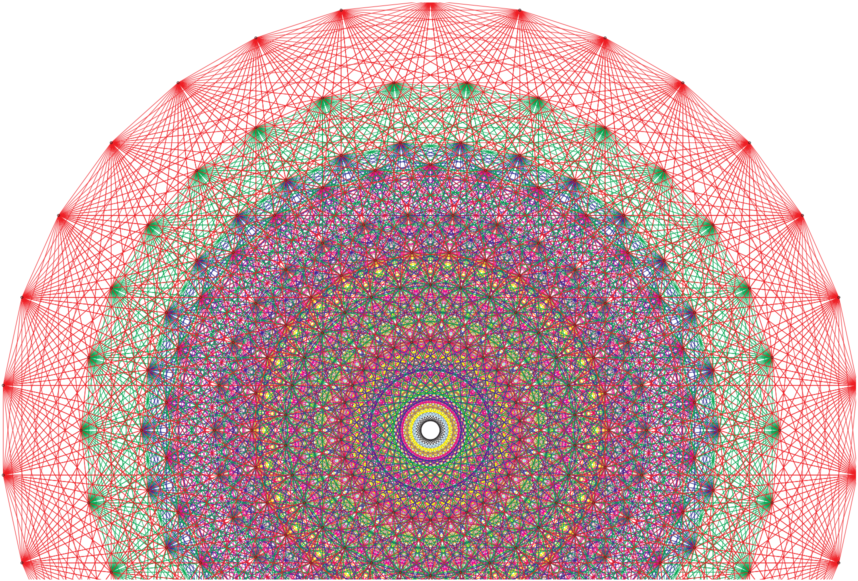
Atlas of Lie Groups and Representations

www.liegroups.org



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Atlas Project Members, AIM, July 2007

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Example: $G = GL(n,\mathbb{R})$ - Vogan (1986)

Known Unitary Duals

red: known black: not known

Type A: $SL(n,\mathbb{R})$, $SL(n,\mathbb{H})$, $SU(n,1)$, $SU(n,2)$, $SL(n,\mathbb{C})$
$SU(p,q)$ $(p,q>2)$

Type B: $SO(2n,1)$, $SO(2n+1,2)$, $SO(2n+1,\mathbb{C})$

$SO(p,q)$ $(p,q\ge 3)$

Type C: $Sp(4,\mathbb{R})$, $Sp(n,1)$, $Sp(2n,\mathbb{C})$

$Sp(p,q)$ $(p,q\ge 2)$

Type D: $SO(2n+1,1)$, $SO(2n,2)$, $SO(2n,\mathbb{C})$

$SO(p,q)$ $(p,q\ge 3)$, $SO^*(2n)$ $(n\ge 4)$

Type E_6: $E_6(F_4)=SL(3,\text{Cayley})$

E_6(Hermitian), E_6(split), E_6(quaternionic), $E_6(\mathbb{C})$

Type F_4: $F_4(B_4)$

F_4(split), $F_4(\mathbb{C})$

Type G_2: G_2(split), $G_2(\mathbb{C})$

E_7/E_8: nothing known

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Atlas of Lie Groups and Representations:

Take this idea seriously

Goals of the Atlas Project

- Tools for education: teaching Lie groups to graduate students and researchers

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- Compute the unitary dual

Outline of the lecture

Constructing representations of Weyl Groups
Computing the signature of a quadratic form
Explicitly computing the admissible dual
KLV polynomials and the E_8 calculation
The Future

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Example: The character table of every Weyl group W is known.

W=Weyl group, simple reflections $s_1, \dots, s_n$

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Fact: can use matrices with integral entries (Springer correspondence)

Character table of $W(E_8)$

Class		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Size		1	1	120	120	3150	3780	3780	37800	37800	113400	2240	4480	89600	268800	15120
Order		1	2	2	2	2	2	2	2	2	2	3	3	3	3	4
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	+	1	1	-1	-1	1	1	1	-1	-1	1	1	1	1	1	1
X.3	+	8	-8	-6	6	0	4	-4	2	-2	0	5	-4	-1	2	0
X.4	+	8	-8	6	-6	0	4	-4	-2	2	0	5	-4	-1	2	0
X.5	+	28	28	14	14	-4	4	4	-2	-2	-4	10	10	1	1	4
X.6	+	28	28	-14	-14	-4	4	4	2	2	-4	10	10	1	1	4
X.7	+	35	35	21	21	3	11	11	5	5	3	14	5	-1	2	-5
X.8	+	35	35	-21	-21	3	11	11	-5	-5	3	14	5	-1	2	-5
X.9	+	50	50	20	20	18	10	10	4	4	2	5	5	-4	5	10
...																
X.100	+	4200	4200	0	0	104	40	40	0	0	8	-120	15	-12	6	-40
X.101	+	4200	4200	420	420	-24	40	40	4	4	8	-30	-30	15	-3	40
X.102	+	4480	4480	0	0	-128	0	0	0	0	0	-80	-44	-20	4	64
X.103	+	4536	-4536	-378	378	0	60	-60	30	-30	0	-81	0	0	0	0
X.104	+	4536	-4536	378	-378	0	60	-60	-30	30	0	-81	0	0	0	0
X.105	+	4536	4536	0	0	-72	-72	-72	0	0	24	0	81	0	0	-24
X.106	+	5600	-5600	0	0	0	-80	80	0	0	0	-10	-100	2	-4	0
X.107	+	5600	-5600	-280	280	0	-80	80	8	-8	0	20	20	11	2	0
X.108	+	5600	-5600	280	-280	0	-80	80	-8	8	0	20	20	11	2	0
X.109	+	5670	5670	0	0	-90	-90	-90	0	0	6	0	-81	0	0	6
X.110	+	6075	6075	405	405	27	-45	-45	-27	-27	-21	0	0	0	0	-45
X.111	+	6075	6075	-405	-405	27	-45	-45	27	27	-21	0	0	0	0	-45
X.112	+	7168	-7168	0	0	0	0	0	0	0	0	-128	16	-32	-8	0

Example: one matrix from a 30-dimensional representation of $W(E_6)$

```
1/2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1/4,0,0,0,0,0,0,0,0,0,
0,-1/8,0,0,0,-15/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3/8,0,0,0,0,0,0,0,0,
0,0,-1/8,0,0,0,-15/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3/8,0,0,0,0,0,0,0,
0,0,0,-1/8,0,0,0,-15/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3/8,0,0,0,0,0,0,
0,0,0,0,-1/8,0,0,0,-15/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3/8,0,0,0,0,0,
0,-3/8,0,0,0,3/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1/8,0,0,0,0,0,0,0,0,
0,0,-3/8,0,0,0,3/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1/8,0,0,0,0,0,0,0,
0,0,0,-3/8,0,0,0,3/8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1/8,0,0,0,0,0,0,
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Construct π by constructing its restriction to a subgroup, and building up.
John Stembridge: $\mathbb{Q}$-models including $W(E_8)$
(for $W(E_8)$, LCD(denominators)$\leq$594)

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2) or all eigenvalues are ≥ 0
3) or det(all principal minors)≥ 0 (2^n of them)

What is wrong with computers

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$$

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$$\frac{11}{3} + \frac{23\,5^{\frac{2}{3}}}{3\left(241 + 9\,i\,\sqrt{34}\right)^{\frac{1}{3}}} + \frac{\left(5\left(241 + 9\,i\,\sqrt{34}\right)\right)^{\frac{1}{3}}}{3}$$

$$\frac{11}{3} - \frac{23\,5^{\frac{2}{3}}\left(1 + i\,\sqrt{3}\right)}{6\left(241 + 9\,i\,\sqrt{34}\right)^{\frac{1}{3}}} - \frac{\left(1 - i\,\sqrt{3}\right)\left(5\left(241 + 9\,i\,\sqrt{34}\right)\right)^{\frac{1}{3}}}{6}$$

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$$=\{10.79 + 0.i,\ -0.34 + 4.44 \times 10^{-16} i,\ 0.54 - 4.44 \times 10^{-16} i\}$$

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p = number of sign changes: $(\dots a_i, 0, \dots, 0, a_j \dots)$ $\quad (a_i a_j < 0)$

z = number of zeroes at the beginning

q = number of sign changes using $f_M(-x)$

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Lemma (Descartes' rule of signs)

$$\sigma(M) = \sigma(f_M)$$

David Saunders, Zhendong Wan (Delaware), A:
Compute the characteristic polynomial mod p + Chinese Remainder Theorem $\rightarrow$ compute $\sigma(M)$

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1,000	3 hours
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Note: Embarassingly parallelizable

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Spherical Unitary Dual

What is wrong with computers II

$\int \sin^{10}(x)\cos(x)\,dx$ = [Mathematica]:

$$\frac{21}{512}\sin(x) - \frac{15}{512}\sin(3x) + \frac{15}{512}\sin(35x)$$
$$-\frac{5}{1024}\sin(7x) + \frac{11}{11264}\sin(9x) + C$$

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Barbasch/Ciubotaru: Also results for exceptional groups; confirmed by atlas computations

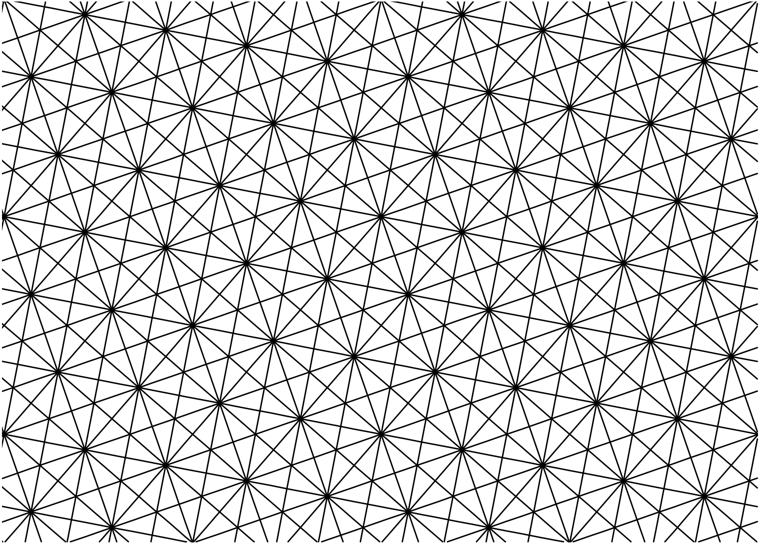
Spherical Unitary dual via atlas

G: split, p-adic

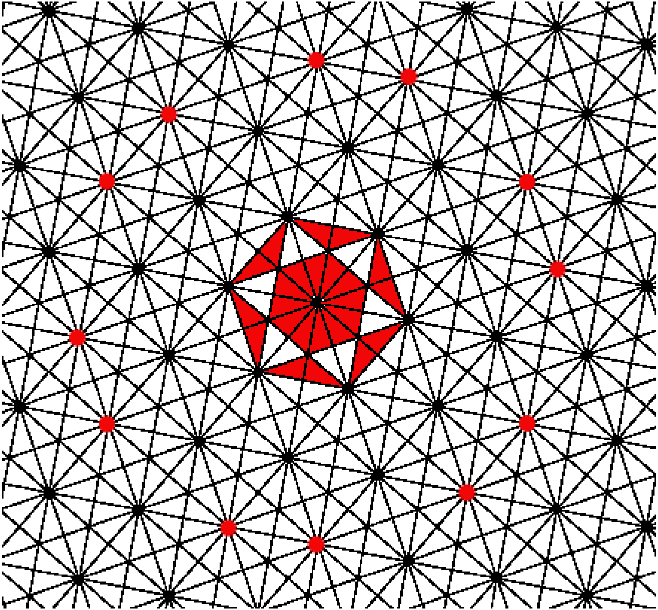
Atlas: computes the spherical unitary dual $\widehat{G}_{sph}$

Example G=G_2

```
(0,0,0)
(-3/8,-3/8,3/4)
(-1/4,-1/2,3/4)
(-1/6,-5/12,7/12)
(-1/2,-1/2,1)
(-1,-2,3)
(0,-1,1)
(-1/3,-1/3,2/3)
```



Example: Hyperplanes in $\mathfrak{a}(\mathbb{R})^*$ for G_2



Example: Spherical unitary dual of G_2 (Vogan, Barbasch, Atlas)

Unitary Dual

G = real reductive group

for example $GL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $Spin(p,q)$, $E_8(split)$,...)

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Hermitian Dual $\widehat{G}_h$: $(\mathfrak{g}, K)$-modules preserving a Hermitian form (not necessarily positive definite)

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Note: $\langle\, ,\, \rangle$ is not the usual one for $-1 \leq \nu \leq 1, \nu \neq 0$

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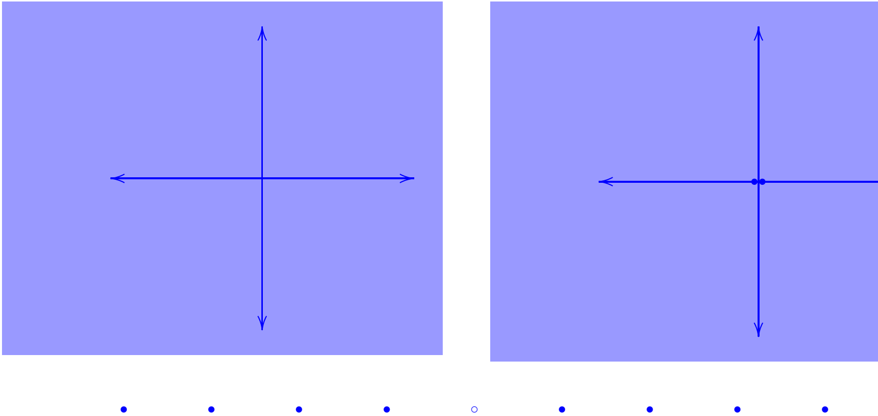

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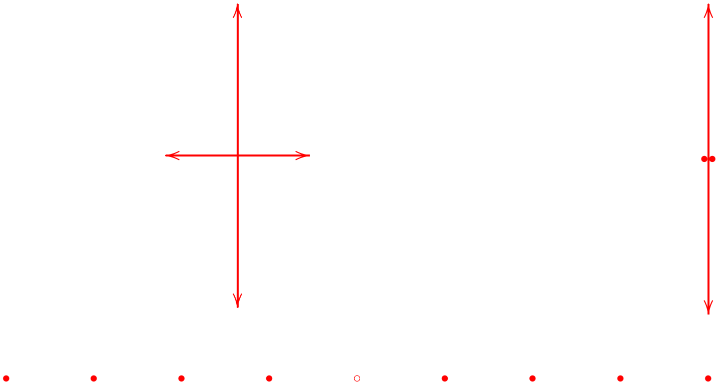

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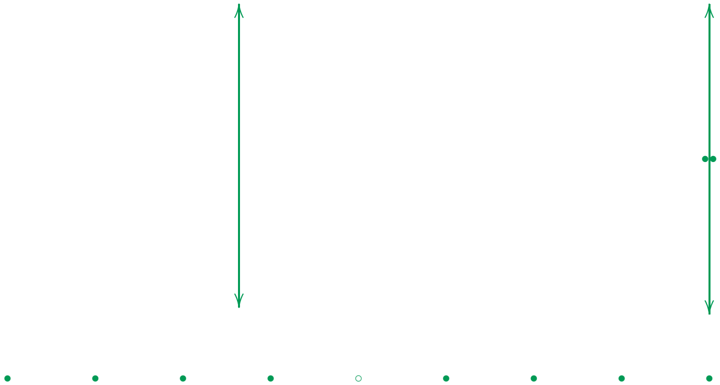

Tempered dual

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Answer: 526,471

(2,157 of them = .41% are unitary)

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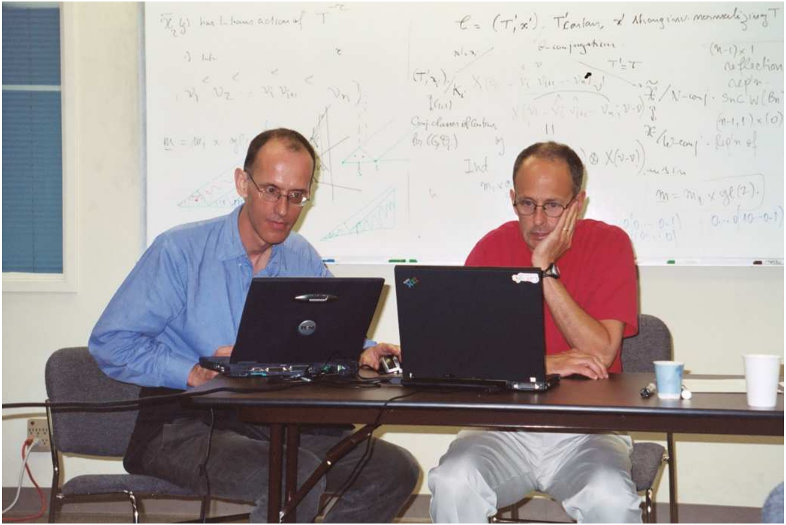
2) natural: make the Kazhdan-Lusztig-Vogan polynomials computable



Fokko du Cloux

What Fokko did

Abstract Mathematics	$\rightarrow$	Algorithm	$\rightarrow$	Software
Lie Groups		Combinatorial Set		C++ code
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Mathematical Structures $\longleftrightarrow$ Data Structures

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Mathematical Structures ⟷ Data Structures

Mathematics ⟵ Computer output

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For now assume G is simply connected, adjoint and $\mathrm{Out}(G) = 1$ (Examples: $G = G_2, F_4$ or E_8)

$K\backslash G/B$

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Problem: Parametrize K-orbits on G/B

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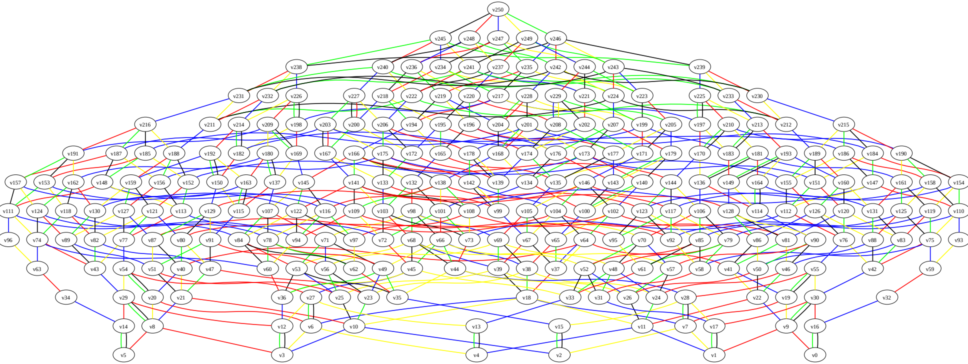
Theorem: There is a natural bijection

$$\mathcal{X} \overset{1-1}{\longleftrightarrow} \coprod_i K_i\backslash\mathcal{B}$$

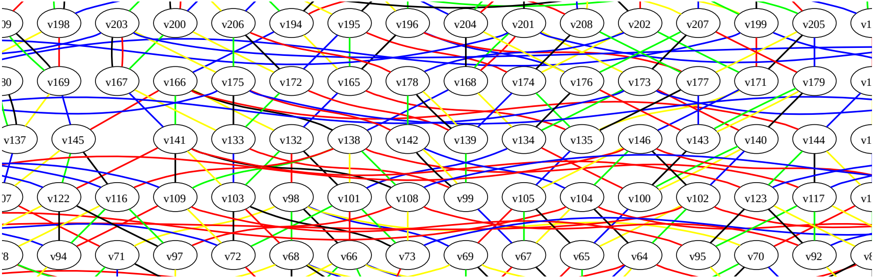
(union over real forms, corresponding $K_1, \dots, K_n$)

Example: $K\backslash G/B$ for $SL(4,\mathbb{R})$:

```
 0:  0  0  [C,n,C]    3   1   3     *   2   *
 1:  0  0  [C,n,C]    4   0   4     *   2   *
 2:  1  1  [C,r,C]    6   2   5     *   *   *  2
 3:  1  0  [C,C,C]    0   7   0     *   *   *  1,3
 4:  1  0  [C,C,C]    1   8   1     *   *   *  1,3
 5:  2  1  [C,C,C]   10   9   2     *   *   *  3,2,1
 6:  2  1  [C,C,C]    2  11  10     *   *   *  1,2,3
 7:  2  0  [n,C,n]    8   3   8    11   *   9  2,1,3,2
 8:  2  0  [n,C,n]    7   4   7    11   *   9  2,1,3,2
 9:  3  1  [n,C,r]    9   5   9    12   *   *  2,1,3,2,1
10:  3  1  [C,n,C]    5  10   6     *  12   *  1,2,3,2,1
11:  3  1  [r,C,n]   11   6  11     *   *  12  1,2,1,3,2
12:  4  2  [r,r,r]   12  12  12     *   *   *  1,2,1,3,2,1
```



$K\backslash G/B$ for $SO(5,5)$



Closeup of $SO(5,5)$ graph

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Theorem: (A/du Cloux) There is a natural bijection:

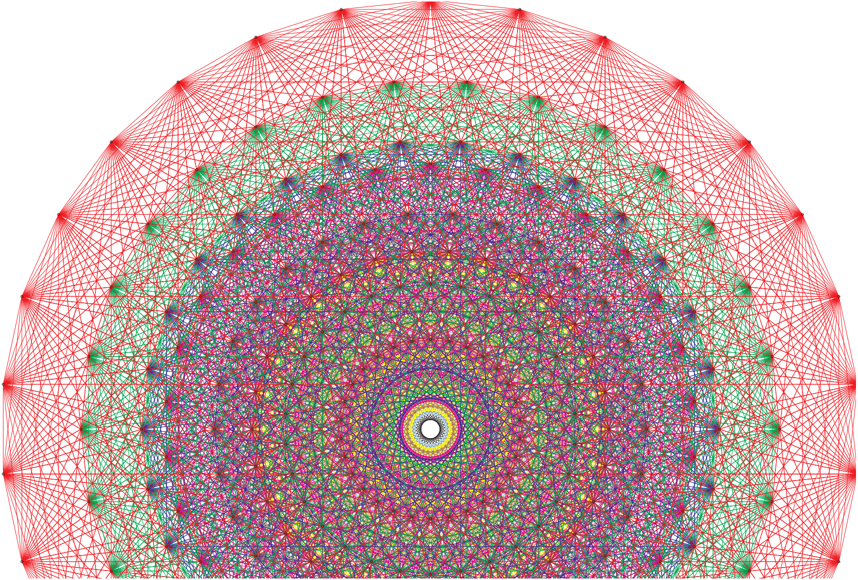
$$\mathcal{Z} \overset{1-1}{\longleftrightarrow} \coprod_{i=1}^{n} \Pi(G(\mathbb{R})_i,\lambda)$$

(union over real forms of G)

$\mathcal{Z}$ = certain subset of

$$\mathcal{X} \times \mathcal{X}^\vee = \coprod_i K_i\backslash\mathcal{B} \times \coprod_j K_j^\vee\backslash\mathcal{B}^\vee$$

Kazhdan-Lusztig-Vogan Polynomials



Fokko du Cloux
December 20, 1954 - November 10, 2006

Marc van Leeuwen
Poitiers
LiE software

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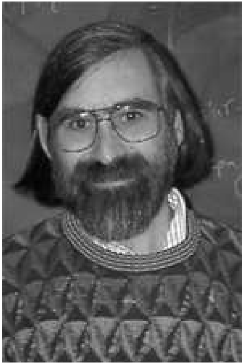

David Vogan
MIT

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Proposition (Langlands, Zuckerman): $\mathcal{M} = \mathbb{Z}\langle I(\gamma)\rangle \quad (\gamma \in \mathcal{Z})$

Kazhdan-Lusztig-Vogan Polynomials

Change of Basis Matrices:

$I(\delta) = \sum_{\delta\in\mathcal{Z}} m(\gamma, \delta)\pi(\gamma)$

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$$\boxed{M(\gamma, \delta) = (-1)^{\ell(\gamma)-\ell(\delta)} P_{\gamma,\delta}(1)}$$

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Note: David Vogan calls the polynomials for $G(\mathbb{R})$ Kazhdan-Lusztig (not Kazhdan-Lusztig-Vogan) polynomials

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Recursive Definition of KLV polynomials

α w.r.t. δ	α w.r.t. γ	$P_{\gamma,\delta} =$
ic/C-/r1 *or* r2	i1 or i2	$v^{-1}P_{\gamma_\alpha,\delta}$ *or* $v^{-1}(P_{\gamma_\alpha^+,\delta} + P_{\gamma_\alpha^-,\delta})$
ic/C-/r1 or r2	C+	$v^{-1}P_{s_\alpha\times\gamma,\delta}$
C-	C-	$vP_{\gamma,s_\alpha\times\delta} + P_{s_\alpha\times\gamma,s_\alpha\times\delta} - U^\alpha_{\gamma,\delta}$
r1 or r2*	r1	$(v - v^{-1})P_{\gamma,\delta_\alpha^+} + P_{\gamma_\alpha^+,\delta_\alpha^+} + P_{\gamma_\alpha^-,\delta_\alpha^+} - U^\alpha_{\gamma,\delta_\alpha^+}$
r1 or r2*	r2	$vP_{\gamma,\delta_\alpha} - v^{-1}P_{s_\alpha\times\gamma,\delta_\alpha} + P_{\gamma_\alpha,\delta_\alpha} - U^\alpha_{\gamma,\delta_\alpha}$

(*): formula is for $P_{\gamma,\delta} + P_{\gamma,s_\alpha\delta}$

Recursive Definition of KLV polynomials

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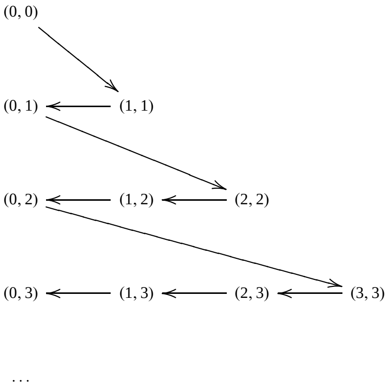
In each case the right formula in boxes involves
$P_{\gamma',\delta'}$ with
1) $\ell(\delta') < \ell(\delta)$ *or*
2) $\ell(\delta') = \ell(\delta), \ell(\gamma') > \ell(\gamma)$

Recursion Relations

$P_{\gamma,\gamma} = 1$

Compute $P_{\gamma,\delta}$ like this:



((i, j) is the $P_{\gamma,\delta}$ with $\ell(\gamma) = i, \ell(\delta) = j$)

Recursion Relations

(0, 0)					
(0, 1)	(1, 1)				
(0, 2)	(1, 2)	(2, 2)			
(0, 3)	(1, 3)	(2, 3)	(3, 3)		
(0, 4)	(1, 4)	(2, 4)	(3, 4)	(4, 4)	
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To compute $P_{\gamma,\delta}$ with $\ell(\gamma) = 3, \ell(\delta) = 5$, need potentially all of the $P_{\gamma,\delta}$ from the blue region.

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(E_8: $U^{\alpha}_{\gamma,\delta}$ has 150 terms on average)

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See:
David Vogan's narrative, October Notices
Marc van Leeuwen's technical discussion
`www.liegroups.org/talks`

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Challenge: Compute KLV for (the large block) of E_8

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$|\mathcal{Z}| = 453,060$ (this is the largest block)
$\deg(P_{\gamma,\delta}) \leq 31$
Big Problem: we did not have a good idea of the size of the answer beforehand.
$a_i \geq 2^{16} = 65{,}535$ (almost certainly)
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Crude estimates: need about 1 terabyte of RAM (=1,000 gigabytes)
(1 gigabyte = 1 billion bytes = RAM in typical home computer)
Typical computational machine (not a cluster): 4-8 gigabytes of RAM

Many of the polynomials are equal for obvious reasons.

Hope: number of distinct polynomials $\leq$ 200 million.

Store only the distinct polynomials (cost of pointers)

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Experiments (Birne Binegar and Dan Barbasch):
About 800 billion distinct polynomials $\rightarrow$ 65 billion bytes

William Stein at Washington lent us SAGE, with 64 gigabytes of RAM (all accessible from one processor)

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This gets us down to about $15 + 4 = 19$ billion bytes

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Date	mod	Status	Result
Dec. 6	251	crash	
Dec. 19	251	complete	16 hours
Dec. 22	256	crash	
Dec. 22	256	complete	11 hours
Dec. 26	255	complete	12 hours
Dec. 27	253	crash	
Jan. 3	253	complete	12 hours

The final result

Combine the answers using the Chinese Remainder Theorem.
Answer is correct if the biggest coefficient is less than 4,145,475,840
Total time (on SAGE): 77 hours

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Size of output: 60 gigabytes

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Polynomial with the maximal coefficient:

$$152q^{22} + 3,472q^{21} + 38,791q^{20} + 293,021q^{19} + 1,370,892q^{18} + 4,067,059q^{17} + 7,964,012q^{16} + 11,159,003q^{15} + 11,808,808q^{14} + 9,859,915q^{13} + 6,778,956q^{12} + 3,964,369q^{11} + 2,015,441q^{10} + 906,567q^{9} + 363,611q^{8} + 129,820q^{7} + 41,239q^{6} + 11,426q^{5} + 2,677q^{4} + 492q^{3} + 61q^{2} + 3q$$

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Number of coefficients in distinct polynomials: 13,721,641,221 (13.9 billion)

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