The Langlands Classification
Background for the Atlas workshop 2017

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1 Representations of $SL(2,\mathbb{R})$

Elsewhere, and in the background reading, we discuss how to go from representations of a group on a Hilbert space, to algebraic representations, or $(\mathfrak{g},K)$-modules. Once in the setting of $(\mathfrak{g},K)$-modules it is possible to do some explicit calculations. The case of $SL(2,\mathbb{R})$ is very informative, and we will treat this in some detail in the lectures. Here is some background.

We take $G=SL(2,\mathbb{R})$, with Lie algebra $\mathfrak{g}_0=\mathfrak{sl}(2,\mathbb{R})$, the two-by-two real matrices of trace 0.

Set $\mathfrak{g}=\mathfrak{g}_0\otimes\mathbb{C}=\mathfrak{sl}(2,\mathbb{C})$. Let $t(e^{i\theta})=\begin{pmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{pmatrix}$, so t is an isomorphism from the circle to the subgroup $K=\{g\in G\mid g\,{}^tg=I\}=SO(2)\subset SL(2,\mathbb{R})$. This is a maximal compact subgroup of G. We identity the irreducible representation $\widehat{K}$ of K with $\mathbb{Z}$. We assume familiarity with representations of compact groups.

Define the following basis of $\mathfrak{g}$:

$$(1.1)\qquad E=\frac{1}{2}\begin{pmatrix}1&i\\i&-1\end{pmatrix},\quad F=\frac{1}{2}\begin{pmatrix}1&-i\\-i&-1\end{pmatrix},\quad H=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

Exercise 1.2 Check that H,E,F satisfy the familiar identities $[H,E]=2E,[H,F]=-2F,[E,F]=H$. Also iH is a basis of $\mathfrak{k}_0=\mathrm{Lie}(K)\subset\mathfrak{g}_0$.

Definition 1.3 *For $\nu\in\mathbb{C},\epsilon\in\mathbb{Z}/2\mathbb{Z}$ define a vector space $I(\nu,\epsilon)$, with representations of $\mathfrak{g}$ and K as follows. It has a basis $\{v_j\mid j\in\epsilon+2\mathbb{Z}\}$. The action of K is:*

$$(1.4)(\mathrm{a})\qquad \pi(t(e^{i\theta}))(v_k)=e^{ik\theta}v_k$$

The action of $\mathfrak{g}$ is given by

$$\pi(H)v_j = jv_j \tag{1.4)(b}$$

$$\pi(E)v_j = 1/2(\nu + (j+1))v_{j+2} \tag{1.4)(c}$$

$$\pi(F)v_j = 1/2(\nu - (j-1))v_{j-2} \tag{1.4)(d}$$

Exercise 1.5 Verify that $I(\nu, \epsilon)$ is a $(\mathfrak{g}, K)$-module. This means:

(1) This is a representation of $\mathfrak{g}$ and K;

(2) Every vector is K-finite: $\dim\langle \pi(k)v \mid k \in K\rangle < \infty$ for all $v \in V$;

(3) The representation of $\mathfrak{g}$, restricted to $\mathfrak{k}_0$, is the differential of the representation of K.

(We have omitted a fourth condition which is not necessary since K is connected.)

We say a $(\mathfrak{g}, K)$-module is irreducible if it does not contain a proper subspace W, invariant under the action of $\mathfrak{g}$ and K.

Lemma 1.6 *$I(\nu, \epsilon)$ is reducible if and only if*

$$\nu \in \epsilon + 2\mathbb{Z} + 1. \tag{1.7}$$

Lemma 1.8 *Suppose $\nu = n \in \epsilon + 2\mathbb{Z} + 1$ with $n \geq 0$. Then $I(\nu, \epsilon)$ has two infinite dimensional subrepresentatios*

$$I_+(\nu, \epsilon) = \langle v_{n+1}, v_{n+3}, \dots \rangle$$
$$I_-(\nu, \epsilon) = \langle v_{-n-1}, v_{-n-3}, \dots \rangle$$

Furthermore $V/(I_+ \oplus I_-)$ is an irreducible finite dimensional representation $I_0(\nu, \epsilon)$ of dimension n with basis $v_{-n+1}, v_{-n+3}, \dots, v_{n-1}$ (the image of these vectors in the quotient).

In other words there is an exact sequence of $(\mathfrak{g}, K)$-modules:

$$0 \to I_+(\nu, \epsilon) \oplus I_-(\nu, \epsilon) \to I(\nu, \epsilon) \to I_0(\nu, \epsilon) \to 0$$

Furthermore:

$$I_\pm(\nu, \epsilon) \simeq I_\pm(-\nu, \epsilon)$$
$$I_0(\nu, \epsilon) \simeq I_0(-\nu, \epsilon)$$

An important special case is $\epsilon = 1, \nu = 0$, in which case $V = I_+ \oplus I_-$ and I_0 vanishes.

Lemma 1.9 *Suppose $\nu = n \in \epsilon + 2\mathbb{Z} + 1$, and $n \leq 0$. Then $I(\nu, \epsilon)$ has a finite dimensional subrepresentation $I_0(\nu, \epsilon) = \langle v_{n+1}, v_{n+3}, \dots, v_{-n-1}\rangle$ of dimension*

n (vanishing if $n = 0$), and V/I_0 is the direct sum of two irreducible, infinite dimensional representations:

$$I_+(\nu, \epsilon) = \langle v_{-n+1}, v_{-n+3}, \dots \rangle$$
$$I_-(\nu, \epsilon) = \langle v_{n-1}, v_{n-3}, \dots \rangle$$

In other words there is an exact sequence

$$0 \to I_0(\nu, \epsilon) \to I(\nu, \epsilon) \to I_+(\nu, \epsilon) \oplus I_-(\nu, \epsilon) \to 0$$

Exercise 1.10 Prove both of these Lemmas. The main point is that $I(\nu, \epsilon)$ is irreducible if and only if $Ev_j \neq 0$ and $Fv_j \neq 0$ for all j. Furthermore $Ev_n = 0$ if and only if $Fv_{-n} = 0$, and the behavior of the submodules/quotients depends on whether ν is positive or negative.

Here are a few useful formulas.

$$\begin{aligned} \pi(E)v_{j-2} &= 1/2(\nu + (j-1))v_j \\ \pi(F)v_{j+2} &= 1/2(\nu - (j+1))v_j \\ \pi(E)\pi(F)v_j &= \frac{1}{4}(\nu^2 - (j-1)^2)v_j \\ \pi(F)\pi(E)v_j &= \frac{1}{4}(\nu^2 - (j+1)^2)v_j \end{aligned} \tag{1.11}$$

Let C be the Casimir element:

$$\begin{aligned} C &= H^2 + 2EF + 2FE + 1 \\ &= H^2 + 2H + 4FE + 1 \\ &= H^2 - 2H + 4EF + 1 \end{aligned} \tag{1.12}$$

This is an element of the universal enveloping algebra of $\mathfrak{g}$, and it acts on a representation using any one of these formulas, for example $\pi(C)(v) = (\pi(H)^2 + 2\pi(E)\pi(F) + 2\pi(F)\pi(E) + 1)v$.

Exercise 1.13 Show that $\pi(C)\pi(X) = \pi(X)\pi(C)$ for all $X \in \mathfrak{g}$, and that given ν, ϵ,

$$\pi(C)(v) = \nu^2 v \quad \text{for all } v \in I(\nu, \epsilon).$$

In fact C is in the center of the universal enveloping algebra.

Theorem 1.14 *Consider the irreducible $(\mathfrak{g}, K)$-modules:*

(a) $I(\nu, \epsilon) \quad (\nu \notin \epsilon + 2\mathbb{Z} + 1)$*;*

(b) $I_0(n, \epsilon) \quad (n = 1, 2, 3, \dots)$

(c) $I_\pm(n, \epsilon) \quad (n = 0, 1, 2, 3 \dots)$

In (a) $I(\nu,\epsilon) \simeq I(-\nu,\epsilon)$; there are no other isomorphisms between these modules. Every irreducible $(\mathfrak{g},K)$ module is isomorphic to one of these.

Most of this is straightforward, except for the fact that every irreducible representation is isomorphic to one of these. See [5, Proposition 1.2.14].

Define an action of K on the algebraic dual space $\mathrm{Hom}(V,\mathbb{C})$ as usual: $\pi(k)(f)(v) = f(\pi(k^{-1})v)$. Let V^* be the K-finite vectors. Then $\mathfrak{g}$ acts on V^* as usual: $\pi(X)(f)(v) = -f(\pi(X)v)$.

Exercise 1.15 Show that V^* is a $(\mathfrak{g},K)$-module. What is $I(\nu,\epsilon)^*$? Include the cases when this module is reducible. What about $I_0(\nu,\epsilon)^*$ and $I_\pm(\nu,\epsilon)^*$?

2 Hermitian forms

Given a $(\mathfrak{g},K)$-module (π,V) for $SL(2,\mathbb{R})$, suppose $(,)$ is a Hermitian form on the complex vector space V. The natural notion of invariance under the action of the Lie algebra is

$$(\pi(X)v,w) + (v,\pi(X)w) = 0 \quad (X \in \mathfrak{g}_0). \tag{2.1)(a}$$

It is essential that X is in the *real* Lie algebra. To give a valid formula on $\mathfrak{g}$ observe that $\pi(iX) = i\pi(X)$, and (a) implies

$$(\pi(iX)v,w) + (v,\pi(-iX)w) = i(\pi(X)v,w) + \overline{-i}(v,\pi(X)w) = 0 \tag{2.1)(b}$$

In other words if σ is complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, then (a) is equivalent to

$$(\pi(X)v,w) + (v,\pi(\sigma(X))w) = 0 \quad (X \in \mathfrak{g}). \tag{2.1)(c}$$

Definition 2.2 *An invariant Hermitian form on a $(\mathfrak{g},K)$-module (π,V) is a Hermitian form satisfying*

$$\begin{aligned}(\pi(k)v,\pi(k)w) &= (v,w) \quad (k \in K)\\ (\pi(X)v,w) + (v,\pi(\sigma(X))w) &= 0 \quad (X \in \mathfrak{g}).\end{aligned}$$

Note that $K = K(\mathbb{R}) = SO(2)$. It is natural to replace K with $K(\mathbb{C}) = \mathbb{C}^*$, in which case the first formula would have an extra σ on the right. We don't do this here.

Exercise 2.3 Show that the invariance condition is equivalent to

$$\begin{aligned}(\pi(E)v,w) &= -(v,\pi(F)w)\\ (\pi(F)v,w) &= -(v,\pi(E)w)\\ (\pi(H)v,w) &= (v,\pi(H)w)\end{aligned} \tag{2.4}$$

Consider the module $I(\nu,\epsilon)$ with $\nu \in \mathbb{R}$, and basis $\{v_j\}$.

Exercise 2.5 Suppose $(,)$ is an invariant Hermitian form on $I(\nu,\epsilon)$. Then $(v_j, v_k) = 0$ for $j \neq k$.

Exercise 2.6 Suppose $I(\nu,\epsilon)$ is irreducible, and $I(\nu,\epsilon)$ has a invariant form. Show that (2.4) and (1.4)(b-d) imply for all j

$$(v_{j+2}, v_{j+2}) = \frac{(-\overline{\nu} + (j+1))}{(\nu + (j+1))}(v_j, v_j). \tag{2.7}$$

Using the fact that $(v,v) \in \mathbb{R}$ for all v implies $I(\nu,\epsilon)$ supports an invariant Hermitian form if and only if $\nu \in i\mathbb{R} \cup \mathbb{R}$.

(1) If $\nu \in i\mathbb{R}$ this form can be taken to be identically 1.

(2) If $\nu \in \mathbb{R}$ and $\epsilon = 0$ this form can be uniquely normalized so that $(v_0, v_0) = 1$.

(3) If $\nu \in \mathbb{R}^*$ and $\epsilon = 1$ then $(v_{-1}, v_{-1}) = -(v_1, v_1)$.

The irreducibility assumption is only to avoid the denominator being 0. With a little more care this can be dropped.

The representations with $\nu \in i\mathbb{R}$ are the tempered, unitary principal series. The important case is $\nu \in \mathbb{R}$ (the case of real infinitesimal character).

The last case in the exercise illustrates a crucial problem: there is no canonical way to normalize the form on the lowest K-types $v_{\pm 1}$ in this case.

The remedy is to use a modification of the invariant Hermitian form. Define

$$\sigma_c(X) = -{}^t\overline{X}$$

Then $\mathfrak{g}^{\sigma_c} = \mathfrak{su}(2)$, the Lie algebra of the compact group $SU(2)$, the subscript stands for compact. Notice that $\sigma_c(\mathrm{Lie}(K)) = \mathrm{Lie}(K)$.

Definition 2.8 *A c-invariant Hermitian form on a $(\mathfrak{g}, K)$-module (π, V) is a Hermitian form $(,)_c$ satisfying*

$$(\pi(k)v, \pi(k)w)_c = (v,w)_c \quad (k \in K)$$
$$(\pi(X)v, w)_c + (v, \pi(\sigma_c(X))w)_c = 0 \quad (X \in \mathfrak{g}).$$

Exercise 2.9 Suppose $I(\nu,\epsilon)$ is irreducible, and $I(\nu,\epsilon)$ has a c-invariant form. Show that (2.4) and (1.4)(b-d) imply for all j

$$(v_{j+2}, v_{j+2})_c = \frac{(\overline{\nu} - (j+1))}{(\nu + (j+1))}(v_j, v_j)_c \tag{2.10}$$

Conclude that if $\nu \in \mathbb{R}$ then $I(\nu,\epsilon)$ supports a unique c-invariant Hermitian form normalized so that $(v_0, v_0)_c = 1$ $(\epsilon = 0)$ or $(v_1, v_1)_c = (v_{-1}, v_{-1})_c = 1$ $(\epsilon = 1)$.

Exercise 2.11 Show that the n-dimensional irreducible representation has a positive definite c-invariant Hermitian form. It supports an invariant Hermitian form, which is not positive definite unless $n = 1$.

Exercise 2.12 Suppose $\nu > 0$, $\nu \notin \mathbb{Z}$.

(1) Show that $I(\nu, 1)$ has an invariant Hermitian form, which is not positive definite.

(2) Show that $I(\nu, 0)$ has an invariant Hermitian form, which is positive definite if and only if $\nu < 1$.

The representations (2) are the complementary series for $SL(2, \mathbb{R})$.

3 Tori

An important role is played by real algebraic tori and their representations. We discuss representations of real algebraic tori. For background on algebraic groups see [4], [2] or [3]. and [1, Section 3] for the representations of tori.

Start with a complex torus $H = H(\mathbb{C}) \simeq \mathbb{C}^{*n}$. Let X^* be the holomorphic characters of H. If $n = 1$ these are the maps $z \to z^n$ ($n \in \mathbb{Z}$). Let X_* be the algebraic co-characters, i.e. holomorphic group homomorphisms $\mathbb{C}^* \to H$. Then X^*, X_* are isomorphic to $\mathbb{Z}^n$, and there is a perfect pairing $\langle , \rangle : X^* \times X_* \to \mathbb{Z}$. If $\alpha(\gamma^\vee(z)) = z^k$ ($z \in \mathbb{C}^*, k \in \mathbb{Z}$) then $\langle \alpha, \gamma^\vee \rangle = k$.

Lemma 3.1 *Suppose θ is a holomorphic involution of $H = H(\mathbb{C})$. Then there is an isomorphism $H \simeq \mathbb{C}^{*n}$ so that*

$$\theta(z_1, \dots, z_a, w_1, \dots, w_b, u_1, v_1, \dots, u_c, v_c) = \\ (z_1, \dots, z_a, w_1^{-1}, \dots, w_b^{-1}, v_1, u_1, \dots, v_c, u_c).$$

Exercise 3.2 Try proving this result. It is *not* elementary. It is equivalent to proving that any involution in $GL(n, \mathbb{Z})$ is conjugate to a matrix with diagonal entries 1, -1, or 2×2 blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Lemma 3.3 *Suppose σ is an anti-holomorphic involution of H. Then there is an isomorphism $H \simeq \mathbb{C}^{*n}$ so that*

$$\sigma(z_1, \dots, z_a, w_1, \dots, w_b, u_1, v_1, \dots, u_c, v_c) = \\ (\overline{z}_1^{-1}, \dots, \overline{z}_a^{-1}, \overline{w}_1, \dots, \overline{w}_b, \overline{v}_1^{-1}, \overline{u}_1^{-1}, \dots, \overline{v}_c^{-1}, \overline{u}_c^{-1}).$$

Write $H(\mathbb{R}) = H^\sigma$. This is a real Lie group. With these coordinates we have:

$$H(\mathbb{R}) = S^{1a} \times \mathbb{R}^{*b} \times \mathbb{C}^{*c} \quad \text{(the real torus)}$$
$$T(\mathbb{R}) = H(\mathbb{R})^\theta = S^{1a} \times (\mathbb{Z}/2\mathbb{Z})^b \times S^{1c} \quad \text{(the maximal compact subgroup)}$$
$$T(\mathbb{C}) = H^\theta = \mathbb{C}^{*a} \times (\mathbb{Z}/2\mathbb{Z})^b \times \mathbb{C}^{*c} \quad \text{(the complexified maximal compact subgroup)}$$

Lemma 3.4 *There is a natural isomorphism* $\widehat{T(\mathbb{R})} \simeq X^*/(1-\theta)X^*$.

Exercise 3.5 Prove the Lemma. First of all, the characters of $T(\mathbb{R})$ are the same thing as the algebraic characters of $T(\mathbb{C}) = H^\theta$. You may assume that any algebraic character of H^θ is the restriction of a character of H.

A character of $H(\mathbb{R})$ may be identified with a $(\mathfrak{h}, H^\theta)$-module.

Proposition 3.6 *The* $(\mathfrak{h}, H^\theta)$ *modules are parametrized by pairs* (ν, κ) *satisfying:*

(1) $\nu \in X^* \otimes \mathbb{C} \simeq \mathfrak{h}^*$;

(2) $\kappa \in X^*/(1-\theta)X^*$;

(3) $(1+\theta)\nu = (1+\theta)\kappa$.

The corresponding character ν *of* $H(\mathbb{R})$ *has differential* ν*, and the restriction of* ν *to* $H(\mathbb{R})^\theta$ *is* κ.

Exercise 3.7 Prove the Proposition. Note that $H(\mathbb{R}) = T(\mathbb{R})\exp(\mathfrak{h}_0)$.

Example 3.8 Suppose $\theta = 1$, so $H(\mathbb{R}) \simeq S^{1n}$ is compact. The characters are parametrized by $\kappa \in X^*$.

If $\theta = -1$, $H(\mathbb{R}) \simeq \mathbb{R}^{*n}$, and characters are parametrized by pairs (ν, κ) with $\nu \in X^* \otimes \mathbb{C} \simeq \mathbb{C}^n$ and $\kappa \in X^*/2X^*(\simeq \mathbb{Z}/2\mathbb{Z})^n$.

Exercise 3.9 Work out the case of $H(\mathbb{R}) \simeq \mathbb{C}^*$. In this case $H(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$, $\sigma(z, w) = (\overline{w}^{-1}, \overline{z}^{-1})$, and $\theta(z, w) = (w, z)$.

3.1 Digression: Characters of compact groups

Let $G = SU(2)$, with $T \simeq S^1$ the diagonal Cartan subgroup. The character of the n-dimensional irreducible representation $(n = 1, 2, \dots)$ is

$$e^{i(-n+1)\theta} + e^{i(-n+3)\theta} + \dots e^{i(n-3)\theta} + e^{i(n-1)\theta} = \frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}$$

This is the Weyl character formula in this case.

Now consider the similar case of $G = SO(3)$. Again a Cartan subgroup T is the circle, for example take $\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ The irreducible representations of G are odd dimensional; the character of the n-dimensional irreducible representation for $(n = 1, 3, 5 \dots)$ is:

$$(3.1.1)(a) \quad e^{i(\frac{-n+1}{2})\theta} + e^{i(\frac{-n+3}{2})\theta} + \dots e^{i(\frac{n-3}{2})\theta} + e^{i(\frac{n-1}{2})\theta} = \frac{e^{in\theta/2} - e^{-in\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}$$

The numerator and denominator of the quotient are not well defined functions on T, because of the $2s$ in the denominator. The quotient is well defined; after multiplying by $\frac{e^{i\theta/2}}{e^{i\theta/2}}$ it can be written

$$\text{(3.1.1)(b)} \qquad \frac{e^{i(\frac{n+1}{2})\theta} - e^{i(\frac{-n+1}{2})\theta}}{1 - e^{-i\theta}}$$

Since n is odd both numerator and denominator are well defined. However (a) is clearly a more symmetric, and therefore preferable, expression.

For a general conneced compact group the Weyl denominator is

$$\prod_{\alpha>0}(e^{-\alpha/2} - e^{-\alpha/2}) = \prod_{\alpha>0}(1 - e^{-\alpha})e^{\rho}$$

Here $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ as usual. The first expression isn't really well defined; it is shorthand for the second. The second expression is well defined if ρ exponentiates to a character of T. This holds for $SL(2)$, but not $PSL(2)$.

The conclusion is it is very useful to introduce a two-fold cover of the torus on which ρ is well defined. These play an important role in the Langlands classification. If ρ exponentiates this cover can be ignored, and it is reasonable to focus on this case first time around.

3.2 Covers of tori

Now fix an element $\gamma \in \frac{1}{2}X^*$. Let

$$\text{(3.2.1)} \qquad H_\gamma = \{(h, z) \in H \times \mathbb{C}^* \mid 2\gamma(h) = z^2\}.$$

This is a two-fold cover of H via the map $(h, z) \to h$; write ζ for the nontrivial element in the kernel of this map. We call this the γ-cover of H. Note that $(h, z) \to z$ is a character of H_γ, and is a canonical square root of 2γ, denoted γ.

Exercise 3.2.2 Prove this cover splits, i.e. $H_\gamma \simeq H \times \mathbb{Z}/2\mathbb{Z}$ if and only if $\gamma \in X^*$.

Now assume H is defined over $\mathbb{R}$, with Cartan involution θ. The γ cover of $H(\mathbb{R})$ is defined to be the inverse image of $H(\mathbb{R})$ in H_γ. A character of $H(\mathbb{R})_\gamma$ is said to be genuine if it is nontrivial on ζ.

Lemma 3.2.3 *The genuine characters of $H(\mathbb{R})_\gamma$ are canonically parametrized by the set of pairs (ν, κ) with $\nu \in \mathfrak{h}^*$, $\kappa \in \gamma + X^*/(1-\theta)X^*$, and satisfying $(1+\theta)\nu = (1+\theta)\kappa$.*

Exercise 3.2.4 Prove the Lemma.

Example 3.2.5 Suppose H is a Cartan subgroup of a reductive group G. Choose a set of positive roots Δ^+ and let $\rho = \frac{1}{2}\sum_{\Delta^+}\alpha$, and consider the cover

H_ρ of H. This is independent of the choice of Δ^+ up to *canonical* isomorphism (prove this!).

Suppose $G = SL(2,\mathbb{R})$. For any real Cartan subgroup $H(\mathbb{R})_\rho \simeq H(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$.

Let $G = PSL(2,\mathbb{R}) \simeq SO(2,1)$ (this group is disconnected). If $H(\mathbb{R}) \simeq S^1$. then $H(\mathbb{R})_\rho \simeq S^1$, with projection map $z \to z^2$.

If $H(\mathbb{R}) \simeq \mathbb{R}^*$ then $H(\mathbb{R})_\rho \simeq \mathbb{R}^* \cup i\mathbb{R}^*$. Note this has an element of order 4. Although H_ρ is a real algebraic group, $H(\mathbb{R})_\rho$ is not its real points; $H_\rho(\mathbb{R}) \simeq \mathbb{R}^*$ is a subgroup of index 2 in $H(\mathbb{R})_\rho$.

Exercise 3.2.6 Think through the final example .

4 Cartan subgroups

If G is compact or complex, its Cartan subgroups are unique up to conjugacy. In a real Lie group there are a finite number of conjugacy classes of Cartan subgroups, and these play an important role in representation theory.

If $G = SL(2,\mathbb{R})$ there are two Cartan subgroups, up to conjugacy. The diagonal $\{\mathrm{diag}(x, \frac{1}{x})\} \simeq \mathbb{R}^*$, and the circle $SO(2) \simeq S^1$.

Exercise 4.1 Show that every semisimple element of $GL(2,\mathbb{R})$ is conjugate to either $\mathrm{diag}(x,y)$ or $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Conclude that $GL(2,\mathbb{R})$ has two Cartan subgroups (up to conjugacy), one $\mathbb{R}^{*2}$ and $\mathbb{C}^*$.

Exercise 4.2 Find representatives of all conjugacy classes of Cartan subgroups in $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$.

References

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