# Cohomological Parabolic Induction in `atlas`

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# Recall: Goals

## Tasks

- *Given a real group $G(\mathbb{R})$, a real parabolic subgroup $P(\mathbb{R})$ of $G(\mathbb{R})$ with Levi factor $L(\mathbb{R})$, and $\pi \in \widehat{L(\mathbb{R})}$, use* `atlas` *to find the composition factors of $\text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi \otimes 1)$.*
- *Given a $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ and $X$ an irreducible representation of $L(\mathbb{R})$, compute the composition series of $\mathcal{R}_{\mathfrak{q}}^{S}(X)$.*

**Some Questions:** Given a group $G(\mathbb{R})$ and a $\theta$-stable parabolic subalgebra $\mathfrak{q}$ with associated Levi subgroup $L$,

- Construct various $A_{\mathfrak{q}}(\lambda)$ modules.
- How does the result change as we vary $\lambda$?
- Reducibility? Unitarity? Non-zero?
- What do the modules and their constituents look like?

# Theta Stable Parabolic Induction

- As for real parabolic induction, there are functions `theta_induce_standard` and `theta_induce_irreducible`.
- The algorithm for `theta_induce_irreducible` is similar to the one for real induction, except:
- Now the unipotent radical of the parabolic matters, in addition to the Levi factor.
- Embedding the parameter for $L(\mathbb{R})$ in $G(\mathbb{R})$ may result in a non-standard parameter; i.e., it may be non-dominant for the imaginary roots. In this case, coherent continuation is used to make the parameter dominant.
- The resulting module may be zero.

# Theta Stable Parabolic Subalgebras

**Recall:**

- Parabolic subgroups and subalgebras for $G(\mathbb{R})$ are parametrized by `KGP` elements, which in `atlas` are pairs `(S,x)` of a set `S` of simple roots defining the complex parabolic subgroup, and a `KGB` element `x` giving an S-equivalence class (defined by cross actions and Cayley transforms through roots in `S`).
- Then `(S,x)` parametrizes a $\theta$-stable parabolic subalgebra of $\mathfrak{g}$ if and only if the equivalence class contains a closed orbit (provided `x` preserves the Levi factor).
- The function `parabolic(lambda,x)` will produce a $\theta$-stable parabolic if `x` is a `KGB` element attached to the fundamental Cartan (and in the distinguished fiber). The weight $\lambda$ must be fixed by the involution $\theta_x$.

# Theta Stable Parabolic Subalgebras

## Example ($U(2,2)$)

Define the $\theta$-stable parabolic subalgebra with $L(\mathbb{R}) = U(2,1) \times U(0,1)$: Choose a "nice" KGB element x_K; then check whether it is suitable by looking at the corresponding compact $\rho$-shift:

```
atlas> set G=U(2,2)
Variable G: RealForm
atlas> simple_roots(G)
Value:
|  1,  0,  0 |
| -1,  1,  0 |
|  0, -1,  1 |
|  0,  0, -1 |
atlas> print_KGB(G)
 0:  0  [n,n,n]    1   2   3    10   8   6  (0,0,0,0)#0 e
 1:  0  [n,c,n]    0   1   4    10   *   7  (1,1,0,0)#0 e
 2:  0  [c,n,c]    2   0   2     *   8   *  (0,1,1,0)#0 e
 3:  0  [n,c,n]    4   3   0    11   *   6  (0,0,1,1)#0 e
 4:  0  [n,n,n]    3   5   1    11   9   7  (1,1,1,1)#0 e
 5:  0  [c,n,c]    5   4   5     *   9   *  (1,0,0,1)#0 e
 ........
```

What is a "nice" x_K?

# Theta Stable Parabolic Induction

## Example ($U(2,2)$ continued)

```
atlas> set x_K=KGB(G,2)
Variable x_K: KGBElt
atlas> rho_c(x_K)
Value: [  1, -1,  1, -1 ]/2

atlas> set P=parabolic([1,1,1,0],x_K)
Parabolic is theta-stable.
Variable P: ([int],KGBElt)
atlas> P
Value: ([0,1],KGB element #2)
atlas> set L=Levi(P)
Variable L: RealForm
atlas> L
Value: connected quasisplit real group with Lie algebra
                                                    'su(2,1).u(1).u(1)'
```

# Theta Stable Parabolic Induction

## Example ($U(2,2)$ continued)

Use this $\mathfrak{q}$ to compute some cohomologically induced modules:

```
atlas> p:=finite_dimensional(L,[2,1,1,-4])
Value: final parameter(x=5,lambda=[3,1,0,-4]/1,nu=[3,0,-3,0]/2)
atlas> dimension(p)
Value: 3

atlas> theta_induce_irreducible(p,G)
Value:
1*parameter(x=15,lambda=[7,3,1,-11]/2,nu=[3,0,-3,0]/2) [23]

atlas> p:=finite_dimensional(L,[2,1,1,4])
Value: final parameter(x=5,lambda=[3,1,0,4]/1,nu=[3,0,-3,0]/2)
atlas> theta_induce_irreducible(p,G)
Value:
1*parameter(x=20,lambda=[7,5,3,1]/2,nu=[3,1,-1,-3]/2) [0]
1*parameter(x=18,lambda=[7,5,3,1]/2,nu=[3,0,0,-3]/2) [3]
1*parameter(x=13,lambda=[7,5,3,1]/2,nu=[0,1,0,-1]/1) [6]
```

# Theta Stable Parabolic Induction

- Of special interest are $A_{\mathfrak{q}}(\lambda)$ modules; i.e., modules that are $\theta$-induced from one-dimensionals of $L(\mathbb{R})$.
- We can define them in `atlas` in two different ways.
- We can use `theta_induce_irreducible` as above, just choosing the parameter on $L(\mathbb{R})$ to be one-dimensional.
- There is also a function command `Aq(x,lambda,lambda_q)` which can be used to define $A_{\mathfrak{q}}(\lambda)$ modules.
- Here `x` is a (fundamental) `KGB` element, `lambda_q` defines the parabolic subalgebra, and `lambda` defines the one-dimensional representation of $L(\mathbb{R})$.
- However, the normalization is different: there is a shift of $\rho(\mathfrak{u})$.
- This also affects the integrality needed for `lambda`: `lambda-rho_u(P)` must be integral.

# Theta Stable Parabolic Induction

## Example ($U(2,2)$ continued)

Continue with $\mathfrak{q}$ with $L(\mathbb{R}) = U(2,1) \times U(0,1)$. Start with $A_{\mathfrak{q}}(0)$:

```
atlas> theta_induce_irreducible(trivial(L),G)
Value:
1*parameter(x=15,lambda=[3,1,-1,-3]/2,nu=[1,0,-1,0]/1) [6]


atlas> rho_u(P)
Value: [  1,   1,   1, -3 ]/2
atlas> Aq(x_K,[1,1,1,-3]/2,[1,1,1,0])
Value: final parameter(x=15,lambda=[3,1,-1,-3]/2,nu=[1,0,-1,0]/1)
```

- Both functions yield the same module. The shift is $\rho(\mathfrak{u})$.
- The `Aq` function returns a parameter, while `theta_induce_irreducible` returns a `ParamPol`. This causes trouble when the module is not irreducible.

Before we continue, recall what Peter said about the relative positions of the parabolic and the representation for this construction.

# Theta Stable Parabolic Induction

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a $\theta$-stable parabolic subalgebra of $\mathfrak{g}$, and let $X$ be an $(\mathfrak{l}, L \cap K)$-module with infinitesimal character $\gamma_L$.

- Recall that $X$ is in the **good** range for $\mathfrak{q}$ if for all $\alpha \in \Delta(\mathfrak{u})$,

$$Re\langle \gamma_L + \rho(\mathfrak{u}), \alpha^\vee \rangle > 0$$

- $X$ is in the **weakly good** range if for all $\alpha \in \Delta(\mathfrak{u})$,

$$Re\langle \gamma_L + \rho(\mathfrak{u}), \alpha^\vee \rangle \geq 0.$$

- In the good range, $\theta$-stable induction takes irreducible modules to irreducible modules, and **preserves unitarity and non-unitarity** of irreducible modules.
- In the weakly good range, $\theta$-stable induction takes an irreducible module to an irreducible module or 0, and **preserves unitarity**.

# Theta Stable Parabolic Induction

## Example ($U(2,2)$)

Recall the parabolic with $L(\mathbb{R}) = U(2,1) \times U(0,1)$, and the two finite dimensional representations of $L(\mathbb{R})$.

```
atlas> p:=finite_dimensional(L,[2,1,1,-4])
Value: final parameter(x=5,lambda=[3,1,0,-4]/1,nu=[3,0,-3,0]/2)
atlas> is_weakly_good(p,G)
Value: true
atlas> goodness(p,G)
Value: "Good"
```

This module was irreducible when $\theta$-induced to $G(\mathbb{R})$.

```
atlas> p:=finite_dimensional(L,[2,1,1,4])
Value: final parameter(x=5,lambda=[3,1,0,4]/1,nu=[3,0,-3,0]/2)
atlas> goodness(p,G)
Value: "None"
```

The $\theta$-induced module had three composition factors.

# Theta Stable Parabolic Induction

For $A_{\mathfrak{q}}(\lambda)$ modules, we can also talk about "fair" and "weakly fair":

- The module $A_{\mathfrak{q}}(\lambda) = \mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}_\lambda)$ is in the **fair** range if for all $\alpha \in \Delta(\mathfrak{u})$,
$$\langle \lambda + \rho(\mathfrak{u}), \alpha^\vee \rangle > 0.$$
- The module $A_{\mathfrak{q}}(\lambda) = \mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}_\lambda)$ is in the **weakly fair** range if for all $\alpha \in \Delta(\mathfrak{u})$,
$$\langle \lambda + \rho(\mathfrak{u}), \alpha^\vee \rangle \geq 0.$$
- In terms of the `atlas` definition `Aq(x,lambda,lambda_q)`, the module is weakly fair if for all $\alpha \in \Delta(\mathfrak{u})$,
$$\langle \lambda, \alpha^\vee \rangle \geq 0.$$
- A weakly fair $A_{\mathfrak{q}}(\lambda)$ module is either 0 or unitary.

# Theta Stable Parabolic Induction

For $A_{\mathfrak{q}}(\lambda)$ modules, the goodness of the module `Aq(x,lambda,lambda_q)` can be determined by the command `goodness(x,lambda,lambda_q)`.

### Example ($Sp(4,\mathbb{R})$)

Consider $A_{\mathfrak{q}}(\lambda)$ modules with $\mathfrak{q}$ the parabolic with $L(\mathbb{R}) = U(1,0) \times SL(2,\mathbb{R})$.

```
atlas> G:=Sp(4,R)
atlas> x:=KGB(G,2)
atlas> P:=parabolic([1,0],x)
Parabolic is theta-stable.
Value: ([1],KGB element #2)
atlas> Levi(P)
Value: connected quasisplit real group with Lie algebra
                                                 'sl(2,R).u(1)'
atlas> rho_u(P)
Value: [ 2, 0 ]/1
```

# Theta Stable Parabolic Induction

**Example ($Sp(4,\mathbb{R})$ continued)**

Start with $A_{\mathfrak{q}}(0)$ (at infinitesimal character $\rho$):

```
atlas> Aq(x,[2,0],[1,0])
Value: final parameter(x=5,lambda=[2,1]/1,nu=[0,1]/1)
atlas> goodness(x,[2,0],[1,0])
Value: "Good"

atlas> Aq(x,[1,0],[1,0])
Value: final parameter(x=5,lambda=[1,1]/1,nu=[0,1]/1)
atlas> goodness(x,[1,0],[1,0])
Value: "Weakly good"

atlas> Aq(x,[0,0],[1,0])
Runtime error:
  Aq is not irreducible. Use Aq_reducible(x,lambda) instead
              ...
Evaluation aborted.
```

When the module is reducible, use the function `Aq_reducible` instead.

# Theta Stable Parabolic Induction

## Example ($Sp(4,\mathbb{R})$ continued)

```
atlas> goodness(x,[0,0],[1,0])
Value: "Weakly fair"

atlas> Aq_reducible(x,[0,0],[1,0])
Value:
1*parameter(x=7,lambda=[3,0]/1,nu=[1,0]/1) [0]
1*parameter(x=2,lambda=[1,0]/1,nu=[0,0]/1) [3]
```

The module is reducible! (Recall from Peter's talk that for $U(p,q)$, weakly fair $A_{\mathfrak{q}}(\lambda)$ modules are either irreducible or zero.)
Let's keep going:

```
atlas> goodness(x,[-1,0],[1,0])
Value: "None"
atlas> set q=Aq(x,[-1,0],[1,0])
Value: final parameter(x=5,lambda=[1,1]/1,nu=[0,1]/1)

atlas> is_unitary(q)
Value: true
```

We'll come back to this module.

# Theta Stable Induction

## Example ($Sp(4, \mathbb{R})$ continued)

Let's go one step further into the bad range:

```
atlas> Aq_reducible(x,[-2,0],[1,0])
Value:
-1*parameter(x=10,lambda=[2,1]/1,nu=[2,1]/1) [0]
1*parameter(x=5,lambda=[2,1]/1,nu=[0,1]/1) [6]
```

Here is what it looks like when the module is zero:

## Example ($Sp(4, \mathbb{R})$: A zero module)

```
atlas> Aq_reducible (x,[0,0],[2,1])
Value: Empty sum of standard modules
```

Here the $\theta$-stable parabolic subalgebra is a Borel (since `lambda_q` is regular). The module is zero because there is a compact simple root that is orthogonal to $\lambda$.

```
atlas> goodness(x,[0,0],[2,1])
Value: "Weakly good"
```

# Detecting $A_{\mathfrak{q}}(\lambda)$ Modules

- Given a parameter $p$ of a group $G(\mathbb{R})$, you might want to know whether it is the parameter for a good or weakly fair $A_{\mathfrak{q}}(\lambda)$ module.
- In the weakly fair case, an irreducible module might be just a constituent of a reducible $A_{\mathfrak{q}}(\lambda)$ module.
- There are three different commands to accomplish this: `is_good_Aq`, `is_weakly_good_Aq`, and `is_weakly_fair_Aq`, all returning "true" or "false".
- The last of the three essentially works by constructing all possible weakly fair $A_{\mathfrak{q}}(\lambda)$ modules with the given infinitesimal character, and comparing. This is obviously quite slow.

# Detecting $A_{\mathfrak{q}}(\lambda)$ Modules

## Example ($Sp(4,\mathbb{R})$)

Recall that we had a weakly fair $A_{\mathfrak{q}}(\lambda)$ module of $Sp(4,\mathbb{R})$ with two constituents. Let's try the function on one of the summands:

```
atlas> set p=parameter(G,7,[3,0],[1,0])
Variable p: Param
atlas> is_weakly_fair_Aq(p)
Value: true
```

In this case, we can find out the parabolic, and the parameter on $L(\mathbb{R})$. There could be more than one, so the output is a list of such pairs:

```
atlas> is_wf_induced_from_one_dim (p)
Weakly fair Aq
Value: [(([1],KGB element #5),
                final parameter(x=2,lambda=[-2,1]/1,nu=[0,1]/1))]
```

# Detecting $A_\mathfrak{q}(\lambda)$ Modules

Recall the “bad” example:

## Example ($Sp(4, \mathbb{R})$)

```
atlas> q:=Aq(x,[-1,0],[1,0])
Value: final parameter(x=5,lambda=[1,1]/1,nu=[0,1]/1)
atlas> goodness(x,[-1,0],[1,0])
Value: "None"

atlas> is_weakly_fair_Aq(q)
Value: true
```

So although the module was not constructed as a weakly fair $A_\mathfrak{q}(\lambda)$, it is equivalent to one. Which one?

# Detecting $A_{\mathfrak{q}}(\lambda)$ Modules

## Example ($Sp(4, \mathbb{R})$ continued)

```
atlas> is_wf_induced_from_one_dim (q)
Weakly fair Aq
Value: [(([1],KGB element #5),
       final parameter(x=2,lambda=[-1,1]/1,nu=[0,1]/1))]
```

Unraveling this information, we see that this module is actually the weakly good one we had earlier:

```
atlas> is_weakly_good_Aq(q)
Value: true
atlas> q=Aq(x,[1,0],[1,0])
Value: true
```

# Detecting $A_{\mathfrak{q}}(\lambda)$ Modules

Deciding whether a parameter corresponds to a (weakly) good $A_{\mathfrak{q}}(\lambda)$ module is much faster.

- If $X$ is a cohomologically induced module in the good range with parameter $p = (\mathtt{x}, \mathtt{lambda}, \mathtt{nu})$, it is easy to determine the parameter $p_L$ on $L(\mathbb{R})$ from...provided that we know which parabolic. Just subtract some $\rho$ shift (and inverse-embed the `KGB` element `inverse_embed_KGB`) .
- It is also not hard to determine a (canonical) parabolic subalgebra for which there is such a parameter ("good range reduction").
- The **support** of a `KGB` element `x` is the (minimal) set $S$ of simple roots so that $\mathtt{x} \sim_S \mathtt{y}$ for a closed `KGB` orbit `y`.
- This is well defined.
- It is clear from the definition that the pair `(S,x)` then defines a $\theta$-stable parabolic subalgebra $\mathfrak{q}(x)$.

# Detecting $A_{\mathfrak{q}}(\lambda)$ Modules

## Example ($Sp(4,\mathbb{R})$)

```
atlas>print_KGB(G)
 0:  0  [n,n]    1   2     4   5  (0,0)#0 e
 1:  0  [n,n]    0   3     4   6  (1,1)#0 e
 2:  0  [c,n]    2   0     *   5  (0,1)#0 e
 3:  0  [c,n]    3   1     *   6  (1,0)#0 e
 4:  1  [r,C]    4   9     *   *  (0,0) 1 1^e
 5:  1  [C,r]    7   5     *   *  (0,0) 2 2^e
 6:  1  [C,r]    8   6     *   *  (1,0) 2 2^e
 7:  2  [C,n]    5   8     *  10  (0,0)#2 1x2^e
 8:  2  [C,n]    6   7     *  10  (0,1)#2 1x2^e
 9:  2  [n,C]    9   4    10   *  (0,0)#1 2x1^e
10:  3  [r,r]   10  10     *   *  (0,0)#3 1^2x1^e
```

The closed orbits are 0, 1, 2, and 3. They all have support the empty set.

```
atlas> support(KGB(G,4))
Value: [0]

atlas> support(KGB(G,5))
Value: [1]

atlas> support(KGB(G,7))
Value: [0,1]
```

# Detecting $A_{\mathfrak{q}}(\lambda)$ Modules

- The support is the set of roots appearing in the last column of `KGB`.
- Some `KGB` elements, e.g., the maximal one, have support the set of all simple roots. The corresponding parabolic subalgebra is then all of $\mathfrak{g}$

**Theorem (Vogan)**

*Let $X$ be the irreducible module given by the parameter $p = (\mathrm{x}, \mathrm{lambda}, \mathrm{nu})$. Let $S$ be the support of* `x`*, and $\mathfrak{q}(x)$ the parabolic given by the pair* `(S,x)`*, with Levi factor $L(\mathbb{R})$. Then $X$ is cohomologically induced, in the weakly good range, from an irreducible module on $L(\mathbb{R})$.*

- The parameter $p_L$ on $L(\mathbb{R})$ can then easily be computed.
- To see whether $X$ is a weakly good $A_{\mathfrak{q}}(\lambda)$ module, check whether $p_L$ is the parameter of a unitary character. For “good”, also check that the infinitesimal character of $X$ is regular.

# Detecting $A_{\mathfrak{q}}(\lambda)$ Modules

## Example ($Sp(4,\mathbb{R})$)

```
atlas> set p1=Aq(x,[2,0],[1,0])
Variable p1: Param
atlas> set p2=Aq(x,[1,0],[1,0])
Variable p2: Param

atlas> is_good_Aq(p1)
Value: true
atlas> is_good_Aq(p2)
Value: false
atlas> is_weakly_good_Aq(p2)
Value: true
```

The function `reduce_good_range` identifies the parabolic and the parameter as given by the theorem (the name `reduce_weakly_good_range` seemed too awkward):

```
atlas> reduce_good_range (p1)
Value: (([1],KGB element #5),
       final parameter(x=2,lambda=[0,1]/1,nu=[0,1]/1))

atlas> reduce_good_range (p2)
Value: (([1],KGB element #5),
       final parameter(x=2,lambda=[-1,1]/1,nu=[0,1]/1))
```

# Good Range Reduction

## Example ($Sp(4,\mathbb{R})$)

Return one more time to the reducible weakly fair $A_{\mathfrak{q}}(\lambda)$ module of $Sp(4,\mathbb{R})$, to get more information about the two constituents:

```
atlas> Aq_reducible(x,[0,0],[1,0])
Value:
1*parameter(x=7,lambda=[3,0]/1,nu=[1,0]/1) [0]
1*parameter(x=2,lambda=[1,0]/1,nu=[0,0]/1) [3]
atlas> set p1=parameter(G,7,[3,0],[1,0])
Variable p1: Param
atlas> reduce_good_range (p1)
Value: (([0,1],KGB element #7),
                 final parameter(x=7,lambda=[3,0]/1,nu=[1,0]/1))
```

This module may not be reduced in the (weakly) good range. (It is attached to `KGB` element 7, which has support $\{0,1\}$, and to the mixed Cartan subgroup.)

```
atlas> set p2=parameter(G,2,[1,0],[0,0])
Variable p2: Param
atlas> reduce_good_range(p2)
Value: (([],KGB element #2),
               final parameter(x=0,lambda=[-1,-1]/1,nu=[0,0]/1))
```

Of course, `KGB` element #2 is a closed orbit, so this is a limit of discrete series representation.

# Applications of Good Range Reduction

Recall that a representation of $G(\mathbb{R})$ with infinitesimal character $\gamma$ is called **strongly regular** if $\gamma$ is real and "at least as regular as $\rho$"; i.e., if for all positive roots $\alpha$,

$$\langle \gamma - \rho, \alpha^\vee \rangle \geq 0.$$

Here $\gamma$ is chosen in the same Weyl chamber as $\rho$.

**Theorem (Salamanca-Riba)**

*If $X$ is strongly regular then $X$ is unitary if and only if it is a good $A_{\mathfrak{q}}(\lambda)$ module.*

- This suggests a quick test (without the heavy machinery of computing hermitian forms) for unitarity in the strongly regular case.
- Both tests (checking for "strongly regular" and `is_good_Aq`) are very fast.

# Applications of Good Range Reduction

So here is an easy unitarity test:

```
set is_unitary_easy(Param p)=string:
   if is_strongly_regular(p) then
     if is_good_Aq(p) then "true"  else "false" fi
       else "This is too hard." fi
```

**Example ($Sp(4,\mathbb{R})$)**

```
atlas> is_unitary_easy(trivial(G))
Value: "true"

atlas> p1:=finite_dimensional(G,[3,1])
Value: final parameter(x=10,lambda=[3,2]/1,nu=[5,2]/1)

atlas> is_unitary_easy(p1)
Value: "false"

atlas> p2:=Aq(x,[1,0],[1,0])
Value: final parameter(x=5,lambda=[1,1]/1,nu=[0,1]/1)

atlas> is_unitary_easy(p2)
Value: "This is too hard."
```

# Applications of Good Range Reduction

More generally, since **the hermitian form is preserved** by induction in the (weakly) good range, good range reduction can reduce this calculation to a computation on the smaller group:

If $X = \mathcal{R}_{\mathfrak{q}}^{S}(X_L)$ in the weakly good range, and `HF_L` is the `ParamPol` representing the hermitian form for $X_L$, then `theta_induce_standard(HF_L,G)` represents the hermitian form on $X$.