Calculating the Hodge Filtration

or

Hermitian Forms and Hodge Theory

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**The Main Result**

Joint with Peter Trapa, David Vogan

$G(\mathbb{R})$: a real form of a connected, complex reductive group

$\pi$: irreducible representation

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Main Theorem

The signature of the c-form on $\pi$ is the reduction mod(2) of the Hodge filtration

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Today:

1. What does this mean?
2. What does this mean?
3. Relationship with the Schmid-Vilonen conjecture

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Hermitian forms $\leftrightarrow$ Hodge theory
K-multiplcilities

$G(\mathbb{C}), G(\mathbb{R}), \theta, K = G^\theta, \mathfrak{g} = \text{Lie}(G)$

$\pi$ admissible $(\mathfrak{g}, K)$-module

$$\pi|_K = \sum_{\mu \in \hat{K}} \text{mult}_\pi(\mu) \mu$$

**Theorem**: There is an algorithm to compute $\text{mult}_\pi(\mu)$

Morally this comes down to the Blattner formula plus parabolic induction. Practically speaking is an entirely different matter (for one thing $K$ is disconnected). This algorithm has been implemented in the Atlas software.
A FEW WORDS ABOUT $\hat{K}$

From now on every representation has real infinitesimal character:

$$\lambda \in X^* \otimes \mathbb{R} \quad (\text{via the Harish-Chandra homomorphism})$$

Suppose $P = MAN$ is a (real) parabolic subgroup, $\pi_M$ is a discrete series of $M$, and $\nu \in \mathfrak{a}^*$.

$\text{Ind}_P^G(\pi_M \otimes \nu)$:

has real infinitesimal character: $\nu \in \mathfrak{a}_0^*$ (real vector space)

is tempered: $\nu \in i\mathfrak{a}_0^*$

is tempered with real infinitesimal character: $\nu = 0$

(countable set)
A FEW MORE WORDS ABOUT $\hat{K}$

$\mathcal{P}_{\text{temp}}$: \{\(\pi \mid \text{irreducible, tempered (real inf. char.)}\}\}

**Theorem (Vogan):**

**Bijection:**

\[
\mathcal{P}_{\text{temp}} \longleftrightarrow \hat{K}
\]

\(\pi \rightarrow \text{lowest K-type of } \pi\)

**Note:** If \(X\) is a \((g, K)\)-module of finite length, then

\[
\text{mult}_X = \sum_{i=1}^{n} a_i \text{mult}_{\pi_i} \quad (a_i \in \mathbb{Z}, \pi_i \in \mathcal{P}_{\text{temp}})
\]
\[ G(\mathbb{R}) = SL(2, \mathbb{R}) \]
\[ K = S^1, \hat{K} = \mathbb{Z} \]
\[ \mathbb{C} = \text{trivial representation of } SL(2, \mathbb{R}): \]

(reducible) spherical principal series = \( \mathbb{C} + DS_+ + DS_- \)

\[ \mathbb{C}|_K = \text{spherical principal series}|_K - DS_+|_K - DS_-|_K \]

\[ 2\mathbb{Z} - \{2, 4, 6, \ldots \} - \{-2, -4, -6, \ldots \} \]

PS: spherical principal series with infinitesimal character 0

DS\( \pm \): holomorphic/antiholomorphic discrete series with infinitesimal character \( \rho \)

\[ \text{mult}_\mathbb{C} = \text{mult}_{PS} - \text{mult}_{DS_+} - \text{mult}_{DS_-} \]
**Signatures of Hermitian forms**

\[ G, \theta, K \ldots G(\mathbb{R}) = G^\sigma \ (\sigma \ \text{antiholomorphic}) \]

Suppose \((\pi, V)\) admits an invariant Hermitian form:

\[ \langle \pi(X)v, w \rangle + \langle v, \pi(\sigma(X))w \rangle = 0 \]

**Theorem:** an irreducible representation \(\pi\) of \(G(\mathbb{R})\) is unitary if and only if its \((g, K)\)-module admits a positive definite invariant Hermitian form.

**Problem:** Describe the Unitary Dual

set of equivalence classes of irreducible unitary representations
**Signatures of Hermitian Forms**

**Problem:** Suppose \((\pi, V)\) supports an invariant Hermitian form \(\langle , \rangle\). Compute the **signature** of \(\langle , \rangle\).

What? \(\langle , \rangle\) is positive definite if \(\langle v, v \rangle > 0\) for all \(v\)

If not, what is the “signature”?

**Definition:** \(\mathcal{W} = \mathbb{Z}[z]/(z^2 - 1) = \mathbb{Z}[s] \ (s^2 = 1)\)

**Definition:** \(\text{sig}_\pi : \hat{K} \to \mathcal{W}:\)

\(\text{sig}_\pi(\mu) = a + bs\) if in the invariant form, restricted to the \(K\)-isotypic, \(\mu\) occurs \(a\) (resp. \(b\)) times with positive (resp. negative) definite form.

**Note:** \(\text{sig}_\pi(\mu)(s = 1) = \text{mult}_\pi(\mu)\)

The question becomes: how to “compute” \(\text{sig}_\pi\)?
**Theorem:** \( \text{sig}_\pi = \sum_{i=1}^n w_i \text{mult}_{\pi_i} \) for some irreducible, tempered representations \( \pi_1, \ldots, \pi_n, \ w_i \in \mathbb{W} \)

The point is this is a **finite** formula.

In other words

\[ \text{sig}_\pi \in \mathbb{W}\langle \text{mult}_\tau \mid \tau \text{ tempered} \rangle \]
**Example:** \( SL(2, \mathbb{R}) \)

\( \pi(\nu) \): spherical principal series with infinitesimal character \( \nu \in \mathbb{R} \)

\( \hat{K} = \mathbb{Z} \)

\( \pi(\nu)|_K = 2\mathbb{Z} = \{ \ldots, -4, -2, 0, 2, 4, \ldots \} \)

\( \pi(\nu) \) is reducible \( \Leftrightarrow \nu \in 2\mathbb{Z} + 1 \)

\( \text{sig}_{\pi(0)} = \text{mult}_{\pi(0)} \) (unitary)

in fact

\( \text{sig}_{\pi(\nu)} = \text{sig}_{\pi(0)} = \text{mult}_{\pi(0)} \quad \nu < 1 \)

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Conclusion:

\[ \text{sign}_{\pi}(1+\epsilon) = \text{mult}_{\pi}(1-\epsilon) + (s-1)(\text{mult}_{\pi}(DS_+) + \text{mult}_{\pi}(DS_-)) \]

=all positive signs...change signs from \( + \) to \( - \)
The c-form

Major fly in the ointment:

a) there may be no invariant Hermitian form on \((\pi, V)\)
b) it may not be unique (up to positive scalar)

Example: odd principal series of \(SL(2, \mathbb{R})\) with \(\nu \neq 0\)

The \(K\)-types 1, \(-1\) have opposite signature \(G(\mathbb{R}), \sigma \circ \sigma_c = \theta\)

Definition The c-form satisfies

\[ \langle \pi(X)\nu, w \rangle_c + \langle \nu, \pi(\sigma_c(X))w \rangle_c = 0 \]

and \(\langle , \rangle_c\) is positive definite on all lowest \(K\)-types
Theorem:
(1) The c-form exists and is unique (up to positive scalar)
(2) The c-form determines the invariant Hermitian form (an explicit formula)

Note: if the group is not equal rank we need the c-form on the extended group

Definition: $\sigma^c_\pi : \hat{K} \to \mathbb{W}$:

$\sigma^c_\pi(\mu) = a + bs$ if in the c-form, restricted to the $K$-isotypic, $\mu$ occurs $a$ (resp. $b$) times with positive (resp. negative) definite form.

Same result as before: $\sigma^c_\pi = \sum_i w_i^c \text{mult}_{\pi_i}$
Fix infinitesimal character $\lambda$

$\mathcal{P}_\lambda$: a set of parameters

$\mathcal{P}_\lambda \ni \gamma \rightarrow I(\gamma)$ (standard module)

$J(\gamma)$ (unique irreducible quotient of $I(\gamma)$)

$\{\text{irreducible representations with infinitesimal character } \gamma\} \leftrightarrow \mathcal{P}_\lambda$

$$I(\gamma) = \text{Ind}^G_{MAN}(\pi_M \otimes \nu \otimes 1) \quad (\nu \in a_0^\ast)$$

Deformation: $\gamma_t \leftrightarrow \text{Ind}^G_{MAN}(\pi_M \otimes t\nu \otimes 1)$
Kazhdan-Lusztig-Vogan polynomials:

\[ P_{\tau, \gamma} \in \mathbb{Z}[q] \]

\[ \{ P_{\tau, \gamma} \mid \tau, \gamma \in \mathcal{P}_\lambda \} \text{ (upper unitriangular matrix)} \]

Inverse matrix \( \{ Q_{\tau, \gamma} \} \) (with signs)

\[
J(\gamma) = \sum_{\tau} (-1)^{\ell(\gamma) - \ell(\tau)} P_{\tau, \gamma}(1) I(\tau)
\]

\[
I(\gamma) = \sum_{\tau} Q_{\tau, \gamma}(1) J(\tau)
\]
**Digression: The Jantzen Filtration**

\[ I(\gamma) = \sum_{\tau} Q_{\tau,\gamma} J(\tau) \]

The **Jantzen filtration** is a canonical filtration of \( I(\gamma) \) by \((g, K)\)-modules.

Jantzen conjecture: if \( Q_{\tau,\gamma} = \sum a_j q^j \), then \( a_r \) is the multiplicity of \( J(\tau) \) in level \( \frac{1}{2}(\ell(\gamma) - \ell(\tau) + r) \) of the Jantzen filtration.

Note: \( Q_{\tau,\gamma}(1) = \sum r \, a_r \) is the multiplicity of \( J(\tau) \) in \( I(\gamma) \).
Suppose $I(\gamma)$ is a reducible standard module (at some $\nu$), and $I(\gamma_t)$ is irreducible for $0 < |1 - t| < \epsilon$.

$$I(\gamma_{1-\epsilon}) \rightarrow I(\gamma_1) \rightarrow I(\gamma_{1+\epsilon})$$

Problem: how does the c-form change as you deform from $I(\gamma_{1-\epsilon})$ to $I(\gamma_{1+\epsilon})$?

**Key fact:** the c-form changes sign on odd levels of the Jantzen filtration at $I(\Gamma)$

(Comes down to: $f(x) = x^n$ changes sign at $x = 0$ if and only if $n$ is odd.)
Algorithm (Deformation of the c-form):

\[
\sigma(\gamma_1 + \epsilon) = \sigma(\gamma_1 - \epsilon) +
(1 - s) \sum_{\phi, \tau} s^{(\ell_0(\gamma) - \ell_0(\tau))/2} P_{\phi, \tau}(s) Q_{\tau, \gamma}(s) \sigma(I(\phi))
\]

\[\phi < \tau < \gamma\]
\[\ell(\gamma) - \ell(\tau) \text{ odd}\]

**Corollary**

There is an inductive algorithm to compute \(\sigma(I(\gamma))\), in terms of \(\sigma(I(\phi))\) where \(I(\phi)\) is (irreducible) tempered.
Saito’s theory of mixed Hodge modules.

Beilinson-Bernstein theory of $\mathcal{D}$-modules, $\mathcal{D}_\lambda$-modules

Global section functor: equivalence of categories $\mathcal{D}_\lambda$-modules and $(\mathfrak{g}, K)$-modules with infinitesimal character $\lambda$. 
Schmid/Vilonen:

**Theorem** If $\pi$ is an irreducible or standard $(\mathfrak{g}, K)$-module $(\pi, V)$ it has the following canonical constructions:

1) Finite, ascending weight filtration by $(\mathfrak{g}, K)$-modules (the Jantzen filtration) $W_0 \subset W_1 \cdots \subset W_n = V$

2) Infinite, ascending Hodge filtration by finite dimensional $K$-modules $F_0 \subset F_1 \subset F_2 \cdots$

Caveat: Schmid and Vilonen have not published a proof of this (need: the global section functor is filtered exact)
The Hodge filtration

$$(\pi, V) \quad 0 \subset F_0 \subset F_1 \subset \ldots$$

$\text{gr}(\pi) = F_p/F_{p-1} \quad \text{(a finite dimensional representation of } K \text{)}$

**Definition**: $\text{hodge}_\pi : \hat{K} \to \mathbb{Z}[v]$

$h\text{odge}_\pi(\mu) = a_0 + a_1 v + \cdots + a_n v^n : \ a_i = \text{mult}_{\text{gr}_i(\pi)}(\mu)$
So:

\[ \text{hodge}_\pi : \hat{K} \to \mathbb{Z}[v] \]
\[ \text{sig}^c_\pi : \hat{K} \to \mathbb{Z}[s] \]
\[ \text{mult}_\pi : \hat{K} \to \mathbb{Z} \]

Note:

\[ \text{hodge}_\pi |_{v=1} = \text{sig}^c_\pi |_{s=1} = \text{mult}_\pi \]
**Examples of the Hodge filtration**

$SL(2, \mathbb{R}), \pi(0) =$ tempered, spherical principal series,
$V = \langle w_k \mid k \in 2\mathbb{Z} \rangle.$

$h_{\text{odge}}I(0)(w_{2k}) = v^{|k|}$

$G(\mathbb{R})$ split, $I(0)$: $I(0)|_K \simeq$ ring of regular functions on $\mathcal{N} \cap \mathfrak{p}$

Discrete series: graded Blattner formula
The Main Result

**Theorem** (Adams/Trapa/Vogan):

\[
\text{hodge}_\pi|_{\nu=s} = \text{sig}_\pi^c
\]

In other words: if \( \mu \in \hat{K} \):

\[
\text{hodge}_\pi(\mu) = a_0 + a_1 \nu + \cdots + a_n \nu^n
\]

implies

\[
\text{sig}_\pi^c(\mu) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n
\]

\[
= (a_0 + a_2 + \cdots) + (a_1 + a_3 + \cdots)s
\]
From earlier:

Suppose $l(\gamma)$ is a reducible standard module (at some $\nu$), and $l(\gamma_t)$ is irreducible for $0 < |1 - t| < \epsilon$.

Problem: how does the c-form change as you deform from $l(\gamma_{1-\epsilon})$ to $l(\gamma_{1+\epsilon})$?

**Key fact** (signature): the c-form changes sign on odd levels of the Jantzen filtration.

Problem: how does the Hodge filtration change as you deform from $l(\gamma_{1-\epsilon})$ to $l(\gamma_{1+\epsilon})$?

**Key fact** (Hodge): a K-type in level $k$ of the Jantzen filtration jumps by $k$ levels in the Hodge filtration.
**Sketch of Proof**

Algorithm (Deformation of the c-form):

\[
sig(\gamma_{1+\epsilon}) = \sig(\gamma_{1-\epsilon}) + (1 - s) \sum_{\phi, \tau \mid \phi < \tau < \gamma, \ell(\gamma) - \ell(\tau) \text{ odd}} s^{(\ell_0(\gamma) - \ell_0(\tau))/2} P_{\phi, \tau}(s) Q_{\tau, \gamma}(s) \sig(I(\phi))
\]

Algorithm (Deformation of the Hodge filtration):

\[
hodge(I(\Gamma_{1+\epsilon})) = hodge(I(\Gamma_{1-\epsilon})) - \sum_{\phi < \Gamma} v^{(\ell_0(\Gamma) - \ell_0(\Phi))/2} \left[ \sum_{\Phi \leq \Xi \leq \Gamma} (-1)^{\ell(\Xi) - \ell(\Phi)} v^{\ell(\Gamma) - \ell(\Xi)} P_{\phi, \Xi}(v) Q_{\Xi, \Gamma}(v^{-1}) \right] hodge(I(\Phi))
\]
The Hodge formula, evaluated at $v = s$, gives the signature formula.

This reduces us to the case of tempered representations.

[This is another story about as long as this one]

Caveat: We haven’t completely finished the tempered part of the argument.

**Note:** This is *theorem*. It *also* gives an algorithm to compute the Hodge filtration.
The Schmid-Vilonen Conjecture

Conjecture 1: The c-form restricted to $F_p$ is non-degenerate.
Assuming this, the c-form induces a form on

$$\text{gr}_p(\pi) = F_p/F_{p-1} \simeq F_p \cap F_{p-1}$$

Conjecture 2: The c-form on $\text{gr}_p(\pi)$ is definite of sign $\epsilon_\pi(-1)^p$
($\epsilon_\pi = \pm 1$ is an elementary sign)

Conjecture 2 implies the Main Theorem
(but NOT vice-versa)

(Main Theorem + Conjecture 1 $\nRightarrow$ Conjecture 2 😞)

Thank You