shows $\rho(K)$ in these coordinates, the height, and the dimension of each K-type:

```
atlas> print_branch_std_long(p,x,8)
rho_K=[  1, -1 ]/2
(1+0s)*(KGB element #2,[ 2, 2 ]) 1     3
(1+0s)*(KGB element #2,[ 2, 0 ]) 3     3
(1+0s)*(KGB element #2,[ 3, 1 ]) 3     6
(1+0s)*(KGB element #2,[  2, -2 ]) 5     6
(1+0s)*(KGB element #2,[  3, -1 ]) 5     7
```

The End

Thank You!