Cells for representations of real groups

or

Carrying coals to Newcastle

Carrying coals to Newcastle:

a) to take something to a place where its kind exists in great quantity.
b) to do something wholly unnecessary.

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Atlas Project

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**Unipotent Representations**

$G$: connected reductive group defined over a local field $F$ of characteristic 0

For simplicity: assume $G(F)$ is (inner to) a split group

$G^\vee$: complex dual group of $G$

$O^\vee$: unipotent orbit in $G^\vee$

**Conjecture** (Arthur): Associated to $O^\vee$ is a finite set $\Pi(O^\vee)$ of irreducible unitary representations of $G(F)$ satisfying certain conditions, including stability...

We call $\Pi(O^\vee)$ a **Weak unipotent Arthur packet**

(Later: honest unipotent Arthur packets)
Arthur did not define $\Pi(\mathcal{O}^\vee)$, and there is no definition in general (that I am aware of). Even if one can give conditions to determine $\Pi(\mathcal{O}^\vee)$ uniquely, computing it might be difficult.

$F = \mathbb{R}$ or $\mathbb{C}$: Barbasch and Vogan gave a definition of $\Pi(\mathcal{O}^\vee)$. Computing $\Pi(\mathcal{O}^\vee)$ is another matter.

Today: Defining and computing $\Pi(\mathcal{O}^\vee)$ for real groups
Representations with fixed infinitesimal character

\( \pi \): irreducible representation of \( G(\mathbb{R}) \), (a \((g,K)\)-module),

\( \chi_\pi \)=infinitesimal character

Fix an infinitesimal character \( \chi \) for \( G \), that of a finite dimensional representation of \( G \)

\( \mathcal{M}_\chi \): Grothendieck group of representations with infinitesimal character \( \chi \)

\[ \mathcal{M}_\chi = \mathbb{Z}\langle \{ J(\gamma) \mid \gamma \in \mathcal{P}_\chi, J(\gamma) \text{ irreducible, } \chi_{J(\gamma)} = \chi \} \rangle \]

where \( \gamma \in \mathcal{P}(\mathcal{M}_\chi) = \) a (finite) set of parameters.

Each \( J(\gamma) \) is the unique irreducible quotient of a standard module \( I(\gamma) \) (\( I \) is for “induced”)

Fact:

\[ \mathcal{M}_\chi = \mathbb{Z}\langle \{ I(\gamma) \mid \gamma \in \mathcal{P}(\mathcal{M}_\chi) \} \rangle \]
**Example:** $\text{SL}(2,\mathbb{R})$

Fix infinitesimal character $\chi$ of the trivial representation

There are 4 irreducible representations: $\mathbb{C}, DS_+, DS_-$ and $PS_-$. These are the trivial representation, two discrete series (one holomorphic, one anti-holomorphic) and $PS_-$ is the irreducible, non-spherical principal series.

$$\mathcal{M}_\chi = \mathbb{Z}\langle \mathbb{C}, DS_+, DS_-, PS_- \rangle$$

Let $PS_+$ be the reducible principal series:

$$PS_+ = \mathbb{C} + DS_+ + DS_-$$

(in the Grothendieck group).

Standard modules:

$$\mathcal{M}_\chi = \mathbb{Z}\langle PS_+, DS_+, DS_-, PS_- \rangle$$

$$\mathbb{C} = PS_+ - DS_+ - DS_-$$
Kazhdan-Lusztig-Vogan polynomials

Change of basis matrix: Kazhdan-Lusztig-Vogan polynomials evaluated at $q = 1$ (up to some elementary signs)

$SL(2, \mathbb{R})$:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Special case: ordinary Kazhdan-Lusztig modules for category $O$ (Verma modules)
**Coherent Continuation**

**Definition (Zuckerman):** There is a natural representation of $W$ on $M_\chi$ (the coherent continuation representation).

**Theorem:** (Lusztig/Vogan) $(M_\chi, \mathcal{P}(M_\chi))$ has a natural structure of $W$-graph in the sense of [Kazhdan-Lusztig, 1979]).

As representations of $W$:

$$M_\chi = B_1 \oplus \cdots \oplus B_n$$

$(B_i$ is a block: $\sim$ generated by $\text{Ext}(X, Y) \neq 0$)

Each block has the structure of a $W$-graph.

$SL(2, \mathbb{R})$:

$$
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$
**Definition:** Given a block $B$, a Harish-Chandra cell is a cell for the $W$-graph of $B$ (as in [KL,1979])

A cell $C$ carries a representation of $W$ on

$$\mathbb{Z}\langle \{J(\gamma) \mid \gamma \in C\}\rangle$$

**Empirical fact** (McGovern, Binegar): If $G(\mathbb{R})$ is a real form of $GL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$ or a simple exceptional group, then every Harish-Chandra cell is isomorphic to a left cell.
Theorem:
1) There is an integer $d(C)$ such that

\[ \text{Hom}_W(\pi_C, \text{Sym}^k(\text{ref})) = \begin{cases} 
0 & k < d(C) \\
1 & k = d(C) 
\end{cases} \]

2) The cell contains a unique special representation $\sigma_C$, which also occurs in $\text{Sym}^{d(C)}(\text{ref})$. 
**Vogan Duality**

\[ G = G(\mathbb{C}), \ G(\mathbb{R}), \ \chi, \ M_\chi \supset B \]

**Theorem (Vogan)**
There exists a real form \( G^\vee(\mathbb{R}) \) of \( G^\vee(\mathbb{C}) \), a block \( B^\vee \), and a bijection

\[ \mathcal{P}(B) \ni \gamma \rightarrow \gamma^\vee \in \mathcal{P}(B^\vee) \]

with the following property: Define a perfect pairing \( B \times B^\vee \) by:

\[ \langle J(\gamma), J(\tau^\vee) \rangle = \delta_{\gamma, \tau} \]

Then:

\[ \langle I(\gamma), I(\tau^\vee) \rangle = \delta_{\gamma, \tau} \]

Equivalently: the matrices of KLV polynomials for \( G(\mathbb{R}) \) and \( G^\vee(\mathbb{R}) \) are inverses.
Vogan duality:
(1) reverses inclusion of primitive ideals;
(2) takes small representations to large ones
(3) interchanges discrete series and (minimal) principal series (of a split group)
(4) takes cells to cells
(5) induces $\sigma \rightarrow \sigma^*$ on $\hat{W} \simeq \hat{W}^\vee$

If $\pi$ is an irreducible representation of $G(\mathbb{R})$ we will write $\pi^\vee$ for the corresponding irreducible representation of some real form of $G^\vee(\mathbb{C})$. 
**Infinitesimal character**

\( \mathcal{O}^\vee \): nilpotent orbit for \( G^\vee \)

Jacobson-Morozov: \( \mathcal{O}^\vee \mapsto \{ H, E, F \} \)

\[
\mathcal{O}^\vee \mapsto \frac{1}{2} H \in \mathfrak{h}^\vee \simeq \mathfrak{h}^* \mapsto \chi(\mathcal{O}^\vee)
\]

For simplicity: assume \( \mathcal{O}^\vee \) is even (\( \Leftrightarrow \chi(\mathcal{O}^\vee) \) is integral).
**Associated Variety**

Associated to an irreducible representation $\pi$ of $G(\mathbb{R})$ is a nilpotent $G(\mathbb{C})$-orbit in $\mathfrak{g}$.

\[ I = \text{gr}(\text{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi)) \]

\[ \pi \mapsto \text{gr}(I) \subset \text{gr}(\mathcal{U}(\mathfrak{g})) \cong S(\mathfrak{g}) \mapsto \mathcal{V}((\text{gr}(I)) \subset \mathfrak{g}^* \]

$\mathcal{V}(\text{gr}(I))$ is $G(\mathbb{C})$-invariant and contained in the nilpotent cone.

Fact: (Borho/Brylinski/Joseph) $\mathcal{V}(\pi)$ is the closure of a single nilpotent orbit $\mathcal{O}$.

**Definition** (Vogan): $\overline{\mathcal{O}}$ (or simply $\mathcal{O}$) is the Associated Variety of the annihilator of $\pi$:

\[ \text{AV}(\text{Ann}(\pi)) = \overline{\mathcal{O}} \]
**Definition of \( \Pi(O^\vee) \)**

Given \( G = G(\mathbb{C}), G(\mathbb{R}), G^\vee(\mathbb{C}) \)
a nilpotent orbit \( O^\vee \) of \( G^\vee(\mathbb{C}) \).

Assume \( \delta^\vee(O^\vee) = O^\vee \) where \( \delta^\vee \) is the involution defining the L-group \( ^LG \). This is automatic if \( G(\mathbb{R}) \) is (inner to) a split group.

**Definition:** \( \Pi(O^\vee) \) is the set of irreducible representations \( \pi \) of \( G(\mathbb{R}) \) satisfying:

1. \( \chi_\pi = \chi(O^\vee) \)
2. \( \text{AV}(\text{Ann}(\pi^\vee)) = \overline{O^\vee} \)
Fix $G(\mathbb{C})$, $G(\mathbb{R})$. For simplicity assume $G(\mathbb{C})$ is simply connected.

0) Compute the conjugacy classes of $\mathcal{W} = \mathcal{W}(G(\mathbb{C}))$.

1) Explicitly compute $\mathcal{M}_\rho$, $\mathcal{P}(\mathcal{M}_\rho)$

2) Compute the blocks in $\mathcal{M}_\rho$, and for each block $\mathcal{B}$ the dual block $\mathcal{B}^\vee$

3) Run over the blocks $\mathcal{B}^\vee$. Compute the KLV polynomials for each $\mathcal{B}^\vee$.

4) Compute the cells $\mathcal{C}_1^\vee, \ldots, \mathcal{C}_n^\vee$ in $\mathcal{B}^\vee$

5) For each cell $\mathcal{C}^\vee$ compute the representation $\pi_{\mathcal{C}^\vee}$ of $\mathcal{W}^\vee$ on $\mathcal{C}^\vee$, and its character $\theta_{\mathcal{C}^\vee} = \text{trace}(\pi_{\mathcal{C}^\vee})$

6) Compute $d = \min\{k \in \mathbb{Z} \mid \langle \theta_{\mathcal{C}^\vee}, \theta_{S^k(\text{ref})} \rangle \neq 0\}$
7) Let $P_{C^\vee} = \sum_{w \in W} \theta_{Sd}(\text{ref})(w)\pi_{C^\vee}(w) \in \text{End}(C^\vee)$ (this is a projection, up to scalar)

8) Compute the representation $\sigma_{C^\vee}$ of $W$ on the image of $P_{C^\vee}$: this is the special representation in the cell $C^\vee$.

Fix a complex even nilpotent orbit $O^\vee$

9) Check if the nilpotent orbit attached to $\sigma_{C^\vee}$ (by the Springer correspondence) is equal to $O^\vee$. If so: translate (apply a Zuckerman translation functor to) the irreducible representations in $C$ to infinitesimal character $\chi(O^\vee)$

10) $\Pi(O^\vee)$ is the set of (non-zero) irreducible representations obtained this way.
Primitive Ideals

11) The columns of $P_C$ correspond to the irreducible representations in the cell. Two such representations have the same primitive ideal $\iff$ the corresponding columns are multiples of each other.
[Interlude: some examples]
An honest unipotent Arthur packet is attached to a homomorphism

$$\Psi : \mathbb{Z}_2 \times SL(2, \mathbb{C}) \rightarrow^L G$$

(Note: $W_\mathbb{R}/W_\mathbb{R}^0 \simeq \mathbb{Z}_2$

So a weak unipotent Arthur packet is the union (not necessarily disjoint) of honest ones.

Recall $\Pi(O^\vee)$ was defined in terms of $AV(Ann(\pi^\vee))$, the closure of single complex nilpotent orbit.

There is a finer invariant $AV(\pi)$ which is a union nilpotent $K(\mathbb{C})$ orbits on $(\mathfrak{g}/\mathfrak{k})^*$ (in bijection with: nilpotent $G(\mathbb{R})$-orbits on $\mathfrak{g}_0$). These “honest” Arthur packets are defined in [Adams/Barbasch/Vogan, 1992].
Another Algorithm: Vogan has given an outline of an explicit algorithm to compute the $K$-equivariant $K$-theory of the nilpotent cone. Assuming this can be made precise:

a) This will prove a version of the Lusztig-Vogan conjecture for real groups (complex case: Lusztig/Bezrukavnikov)

b) This gives an effective algorithm to compute $AV(\pi)$, and therefore honest unipotent Arthur packets.