Galois Cohomology of Real Groups

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1 Introduction

Suppose G is a connected reductive algebraic group, defined over $\mathbb{R}$. Thus we identify G with its complex points $G(\mathbb{C})$, we are given an antiholomorphic involution σ of G, and $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ is a real Lie group. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and write $H^i(\Gamma, G)$ for the Galois cohomology of G. If we want to specify how the nontrivial element of Γ acts we will write $H^i_\sigma(\Gamma, G)$. The real forms of G, which are inner to σ (see Section 6), are parametrized by $H^1(\Gamma, G_{\mathrm{ad}})$ where G_{ad} is the adjoint group.

On the other hand Cartan classified the real forms of G in terms of holomorphic involutions as follows. Associated to σ is a Cartan involution θ of G. This is a holomorphic involution, commuting with σ, such that $K(\mathbb{R}) = G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$. Then $K(\mathbb{R})$ is the real points of the complex, reductive, (possibly disconnected) group $K = G^\theta$. Conversely σ is determined by θ, and the real forms inner to σ can also be parametrized by involutions inner to θ. See Example 4.16 for a precise statement.

Let $H^i(\mathbb{Z}_2, G)$, or $H^i_\theta(\mathbb{Z}_2, G)$ be group cohomology where the nontrivial element of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts by θ. Then conjugacy classes of involutions which are inner to θ are parametrized by $H^1_\theta(\mathbb{Z}_2, G_{\mathrm{ad}})$. Thus the equivalence of the two classifications of real forms amounts to an isomorphism $H^1_\sigma(\Gamma, G_{\mathrm{ad}}) \simeq H^1_\theta(\mathbb{Z}_2, G_{\mathrm{ad}})$.

It is natural to ask if there is a relationship between $H^1_\sigma(\Gamma, G)$ and $H^1_\theta(\mathbb{Z}_2, G)$ in general.

Theorem 1.1 *Suppose G is defined over $\mathbb{R}$, given by an antiholomorphic involution σ. Let θ be a corresponding Cartan involution. Then there is a canonical isomorphism $H^1_\sigma(\Gamma, G) \simeq H^1_\theta(\mathbb{Z}_2, G)$.*

The interplay between the σ and θ pictures plays a fundamental role in the representation theory of real groups, going back at least to Harish Chandra's

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formulation of the representation theory of $G(\mathbb{R})$ in terms of $(\mathfrak{g}, K)$-modules. The theorem is an aspect of this, and we give several applications.

Matsuki proved there is a bijection between $G(\mathbb{R})$-conjugacy classes of Cartan subgroups of $G(\mathbb{R})$ and K-conjugacy classes of θ-stable Cartan subgroups of G [11]. In Section 5 we give a simple proof of this based on Theorem 1.1.

The fact that $H^1(\Gamma, G_{\mathrm{ad}})$ parametrizes the real forms of G in a given inner class reduces the computation of the cohomology to the classification of real forms, which can be accomplished in a number of ways. We seek an analogous description of $H^1(\Gamma, G)$ in general.

A formula for $H^1(\Gamma, G)$ is in [7]. We start with this, and modify it in two steps. First of all we replace $H^1(\Gamma, G)$ with $H^1(\mathbb{Z}_2, G)$. This a more elementary object. For example $G(\mathbb{R})$ is compact if and only if $\theta = 1$, in which case $H^1(\mathbb{Z}_2, G)$ is the conjugacy classes of involutions of G. See Example 2.1. Next, we bring in the theory of strong real forms [3], [4]. The strong real forms of a group G map surjectively to the real forms (see Lemma 6.5), and bijectively if G is adjoint.

Let Z be the center of G. Associated to σ is its central invariant, denoted $\mathrm{inv}(\sigma) \in Z^\Gamma/(1+\sigma)Z$. In addition there is a notion of central invariant of a strong real form, which is an element of Z^Γ. See Section 6 for these definitions.

Theorem 1.2 (Proposition 6.13) *Suppose σ is a real form of G. Choose a representative $z \in Z^\Gamma$ of $\mathrm{inv}(\sigma) \in Z^\Gamma/(1+\sigma)Z$. Then there is a bijection*

$$H^1(\Gamma, G) \overset{1-1}{\longleftrightarrow} \textit{the set of strong real forms with central invariant } z$$

This bijection is useful in both directions. On the one hand it is not difficult to compute the right hand side, thereby computing $H^1(\Gamma, G)$. Over a p-adic field $H^1(\Gamma, G) = 1$ if G is simply connected by (Kneser's theorem). Over $\mathbb{R}$ this is not the case, and we use Theorem 1.1 to compute $H^1(\Gamma, G)$ for all such G. See Section 6 and the tables in Section 9. We used the Atlas of Lie Groups and Representations software for some of these calculations.

On the other hand the notion of strong real form is important in formulating a precise version of the local Langlands conjecture. In that context it would be more natural if strong real forms were described in terms of classical Galois cohomology. The theorem provides such an interpretation. See Corollary 7.5.

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2 Preliminaries on Group Cohomology

See [13] for an overview of group cohomology.

For now suppose τ is an involution of an abstract group G. Define $H^i(\mathbb{Z}_2, G)$ to be the group cohomology where the nontrivial element of $\mathbb{Z}_2$ acts by τ. If G

is abelian these are groups and are defined for all $i \geq 0$. Otherwise these are pointed sets, and defined only for $i = 0, 1$. We have the standard identifications

$$H^0(\mathbb{Z}_2, G) = G^\tau,\ H^1(\mathbb{Z}_2, G) = G^{-\tau}/[g \to xg\tau(x^{-1})]$$

where $G^{-\tau} = \{g \in G \mid g\tau(g) = 1\}$. For $g \in G^{-\tau}$ let $[g]$ be the corresponding class in $H^1(\mathbb{Z}_2, G)$.

If G is abelian we also have the Tate cohomology groups $\widehat{H}^i(\mathbb{Z}_2, G)$ $(i \in \mathbb{Z})$. These satisfy

$$\widehat{H}^0(\mathbb{Z}_2, G) = G^\tau/(1+\tau)G, \quad \widehat{H}^1(\mathbb{Z}_2, G) = H^1(\mathbb{Z}_2, G),$$

and (since $\mathbb{Z}_2$ is cyclic), $\widehat{H}^i(\mathbb{Z}_2, G) \simeq \widehat{H}^{i+2}(\mathbb{Z}_2, G)$ for all i.

Now suppose G is a connected reductive algebraic group. We say G is defined over $\mathbb{R}$ if we are given a Galois action on G. This means that the nontrivial element of Γ acts on $G(\mathbb{C})$ by an antiholomorphic involution σ. Then we recover the usual Galois cohomology $H^i(\Gamma, G)$. (We identify G with its complex points $G(\mathbb{C})$, but sometimes write $G(\mathbb{C})$ for emphasis.) Then $H^0(\Gamma, G) = G(\mathbb{R})$, and if G is abelian then $\widehat{H}^0(\Gamma, G) = \pi_0(G(\mathbb{R}))$.

By definition a real form of G is a conjugacy class of antiholomorphic involutions, and $H^1_\sigma(\Gamma, G)$ depends only on the conjugacy class of σ. When there is no danger of confusion we refer to σ itself as a real form.

On the other hand suppose θ is a holomorphic involution of $G = G(\mathbb{C})$, and let $K = K(\mathbb{C}) = G^\theta$. Then $H^0(\mathbb{Z}_2, G) = K$, and if G is abelian $\widehat{H}^0(\mathbb{Z}_2, G) = \pi_0(K)$.

Since we will be using cohomology defined with respect to both holomorphic and anti-holomorphic involutions we write $H^i(\Gamma, G)$ for Galois cohomology, and $H^i(\mathbb{Z}_2, G)$ for cohomology with respect to a holomorphic involution. When there are multiple involutions being considered we will specify these by writing $H^i_\sigma(\Gamma, G)$ or $H^i_\theta(\mathbb{Z}_2, G)$.

Fix an antiholomorphic involution σ, and a holomorphic involution θ. We say θ corresponds to σ if θ is a Cartan involution of $G(\mathbb{R}) = G(\mathbb{C})^\sigma$. In other words: σ and θ commute, and $G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$. Equivalently $\theta = \sigma\sigma_c$ where σ, σ_c commute, and $G(\mathbb{C})^{\sigma_c}$ is compact. See Example 4.16 for the relationship with the classification of real forms.

Example 2.1 The group $G(\mathbb{R}) = G^\sigma$ is compact if and only if $\theta = 1$. Then $H^1_\theta(\mathbb{Z}_2, G) = \{g \in G \mid g^2 = 1\}/[g \to xgx^{-1}]$, i.e. $H^1_\theta(\mathbb{Z}_2, G)$ is the set of conjugacy classes of involutions of G. Therefore, if we fix a Cartan subgroup H, with Weyl group W, then

$$H^1_\theta(\mathbb{Z}_2, G) \simeq H_2/W$$

where $H_2 = \{h \in H \mid h^2 = 1\}$. See Example 4.15.

2.1 Twisting

We make repeated use of twisting in nonabelian cohomology [13, Section III.4.5]. We describe how this works in our situation. Return to the setting of an abstract group G together with an involution τ, and consider $H^1_\tau(\mathbb{Z}_2, G)$.

For $g \in G$ let $\mathrm{int}(g)(x) = gxg^{-1} \quad (x \in G)$. Suppose $g \in G$, and $g\tau(g)$ is in the center $Z = Z(G)$ of G. Then $\tau' = \mathrm{int}(g) \circ \tau$ is an involution, and $H^i_{\tau'}(\mathbb{Z}_2, G)$ is defined. This is not necessarily isomorphic to $H^i_\tau(\mathbb{Z}_2, G)$, unless $g \in G^{-\tau}$, in which case the map $x \to gx$ induces an isomorphism $H^1_\tau(\mathbb{Z}_2, G) \simeq H^1_{\tau'}(\mathbb{Z}_2, G)$.

3 Cohomology of Tori

Suppose H is a torus, defined over $\mathbb{R}$, with Galois action σ. Let θ be the Cartan involution corresponding to σ: $\theta = \sigma\sigma_c$ where σ_c is the compact real form of H (σ_c and θ are unique). Then $H(\mathbb{R})^\theta$ is the maximal compact subgroup of $H(\mathbb{R})$.

Proposition 3.1 *For all i there is a canonical isomorphism $\widehat{H}^i_\sigma(\Gamma, H) \simeq \widehat{H}^i_\theta(\mathbb{Z}_2, H)$.*

We only need to consider $i = 0, 1$.

Remark 3.2 From the structure of real tori it is easy to see there are isomorphisms as indicated, although not necessarily canonical ones. It is well known that $H(\mathbb{R}) \simeq S^{1a} \times \mathbb{R}^{*b} \times \mathbb{C}^{*c}$, in which case $K = K(\mathbb{C}) = \mathbb{C}^{*(a+c)} \times \mathbb{Z}_2^b$.

Then $\widehat{H}^0_\sigma(\Gamma, H) = \pi_0(H(\mathbb{R})) \simeq \mathbb{Z}_2^b$, and $\widehat{H}^0_\theta(\mathbb{Z}_2, H) = \pi_0(K) = \mathbb{Z}_2^b$.

Furthermore it is easy to see $H^1(\Gamma, H)$ and $H^1(\mathbb{Z}_2, H)$ are trivial if if $H(\mathbb{R}) = \mathbb{R}^*$ or $\mathbb{C}^*$ (for Γ this is Hilbert's theorem 90), or $\mathbb{Z}_2$ if $H(\mathbb{R}) = S^1$. Therefore $H^1_\sigma(\Gamma, H) \simeq H^1_\theta(\mathbb{Z}_2, H) \simeq \mathbb{Z}_2^a$.

Proof. Consider the exact sequence

$$1 \to H_2 \to H \stackrel{z^2}{\to} H \to 1$$

This gives the long exact sequence in Tate cohomology:

$$\widehat{H}^0_\sigma(\Gamma, H) \stackrel{\alpha}{\to} \widehat{H}^0_\sigma(\Gamma, H) \to \widehat{H}^1_\sigma(\Gamma, H_2) \to \widehat{H}^1_\sigma(\Gamma, H) \stackrel{\beta}{\to} \widehat{H}^1_\sigma(\Gamma, H)$$

It is easy to see α, β, which are induced by the square map, are trivial: $\sigma(t) = t$ implies $t^2 = t\sigma(t)$, and $t\sigma(t) = 1$ implies $t^2 = (t^{-1})^{-1}\sigma(t^{-1})$. So we have an exact sequence

$$1 \to \widehat{H}^0_\sigma(\Gamma, H) \to \widehat{H}^1_\sigma(\Gamma, H_2) \to \widehat{H}^1_\sigma(\Gamma, H) \to 1$$

The analogous sequence holds for θ:

$$1 \to \widehat{H}^0_\theta(\mathbb{Z}_2, H) \to \widehat{H}_\theta(\mathbb{Z}_2, H_2) \to \widehat{H}_\theta(\mathbb{Z}_2, H) \to 1$$

Let $H_c(\mathbb{R}) = H^{\sigma_c}$ be the compact real form of H. This is the closure of the subgroup of $H(\mathbb{C})$ of all elements of finite order, so $H_2 \subset H_c(\mathbb{R})$. Since $\theta = \sigma\sigma_c$,

θ and σ agree on H_2, so the identity map on H_2 induces an isomorphism ϕ : $\widehat{H}^1_\sigma(\Gamma, H_2) \simeq \widehat{H}^1_\theta(\mathbb{Z}_2, H_2)$.

It is well known, and easy to compute directly, that $\widehat{H}^0_\sigma(\Gamma, H) = \pi_0(H(\mathbb{R}))$ is isomorphic to $\widehat{H}^0_\theta(\mathbb{Z}_2, H) = \pi_0(H^\theta)$. An explicit calculation using any isomorphism $H(\mathbb{R}) \simeq S^{1a} \times \mathbb{R}^{*b} \times \mathbb{C}^{*c}$ shows that the images of $\widehat{H}^0_\sigma(\Gamma, H)$ in $H^1_\sigma(\Gamma, H_2)$ and $\widehat{H}^0_\theta(\mathbb{Z}_2, H)$ in $H^1_\theta(\mathbb{Z}_2, H_2)$ agree under the the isomorphism ϕ. In other words the left hand box of

$$\begin{tikzcd}
1 \arrow[r] & \widehat{H}^0_\sigma(\Gamma, H) \arrow[r] \arrow[d, "\simeq"'] & H^1_\sigma(\Gamma, H_2) \arrow[r] \arrow[d, "\phi"] & H^1_\sigma(\Gamma, H) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \widehat{H}^0_\theta(\mathbb{Z}_2, H) \arrow[r] & H^1_\theta(\mathbb{Z}_2, H_2) \arrow[r] & H^1_\theta(\mathbb{Z}_2, H) \arrow[r] & 1
\end{tikzcd}$$

commutes. It follows that there is a unique isomorphism $H^1_\sigma(\Gamma, H) \simeq H^1_\theta(\mathbb{Z}_2, H)$ making the entire diagram commute. $\square$

4 Cohomology of reductive groups

Now suppose $G = G(\mathbb{C})$ is a connected reductive group, defined over $\mathbb{R}$, with Galois action σ, and corresponding Cartan involution θ (see Section 2).

If H is a Cartan subgroup of G let $N = \mathrm{Norm}_G(H)$, and $W = N/H$. If H is σ-stable then σ acts on N and W. The short exact sequence $1 \to H \to N \to W \to 1$ gives rise to the exact cohomology sequence

(4.1) $H^0(\Gamma, H) \to H^0(\Gamma, N) \to H^0(\Gamma, W) \to H^1(\Gamma, H) \to H^1(\Gamma, N) \to H^1(\Gamma, W)$

The third map takes $H^0(\Gamma, W) = W^\sigma$ to a subgroup of $H^1(\Gamma, H)$, and thereby acts by conjugation.

Lemma 4.2 *Suppose $w \in W^\sigma$ and $h \in H^{-\sigma}$. Choose $n \in N$ mapping to w. Then the action of w on $H^1(\Gamma, H)$ is $w : [h] \to [nh\sigma(n^{-1})]$; this is well defined, independent of the choices involved.*

The image of $H^1(\Gamma, H)$ in $H^1(\Gamma, N)$ is isomorphic to $H^1(\Gamma, H)/W^\sigma$.

This is immediate.

We say a root α of H in G is imaginary, real, or complex if $\sigma(\alpha) = -\alpha$, $\sigma(\alpha) = \alpha$, or $\sigma(\alpha) \neq \pm\alpha$, respectively. Let $W_i \subset W^\sigma$ be the Weyl group of the root system of imaginary roots.

Lemma 4.3 $H^1(\Gamma, H)/W^\sigma = H^1(\Gamma, H)/W_i$.

Proof. Write $W^\sigma = (W_C)^\sigma \ltimes [W_i \times W_r]$ as in [16, Proposition 4.16]. Here W_r is Weyl group of the real roots, and $(W_C)^\sigma$ is a certain Weyl group, generated by terms of the form $s_\alpha s_{\sigma\alpha}$ where $\alpha, \sigma\alpha$ are orthogonal. It is easy to see that W_r acts trivially on $H^1(\Gamma, H)$, and $(W_C)^\sigma$ does as well [4, Proposition 12.16]. $\square$

Remark 4.4 Note that $H^1_\sigma(\Gamma, H)$ only depends on the restriction of σ to H, as does the notion of imaginary root. However this is not the case for the action of W_i on $H^1_\sigma(\Gamma, H)$ of Lemma 4.2, which is sensitive to the restriction of σ to N.

We say a Cartan subgroup H is fundamental if it is of minimal split rank. Such a Cartan subgroup is the centralizer of a Cartan subgroup of the identity component of K.

Proposition 4.5 *The map $H^1_\sigma(\Gamma, H) \to H^1_\sigma(\Gamma, G)$ factors through the quotient by W_i, and induces an injection*

$$\phi : H^1_\sigma(\Gamma, H)/W_i \hookrightarrow H^1_\sigma(\Gamma, G). \tag{4.6}$$

If H is fundamental this is an isomorphism.

Injectivity amounts to the fact that for σ-stable Cartan subgroups of G, conjugacy is equivalent to stable conjugacy [14, Corollary 2.3] (see below). This in turn follows from the analogous statement for parabolic subgroups (true over any field), and $G(\mathbb{R})$-conjugacy of compact Cartan subgroups. Surjectivity for a fundamental Cartan subgroup H is in [10, Lemma 10.2], For the fundamental Cartan subgroup injectivity and surjectivity are proved in [7] (with W^σ in place of W_i). We give complete proofs, for the convenience of the reader, and because we need to repeat the arguments in the setting of $H^1(\mathbb{Z}_2, G)$.

Lemma 4.7 *Suppose P, P' are σ-stable parabolic subgroups of G, with σ-stable Levi factors M, M'. If M, M' are conjugate then they are $G(\mathbb{R})$-conjugate.*

Proof. We first show that if P, P' are conjugate then they are $G(\mathbb{R})$-conjugate. The parabolic subgroups conjugate to P are in bijection with G/P, via the map $gP \to gPg^{-1}$. The σ-stable parabolic subgroups in this set are given by $(G/P)(\mathbb{R})$. The map $G(\mathbb{R}) \to (G/P)(\mathbb{R})$, obtained by taking σ-fixed points of the projection $G \to G/P$, is surjective [6, Theorem 4.13(a)]. If $g \in G(\mathbb{R})$ maps to $Q \in (G/P)(\mathbb{R})$ then $Q = gPg^{-1}$.

Therefore, after conjugating by $G(\mathbb{R})$, we may assume $P = P'$. Suppose M, M' are σ-stable Levi factors of P. All Levi factors of P are U-conjugate, where U is the unipotent radical of P. Write $M' = uMu^{-1}$ $(u \in U)$. Then M' is σ-stable if and only if $u^{-1}\sigma(u) \in M$, but U is σ-stable and $U \cap M = 1$, so $u = \sigma(u)$.

□

We say two σ-stable Cartan subgroups H, H' of G are stably conjugate if there exists $g \in G(\mathbb{C})$ such that $gHg^{-1} = H'$ and $\mathrm{int}(g) : H \to H'$ is defined over $\mathbb{R}$.

Lemma 4.8 ([14], Corollary 2.3) *Two σ-stable Cartan subgroups are stably conjugate if and only if they are $G(\mathbb{R})$-conjugate.*

Proof. Suppose H, H' are σ-stable and stably conjugate. Choose $g \in G$ so that $\mathrm{int}(g) : H \to H'$ is defined over $\mathbb{R}$.

Write $H = TA$ where T (resp. A) is the maximal compact (resp. split) subtorus of H. Let $M = \mathrm{Cent}_G(A)$, this is σ-stable and contained in a σ-stable parabolic subgroup P (define the roots of H in U to be $\{\alpha \mid \mathrm{Re}(\alpha(\gamma)) > 0\}$ for γ a regular element of the Lie algebra of A). Define $M' \supset H'$ similarly.

The fact that $\mathrm{int}(g)|_H$ is defined over $\mathbb{R}$ implies $\mathrm{int}(g)(M) = M'$, so by the previous Lemma choose $y \in G(\mathbb{R})$ so that $\mathrm{int}(y)(M') = M$. Then T and $\mathrm{int}(y)(T')$ are compact Cartan subgroups of the derived group of M. Therefore there exists $m \in M(\mathbb{R})$ such that $\mathrm{int}(my)(T') = T$. Then $my \in G(\mathbb{R})$ and $\mathrm{int}(my)(H') = H$. $\square$

Proof of Proposition 4.5. For injectivity suppose $h, h' \in H^{-\sigma}$ and $[h], [h']$ have the same image in $H^1(\Gamma, G)$. Thus $h = gh'\sigma(g^{-1})$ for some $g \in G$. Let $\sigma' = \mathrm{int}(h) \circ \sigma$; this is a real form of G. It is immediate that $\mathrm{int}(g) : H \to gHg^{-1}$ commutes with σ'. By the previous lemma $gHg^{-1} = yHy^{-1}$ for some $y \in G^{\sigma'}$. Let $n = g^{-1}y \in N$. Then $n^{-1}h'\sigma(n) = y^{-1}gh'\sigma(g^{-1})\sigma(y) = y^{-1}h\sigma(y) = h$ (the last equality follows from the condition $y \in G^{\sigma'}$).

It is easy to see the image of n in W is contained in W^σ. By Lemma 4.3 we can replace n with an element mapping to W_i. This proves injectivity.

For surjectivity we follow [7]. Suppose $g \in G^{-\sigma}$. Write the Jordan decomposition of g as $g = su$. Then $u^{-1}\sigma(u^{-1}) = s\sigma(s)$, so $u\sigma(u) = s\sigma(s) = 1$. Then $u = v^2$ where $v\sigma(v) = 1$ and v commutes with s. It follows that $vg\sigma(v^{-1}) = s$, so without loss of generality we may assume g is semisimple. Furthermore $g\sigma(g) = 1$ implies g commutes with $\sigma(g)$, so without loss of generality g is contained in a σ-stable torus H' (not necessarily the same as H).

Write $H' = T'A'$ (T' and A' are compact and split, respectively) and $g = ta$ accordingly. Then $t\sigma(t) = (a\sigma(a))^{-1}$, so this element is in $T'(\mathbb{R}) \cap A'(\mathbb{R})$, which is trivial. Therefore $a\sigma(a) = 1$. Since $H^1(\Gamma, A') = 1$ choose $b \in A'$ so that $b\sigma(b^{-1}) = a$, and $bg\sigma(b^{-1}) = t$. So we may assume $g \in T'$. But T', being a compact torus, is conjugate to a subtorus of any fundamental torus. This proves surjectivity. $\square$

The analogous result, with essentially the same proof, holds with θ in place of σ. As in Lemma 4.2, W^θ acts on $H^1_\theta(\mathbb{Z}_2, H)$.

Proposition 4.9 *The map $H^1(\mathbb{Z}_2, H) \to H^1(\mathbb{Z}_2, G)$ factors through the quotient by W_i, and induces an injection*

$$\phi : H^1_\theta(\mathbb{Z}_2, H)/W_i \hookrightarrow H^1(\mathbb{Z}_2, G). \tag{4.10}$$

If H is fundamental this is an isomorphism.

First we first need versions of Lemmas 4.7 and 4.8. Let $K = G^\theta$.

Lemma 4.11 *Suppose Q, Q' are θ-stable parabolic subgroups of G, with θ-stable Levi factors L, L'. If L, L' are conjugate then they are K-conjugate.*

Two θ-stable Cartan subgroups are conjugate if and only if they are K-conjugate.

Proof. The proof of the first part is similar to that of Lemma 4.7, using the fact that $P \simeq K/K \cap P$. The proof of Lemma 4.8 also carries over to this situation; in this setting we use the fact that any two θ-stable split Cartan subgroups of L are K-conjugate. We leave the details to the reader. □

Proof of Proposition 4.9 (sketch). Write $H = TA$ as before, and let $L = \mathrm{Cent}_G(T)$. This is contained in a θ-stable Cartan subgroup Q. Write $Q = LV$ with V unipotent. Injectivity follows as before with (Q, L, V, θ) in place of (P, M, U, σ).

In the proof of surjectivity write $g = ta$. Since θ acts by inverse on A, and (since A is connected) any element of A is of the form $b\theta(b^{-1})$ for some $b \in A$, after conjugating by b we may assume $g \in T$. Finally T is K-conjugate to a subtorus of a fixed fundamental Cartan subgroup.

We leave the few remaining details to the reader. □

Lemma 4.12 *Suppose H is both σ and θ stable. Then the actions of σ and θ on W agree, and the isomorphism of Proposition 3.1 commutes with the actions of $W^\sigma = W^\theta$.*

Proof. Write $\theta = \sigma\sigma_c$ where σ, σ_c commute and $G_c(\mathbb{R}) = G^{\sigma_c}$ is compact. Then W is isomorphic to $\mathrm{Norm}_{G_c(\mathbb{R})}(H^{\sigma_c})$, i.e. every Weyl group element has a representative in $G_c(\mathbb{R})$. The result follows. □

Proof of Theorem 1.1. Fix a σ, θ-stable fundamental Cartan subgroup H_f and consider the diagram

(4.13)
$$\begin{tikzcd} H^1_\sigma(\Gamma, H_f)/W_i \arrow[d, "\simeq"'] & H^1_\sigma(\Gamma, G) \arrow[l, "(4.6)"', "\simeq"] \arrow[d, "\simeq"] \\ H^1_\theta(\mathbb{Z}_2, H_f)/W_i \arrow[r, "\simeq", "(4.10)"'] & H^1_\theta(\mathbb{Z}_2, G) \end{tikzcd}$$

The left arrow is the one from Proposition 3.1, together with Lemma 4.12. Define the right arrow to be the composition of the other three. This is an isomorphism, depending only on the choice of H_f. Any two fundamental Cartan subgroups are conjugate. It is easy to see this changes the induced map by twisted conjugation $g \to xg\sigma(x^{-1})$ and $g \to xg\theta(x^{-1})$. These are absorbed in the quotients defining the cohomology, so the resulting isomorphism is independent of the choice of H_f. □

The isomorphism of Theorem 1.1 is made explicit as follows. Suppose $\gamma \in H^1(\Gamma, G)$. Chooose $h \in H^{-\sigma}$ so that $\gamma = [h]$. Furthermore choose, as is possible, $x \in H$ so that $h' = xh\sigma(h)^{-1} \in H_2$. Then take γ to $[h'] \in H^1(\mathbb{Z}_2, G)$.

Recall (Lemma 4.2) $H^1_\sigma(\Gamma, H)/W_i$ is the image of $H^1(\Gamma, H)$ in $H^1(\Gamma, N)$. We bring N into the picture in Section 5.

Remark 4.14 By Proposition 3.1 if G is a torus, in addition to the isomorphism $H^1(\Gamma, G) \simeq H^1(\mathbb{Z}_2, G)$, we have $\widehat{H}^0_\sigma(\Gamma, G) \simeq \widehat{H}^0_\theta(\mathbb{Z}_2, G)$, i.e. $\pi_0(G(\mathbb{R})) = \pi_0(G^\theta)$ (see Section 3). It is well known that $\pi_0(G(\mathbb{R})) \simeq \pi_0(G^\theta)$ for G reductive. It would be interesting if one could define "non-commutative Tate cohomology" in such a way that $\widehat{H}^0_\theta(\mathbb{Z}_2, G) = \pi_0(G^\theta)$ and $\widehat{H}^0_\sigma(\Gamma, G) = \pi_0(G(\mathbb{R}))$, and periodicity holds, so that Proposition 3.1 holds for G reductive.

Example 4.15 If $G(\mathbb{R})$ is compact θ is the identity, and $H^1(\mathbb{Z}_2, G)$ is the set of conjugacy classes of involutions of $G = G(\mathbb{C})$. This is in bijection with H_2/W, where $H = H(\mathbb{C})$ is any Cartan subgroup and W is the Weyl group (see Example 2.1).

On the other hand $H^1(\mathbb{Z}_2, G(\mathbb{R}))$ (with the trivial action) is the set of conjugacy classes of involutions in $G(\mathbb{R})$, i.e. $H(\mathbb{R})_2/W$. Since $H(\mathbb{R})$ is compact this is equal to $H(\mathbb{C})_2/W$. So we recover [13, Theorem 6.1]: $H^1(\Gamma, G) \simeq H^1(\mathbb{Z}_2, G(\mathbb{R})) = H(\mathbb{R})_2/W$.

Example 4.16 Define

$$\overline{\mathrm{Invol}}(G) = \{\text{antiholomorphic involutions of } G\}$$

and if $\sigma \in \overline{\mathrm{Invol}}(G)$ let

$$\overline{\mathrm{Invol}}_\sigma(G) = \{\mu \in \overline{\mathrm{Invol}}(G) \mid \mu^{-1} \circ \sigma \text{ is an inner automorphism of } G\}.$$

We view $\overline{\mathrm{Invol}}_\sigma(G)$ as a pointed set with distinguished element σ.

By definition the set of real forms of G is $\overline{\mathrm{Invol}}(G)/G$ (with G acting by conjugation by inner automorphisms), and $\overline{\mathrm{Invol}}_\sigma(G)/G$ is the real forms inner to σ (see Section 6).

There is a canonical isomorphism (of pointed sets)

$$H^1_\sigma(\Gamma, G_{\mathrm{ad}}) \xleftrightarrow{1-1} \overline{\mathrm{Invol}}_\sigma(G)/G. \tag{4.17)(a}$$

The map takes $[g]$ to the conjugacy class of $\mathrm{int}(g) \circ \sigma$.

On the other hand let $\mathrm{Invol}(G)$ be the holomorphic involutions of G, and $\mathrm{Invol}_\theta(G)$ to be those which are inner to θ. Then there is a canonical isomorphism

$$H^1_\theta(\mathbb{Z}_2, G_{\mathrm{ad}}) \xleftrightarrow{1-1} \mathrm{Invol}_\theta(G)/G, \tag{4.17)(b}$$

taking $[g]$ to $\mathrm{int}(g) \circ \theta$. We have a commutative diagram:
(4.17)(c)

$$\begin{tikzcd}
H^1_\sigma(\Gamma, G_{\mathrm{ad}}) \arrow[r, leftrightarrow, "(a)"] \arrow[d, leftrightarrow, "\text{Theorem 1.1}"'] & \overline{\mathrm{Invol}}_\sigma(G)/G \arrow[r, equal] \arrow[d, leftrightarrow] & \{\text{real forms of } G \text{ inner to } \sigma\} \\
H^1_\theta(\mathbb{Z}_2, G_{\mathrm{ad}}) \arrow[r, leftrightarrow, "(b)"] & \mathrm{Invol}_\theta(G)/G &
\end{tikzcd}$$

All the arrows are isomorphisms (of pointed sets), and the equality is a definition. The right hand vertical arrow is Cartan's description of real forms in terms of holomorphic involutions.

Example 4.18 Suppose $G = PSL(2,\mathbb{C})$. This has two real forms, $PGL(2,\mathbb{R}) \simeq SO(2,1)$ and $SO(3)$. Since G is adjoint $|H^1(\Gamma, G)| = 2$ for either real form.

Now let $G = SL(2,\mathbb{C})$. From Example 4.15 if $G(\mathbb{R}) = SU(2)$ then $|H^1(\Gamma, G)| = 2$. On the other hand if $G(\mathbb{R}) = SL(2,\mathbb{R})$ then it is well known that $H^1(\Gamma, G) = 1$. To see this using Theorem 1.1, take H to be the diagonal Cartan subgroup, and $\theta_c = 1$, $\theta_s = \mathrm{int}(\mathrm{diag}(i, -i))$ (the Cartan involutions for $SU(2)$ and $SL(2,\mathbb{R})$, respectively). In both cases $H_2 = \pm I$. What is different is the twisted action of W_i, which is trivial if $\theta = \theta_c$, whereas if g represents the nontrivial element of the Weyl group then $gI\theta_s(g^{-1}) = -I$.

Note that, in contrast to the adjoint case, although $SL(2,\mathbb{R})$ and $SU(2)$ are inner forms of each other, their cohomology is different. See Lemma 6.21.

Corollary 4.19 *Suppose H_f is a fundamental Cartan subgroup. Let A_f be the the identity component of the (complex) maximal split subtorus and let $M_f = \mathrm{Cent}_G(A_f)$. Then*

$$H^1_\sigma(\Gamma, G) \simeq H^1_\sigma(\Gamma, M_f) \simeq H^1_\theta(\mathbb{Z}_2, M_f) \simeq H^1_\theta(\mathbb{Z}_2, G).$$

Note that $A_f \subset Z \Leftrightarrow M_f = G \Leftrightarrow$ the derived group of G is of equal rank.

5 Conjugacy classes of Cartan subgroups

We continue in the setting of the previous section, with a Galois action σ and a corresponding Cartan involution θ.

Proposition 5.1 *Let N be the normalizer of a θ, σ-stable Cartan subgroup. Let Φ be the isomorphism of Theorem 1.1. There is a canonical isomorphism Ψ making the following diagram commute:*

(5.2)
$$\begin{tikzcd} H^1_\sigma(\Gamma, N) \arrow[r] \arrow[d, "\Psi"'] & H^1_\sigma(\Gamma, G) \arrow[d, "\Phi"] \\ H^1_\theta(\mathbb{Z}_2, N) \arrow[r] & H^1_\theta(\mathbb{Z}_2, G) \end{tikzcd}$$

Proof. The map $N \to W$ induces a map $H^1_\sigma(\Gamma, N) \to H^1_\sigma(\Gamma, W)$. For $\xi \in H^1_\sigma(\Gamma, W)$ let $F_\sigma(\xi, N)$ be the fiber over ξ, so $H^1_\sigma(\Gamma, N)$ is the disjoint union of the $F_\sigma(\xi, N)$. For $\xi \in H^1_\theta(\mathbb{Z}_2, W)$ define $F_\theta(\xi, N) \subset H^1_\theta(\mathbb{Z}_2, N)$ similarly.

We proceed one fiber at a time. It is enough to show that for all $\xi \in H^1_\sigma(\Gamma, W)$ such that $F_\sigma(\xi, N)$ is nonempty, there is an isomorphism Ψ_ξ making this diagram commute:

(5.3)
$$\begin{tikzcd} F_\sigma(\xi, N) \arrow[r, "\iota_\sigma"] \arrow[d, "\Psi_\xi"'] & H^1_\sigma(\Gamma, G) \arrow[d, "\Phi"] \\ F_\theta(\xi, N) \arrow[r, "\iota_\theta"] & H^1_\theta(\mathbb{Z}_2, G) \end{tikzcd}$$

It is enough to show the two fibers on the left are isomorphic, and the images of $\Phi \circ \iota_\sigma$ and ι_θ agree.

First take $\xi = 1$. By Proposition 4.5, and the discussion preceding it,

$$F_\sigma(1, N) \simeq H^1_\sigma(\Gamma, H)/W_i,\ F_\theta(1, N) \simeq H^1_\theta(\mathbb{Z}_2, H)/W_i.$$

These are isomorphic by Proposition 3.1 and Lemma 4.12. The commutativity of the diagram is clear since the horizontal maps are induced by inclusion.

We treat the general fiber by twisting. Suppose $w \in W^{-\sigma}$ and assume $F_\sigma([w], N)$ is nonempty. Therefore there exists $y \in N^{-\sigma}$ mapping to w. Let $\sigma' = \mathrm{int}(y) \circ \sigma$. Twisting by y (see Section 2.1) defines an isomorphism $H^1_{\sigma'}(\Gamma, N) \simeq H^1_\sigma(\Gamma, N)$, taking $F_{\sigma'}(1, N)$ to $F_\sigma(w, N)$. It also gives an isomorphism $H^1_{\sigma'}(\Gamma, G) \simeq H^1_\sigma(\Gamma, G)$.

Similar comments apply in the θ setting. Putting these together we have the following commutative diagram, where the central square comes from the previous discussion with σ', θ' in place of σ, θ, and Ψ_ξ is defined to make the diagram commute.

$$\begin{tikzcd}
F_\sigma([w], N) \arrow[r, "\simeq"] \arrow[d, "\Psi_\xi"'] & F_{\sigma'}(1, N) \arrow[r, "\iota_\sigma"] \arrow[d] & H^1_{\sigma'}(\Gamma, G) \arrow[r, "\simeq"] \arrow[d] & H^1_\sigma(\Gamma, G) \arrow[d, "\Phi"] \\
F_\theta([w], N) \arrow[r, "\simeq"] & F_{\theta'}(1, N) \arrow[r, "\iota_\theta"] & H^1_{\theta'}(\mathbb{Z}_2, G) \arrow[r, "\simeq"] & H^1_\theta(\mathbb{Z}_2, G)
\end{tikzcd} \tag{5.4}$$

□

Let $H^1_\sigma(\Gamma, N)_0$ be the kernel of the map $H^1_\sigma(\Gamma, N) \to H^1_\sigma(\Gamma, G)$. It is well known that that the set of $G(\mathbb{R})$-conjugacy classes of Cartan subgroups defined over $\mathbb{R}$ is parametrized by $H^1_\sigma(\Gamma, N)_0$ as follows. Suppose $n \in N^{-\sigma}$, and $[n]$ is trivial in $H^1_\sigma(\Gamma, G)$. Write $n = g\sigma(g^{-1})$ for some $g \in G$; the map takes $[n]$ to gHg^{-1}. It is straightforward to see this gives the stated bijection.

Analogous statements hold for θ: the kernel $H^1_\theta(\mathbb{Z}_2, H)_0$ of $H^1_\theta(\mathbb{Z}_2, H) \to H^1_\theta(\mathbb{Z}_2, G)$ parametrizes the θ-stable Cartan subgroups.

Matuski's result on Cartan subgroups ([11],[5, Proposition 6.18]) is now immediate.

Proposition 5.5 *There is a bijection between $G(\mathbb{R})$-conjugacy classes of σ-stable Cartan subgroups, and K-conjugacy classes of θ-stable Cartan subgroups.*

Proof. By Proposition 5.1 the isomorphism Ψ of Proposition 5.1 restricts to an isomorphism $H^1_\sigma(\Gamma, N)_0 \simeq H^1_\theta(\mathbb{Z}_2, N)_0$. □

As a corollary of the proof of Proposition 5.1 we obtain a description of $H^1(\Gamma, N)$.

Proposition 5.6 *Suppose H is a σ-stable Cartan subgroup. Let S be a set of representatives of the $G(\mathbb{R})$-conjugacy classes of σ-stable Cartan subgroups. For*

$H' \in S$ let $W(H')_i$ be the imaginary Weyl group of H'.

$$H^1_\sigma(\Gamma, N) \simeq \bigcup_{H' \in S} H^1_\sigma(\Gamma, H')/W(H')_i$$

If H is θ-stable the analogous result holds:

$$H^1_\theta(\mathbb{Z}_2, N) \simeq \bigcup_{H' \in S} H^1_\theta(\mathbb{Z}_2, H')/W(H')_i$$

Using the bijection of Proposition 5.5 these two sets are termwise isomorphic.

6 Strong real forms and $H^1(\Gamma, G)$

Suppose G is defined over $\mathbb{R}$. Part of the long exact cohomology sequence, associated to the exact sequence $1 \to Z \to G \to G_{\mathrm{ad}} \to 1$, is

$$H^1(\Gamma, G) \to H^1(\Gamma, G_{\mathrm{ad}}) \to H^2(\Gamma, Z). \tag{6.1}$$

Recall (Example 4.16)

$$H^1(\Gamma, G_{\mathrm{ad}}) = \overline{\mathrm{Invol}}_\sigma(G)/G = \{\text{real forms of } G \text{ inner to } \sigma\}.$$

We seek a similar description of $H^1(\Gamma, G)$. This is straightforward if the the map to $H^1(\Gamma, G_{\mathrm{ad}})$ in (6.1) is surjective. In general we need to to replace $H^1(\Gamma, G)$ with a bigger space which maps surjectively to $H^1(\Gamma, G_{\mathrm{ad}})$. This is provided by the theory of strong real forms [3]. We follow the equivalent version described in [4] (see Remark 6.19). We work in the context of $H^1(\mathbb{Z}_2, G)$, and then use Theorem 1.1 to state the results in terms of $H^1(\Gamma, G)$.

For now assume we are given only a complex reductive group G. We make use of the exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 \tag{6.2}$$

where $\mathrm{Aut}(G)$ is the (holomorphic) automorphisms of G, $\mathrm{Int}(G)$ are the inner ones, and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$. We say two automorphisms are inner to each other, or in the same inner class, if they have the same image in $\mathrm{Out}(G)$. Thus an inner class is determined by an involution $\tau \in \mathrm{Out}(G)$, and we refer to this as the inner class of τ. Note that the action of $\mathrm{Aut}(G)$ on Z factors to $\mathrm{Out}(G)$.

We say two real forms σ, σ' are in the same inner class if $\sigma' \circ \sigma^{-1} \in \mathrm{Int}(G)$. Equivalently, if θ, θ' are corresponding Cartan involutions, then θ and θ' are in the same inner class. See example 4.16.

We take as our starting point an involution $\tau \in \mathrm{Out}(G)$, which determines an inner class of real forms. Let $\mathrm{Invol}_\tau(G) \subset \mathrm{Invol}(G)$ be the involutions in the inner class of τ. There are two natural choices for a basepoint making $\mathrm{Invol}_\tau(G)/G$ a pointed set. One is the quasisplit (most split) real form in the inner class. Because of our focus on θ, rather than σ, we prefer to choose the quasicompact (most compact) form: a real form is said to be quasicompact if its Cartan involution fixes a pinning datum $(H, B, \{X_\alpha\})$ [15], [4, Section 2.1]. Each inner class contains a unique quasicompact real form, whose Cartan involution is denoted θ_{qc}. The inner class is said to be of equal rank, or compact, if any of the following equivalent conditions hold: $\tau = 1$; $\theta_{\mathrm{qc}} = 1$; the inner class contains the compact form of G; $\mathrm{rank}(G) = \mathrm{rank}(G^\theta)$ for all $\theta \in \mathrm{Invol}_\tau(G)$.

We fix once and for all a pinning datum $(H, B, \{X_\alpha\})$ This defines a splitting of (6.2), taking $\mathrm{Out}(G)$ to the elements of $\mathrm{Aut}(G)$ which preserve the pinning. Then θ_{qc} is the image of τ under the splitting. The associated real form is quasicompact.

Let

$$^\delta = G \rtimes \mathbb{Z}_2 = \langle G, \delta\rangle \quad (\delta^2 = 1,\, \delta g \delta^{-1} = \theta_{\mathrm{qc}}(g)).$$

Definition 6.3 *A strong involution is an element $x \in G\delta$ satisfying $x^2 \in Z$. A strong real form is a G-conjugacy class of strong involutions. Let $\mathrm{SRF}_\tau(G)$ be the set of strong real forms (in the inner class of τ):*

$$\mathrm{SRF}_\tau(G) = \{x \in G\delta \mid x^2 \in Z(G)\}/G. \tag{6.4}$$

Set $\mathrm{inv}(x) = x^2 \in Z^\tau$. *This is invariant under conjugation and so defines a map* $\mathrm{inv} : \mathrm{SRF}_\tau(G) \to Z^\tau$. *We refer to* inv *as the central invariant of a strong real form.*

If x is a strong involution define $\theta_x \in \mathrm{Invol}(G)$ by $\theta_x(g) = xgx^{-1}$ $(g \in G)$. The map $x \to \theta_x$ factors to a surjection

$$\mathrm{SRF}_\tau(G) \twoheadrightarrow \mathrm{Invol}_\tau(G)/G = \{\text{real forms in the inner class of } \tau\}.$$

The surjectivity statement of Proposition 4.9 amounts to the same being true when restricted to strong involutions in $H\delta$.

Lemma 6.5 *If $\theta \in \mathrm{Invol}_\tau(G)$ then $\theta = \mathrm{int}(x)$ for some strong involution x. A conjugate of θ is equal to θ_x for some $x \in H\delta$.*

Therefore is enough to compute $H^1_{\theta_x}(\mathbb{Z}_2, G)$ for all strong involutions $x \in H\delta$. By Proposition 4.9 (using the fact that H is a fundamental Cartan subgroup with respect to θ_x)

$$\begin{aligned} H^1_{\theta_x}(\mathbb{Z}_2, G) &\simeq H^1_{\theta_x}(\mathbb{Z}_2, H)/W_i \\ &= [H^{-\theta_x}/(1+\theta_x)H]/W_i \end{aligned} \tag{6.6(a)}$$

Note that, since $\theta_x|_H = \theta_{\mathrm{qc}}$, the numerator is is independent of x, although the action of W_i depends on x (see Remark 4.4).

We rewrite this expression by twisting by x. This is analogous to twisting of cohomology as in Section 2.1 (that this is more than an analogy is explained in Section 7). Let $z = \mathrm{inv}(x) \in Z^\tau$, and consider the map $h \to hx \in H\delta$. It is easy to see this gives a bijection between (a) and

$$\text{(6.6)(b)} \qquad [\{y \in H\delta \mid y^2 = z\}/\sim H]/W_i$$

The main point is that now H, and especially W_i, are acting on $H\delta$ by ordinary conjugation. We make the action of W_i explicit. If $w \in W_i$, choose $n \in N$ mapping to w, and take $y \in H\delta$ to nyn^{-1}. Writing $y = h\delta$, $nyn^{-1} = (nhn^{-1})n\delta(n^{-1})\delta$, which is in $H\delta$ since $w \in W_i$. Furthermore this is easily seen to be independent of the choice of n. This proves the following Proposition.

Lemma 6.7 *Suppose $x \in H\delta$ is a strong involution. Let $z = \mathrm{inv}(x) \in Z^\tau$. Then there is a bijection*

$$H^1_{\theta_x}(\mathbb{Z}_2, G) \leftrightarrow [\{y \in H\delta \mid y^2 = z\}/\sim H]/W_i$$

The right hand side is precisely the strong real forms with central invariant z. This only depends on z, and its order only depends only on the image of z in $Z^\tau/(1+\tau)Z$.

Tracing through the construction we see the map from right to left takes $y \in H\delta$ to $[yx^{-1}]$.

This result is not optimal because, given a Cartan involution θ, to compute $H^1_\theta(\mathbb{Z}_2, G)$ we need to choose a strong involution x so that θ is conjugate to θ_x. On the other hand the right hand side of the Lemma only depends on $z \in Z^\tau$. What is missing is a definition of invariant of a real form (compare Definition 6.3).

Definition 6.8 *Define* $\mathrm{inv} : \mathrm{Invol}_\tau(G)/G \to Z^\tau/(1+\tau)Z$ *by the composition of maps:*

$$\text{(6.9)} \qquad \mathrm{Invol}_\tau(G)/G \leftrightarrow H^1_{\theta_{qc}}(\mathbb{Z}_2, G_{\mathrm{ad}}) \to H^2_{\theta_{qc}}(\mathbb{Z}_2, Z) \simeq Z^\tau/(1+\tau)Z.$$

We refer to inv *as the central invariant of a real form.*

Note that the central invariant of the quasicompact real form is the identity.

For the first map, it is straightforward to see that the map $g \to \mathrm{int}(g) \circ \theta_{\mathrm{qc}}$ induces a bijection $H^1_{\theta_{\mathrm{qc}}}(\mathbb{Z}_2, G_{\mathrm{ad}}) \longleftrightarrow \mathrm{Invol}_\tau(G)/G$. The second map is from the long exact cohomology sequence associated to the exact sequence $1 \to Z \to G \to G_{ad} \to 1$. Since $\mathbb{Z}_2$ is cyclic there is an isomorphism in Tate cohomology $H^2_\theta(\mathbb{Z}_2, Z) = \widehat{H}^2_\theta(\mathbb{Z}_2, Z) \simeq \widehat{H}^0_\theta(\mathbb{Z}_2, Z)$, and the final isomorphism is the standard description of this cohomology.

Note that the invariant of a real form is an element of $Z^\tau/(1+\tau)Z$, whereas the invariant of a strong real form (Definition 6.3) is an element of Z^τ. Now we can restate Lemma 6.7.

Lemma 6.10 *Suppose $\theta \in \mathrm{Invol}(G)$. Choose a representative $z \in Z^\tau$ of $\mathrm{inv}(\theta) \in Z^\tau/(1+\tau)Z$. Then there is a bijection*

$$H^1_\theta(\mathbb{Z}_2, G) \longleftrightarrow \textit{the strong real forms with central invariant } z$$

Finally we pass back to the antiholomorphic picture to get Theorem 1.2. We first need a version of invariant in this setting. Let σ_{qc} be an anitholomorphic involution corresponding to θ_{qc}, so $G(\mathbb{R}) = G^{\sigma_{\mathrm{qc}}}$ is quasicompact. Every real form in this inner class has the same restriction to Z, which we denote σ. As in (6.9) define a map $\mathrm{inv} : \overline{\mathrm{Invol}}_\tau(G)/G \to Z^\Gamma/(1+\sigma)Z$ by:

$$\overline{\mathrm{Invol}}_\tau(G)/G \leftrightarrow H^1_{\sigma_{\mathrm{qc}}}(\Gamma, G_{\mathrm{ad}}) \to H^2_{\sigma_{\mathrm{qc}}}(\Gamma, Z) \simeq Z^\Gamma/(1+\sigma)Z. \tag{6.11}$$

This takes the quasicompact form to the identity.

Lemma 6.12 *There is a canonical bijection between $Z^\tau/(1+\tau)Z$ and $Z^\Gamma/(1+\sigma)Z$. This makes the following diagram commute:*

$$\begin{tikzcd}
H^1_{\theta_{qc}}(\mathbb{Z}_2, G_{\mathrm{ad}}) \arrow[r, "(6.9)"] \arrow[d, "\textit{Theorem 1.1}"'] & Z^\tau/(1+\tau)Z \arrow[d] \\
H^1_{\sigma_{qc}}(\Gamma, G_{\mathrm{ad}}) \arrow[r, "(6.11)"'] & Z^\Gamma/(1+\sigma)Z
\end{tikzcd}$$

This comes down to the fact that we can choose representatives of $Z^\tau/(1+\tau)Z$ and $Z^\Gamma/(1+\sigma)Z$ of finite order, and θ and $\sigma = \theta\sigma_c$ agree on these.

Proposition 6.13 *Suppose σ is a real form of G, and choose a representative $z \in Z^\Gamma$ of $\mathrm{inv}(\sigma) \in Z^\Gamma/(1+\sigma)Z$. Then there is a bijection*

$$H^1(\Gamma, G) \longleftrightarrow \textit{the set of strong real forms of central invariant } z.$$

Corollary 6.14 *Suppose $G(\mathbb{R})$ is an equal rank real form. Choose $x \in G$ so that $\mathrm{Cent}_G(x)$ is a complexified maximal compact subgroup of $G(\mathbb{R})$, and let $z = x^2 \in Z(G)$. Then*

$$H^1(\Gamma, G) \longleftrightarrow \textit{the set of conjugacy classes of } G \textit{ with square equal to } z$$

If H is any Cartan subgroup, with Weyl group W, then this is equal to

$$\{h \in H \mid h^2 = z\}/W \tag{6.15}$$

Example 6.16 Taking $x = z = I$ gives $G(\mathbb{R})$ compact and recovers [13, Theorem 6]: $H^1(\Gamma, G)$ is the set of conjugacy classes of G of involutions of G. See Example 4.15.

Example 6.17 Let $G(\mathbb{R}) = Sp(2n, \mathbb{R})$. We can take $x = iI$, $z = -I$. Every element of G whose square is $-I$ is conjugate to x. This gives the classical result $H^1(\Gamma, G) = 1$, which is equivalent to the classification of nondegenerate symplectic forms [12, Chapter 2].

Example 6.18 Let $G(\mathbb{R}) = SO(p, q)$. If pq is even we can take $z = I$, Corollary 6.14 applies, and the set (6.15) is equal to $\{\mathrm{diag}(I_r, -I_s) \mid r+s = p+q;\ s \text{ even}\}$.

Suppose p and q are odd. Apply Corollary 4.19 with $M_f(\mathbb{R}) = SO(p-1, q-1) \times GL(1, \mathbb{R})$. By the previous case we conclude $H^1(\Gamma, G)$ is parametrized by $\{\mathrm{diag}(I_r, I_s) \mid r + s = p + q - 2;\ r, s \text{ even}\}$. Adding $(1, 1)$ this is the same as $\{\mathrm{diag}(I_r, -I_s) \mid r + s = p + q;\ s \text{ odd}\}$.

In all cases we recover the classical fact that $H^1(\Gamma, G)$ is the set of equivalence classes of quadratic forms of dimension $p+q$ and discriminant $(-1)^q$ [12, Chapter 2].

Remark 6.19 In [3] and [17] strong real forms are defined in terms of the Galois action, as opposed to the Cartan involution as in [4] (and elsewhere, including [1]). The preceding discussion shows that these two theories are indeed equivalent. However the choices of basepoints in the two theories are different. In the Galois setting we choose the quasisplit form, and in the algebraic setting we use the quasicompact one.

In [17] the invariant of a real form is given by [17, (2.8)(c)]. This differs from the normalization here by multiplication by $\exp(2\pi i\rho^\vee) \in Z$. Note that the "pure" rational forms, which are parametrized by $H^1(\Gamma, G)$, include the quasisplit one [17, Proposition 2.7(c)], rather than the quasicompact one.

Remark 6.20 Kottiwtz relates $H^1(\Gamma, G)$ to the center of the dual group [10, Theorem 1.2]. This is a somewhat different type of result. For example this result identifies the kernel of the map from $H^1(\Gamma, G_{\mathrm{sc}})$ to $H^1(\Gamma, G)$, but if G is simply connected this gives no information.

Contrary to the adjoint case, inner forms do not necessarily have isomorphic cohomology, as was illustrated in Example 4.18. Proposition 6.13 clarifies the situation.

Corollary 6.21 *Suppose σ, σ' are inner forms of G. If $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma')$ then $H^1_\sigma(\Gamma, G) \simeq H^1_{\sigma'}(\Gamma, G)$.*

7 Relation with Rigid Rational forms

The space of strong rational forms can naturally be thought of as a cohomological object. From this point of view the proof of Lemma 6.7 amounts to the standard twisting argument.

Vogan [17, Problem 9.3] has conjectured that there should be a notion of strong rational form in the p-adic case, generalizing the real case, and gave a number of properties this definition should satisfy. Kaletha has found a definition which satisfies these conditions [8], and the relationship between Galois cohomology and rigid rational forms carries over to that setting. We confine ourselves to the real case, and refer the reader to [8] for more details, and the p-adic case.

Recall $\mathrm{SRF}_\tau(G) = \{x \in G\delta \mid x^2 \in Z\}/G$. It is easy to see there is an exact sequence of pointed sets

$$1 \to H^1_{\theta_{\mathrm{qc}}}(\mathbb{Z}_2, G) \to \mathrm{SRF}_\tau(G) \xrightarrow{\mathrm{inv}} Z^\tau \tag{7.1}$$

The first map takes $[g]$ ($g \in G^{-\theta_{\mathrm{qc}}}$) to $g\delta$, and $\mathrm{inv}(x) = x^2$ (see the discussion following Definition 6.8, and Definition 6.3).

By definition the strong real forms of invariant $z \in Z^\tau$ are the fiber over z of the invariant map, and the exact sequence identifies $H^1_{\theta_{\mathrm{qc}}}(\mathbb{Z}_2, G)$ as the strong real forms of invariant 1.

Proposition 7.2 *Suppose the fiber over $z \in Z^\tau$ is nonempty. Choose $x \in G\delta$ so that $x^2 = z$. Then we may identify the fiber over z with $H^1_{\theta_x}(\mathbb{Z}_2, G)$.*

Since the fiber over z is, by definition, the strong real forms of invariant z, this gives Lemma 6.7.

Proof. The proof is by twisting (and is essentially equivalent to the proof of Lemma 6.7). For this we generalize the definition of $\mathrm{SRF}_\tau(G)$. Given $\theta \in \mathrm{Invol}(G)$ consider the group

$$G \rtimes \mathbb{Z}_2 = \langle G, \delta_\theta \rangle \quad (\delta_\theta^2 = 1,\ \delta_\theta g \delta_\theta^{-1} = \theta(g))$$

and

$$\mathrm{SRF}(\theta, G) = \{x \in G\delta_\theta \mid x^2 \in Z\}/G.$$

We view this is a pointed set with distinguished element δ_θ. With this notation $\mathrm{SRF}_\tau(G) = \mathrm{SRF}(\theta_{\mathrm{qc}}, G)$.

Just as in (7.1) there is an exact sequence

$$1 \to H^1_\theta(\mathbb{Z}_2, G) \to \mathrm{SRF}(\theta, G) \xrightarrow{\mathrm{inv}} Z^\tau \tag{7.3}$$

Suppose $x \in G\delta_\theta$ satisfies $z = x^2 \in Z$, and let $\theta' = \mathrm{int}(x) \circ \theta$. Unless $z = 1$ twisting by x is not defined at the level of $H^1(\mathbb{Z}_2, G)$, so $H^1_\theta(\mathbb{Z}_2, G)$ and $H^1_{\theta'}(\mathbb{Z}_2, G)$ are not necessarily isomorphic. See Section 2.1. However twisting by x makes sense within the larger set $\mathrm{SRF}(\theta, G)$, and defines an isomorphism of $\mathrm{SRF}(\theta, G)$ and $\mathrm{SRF}(\theta', G)$.

More precisely, there is a natural bijection $t_x : \mathrm{SRF}_\tau(G) \to \mathrm{SRF}(\theta_x, G)$ essentially given by multiplying on the right by x^{-1}. We need to account for the fact that we have two different groups $\langle G, \delta_{\theta_x} \rangle$ and $\langle G, \delta \rangle$. To be precise the map $G\delta \ni y \to (yx^{-1})\delta_{\theta_x} \in G\delta_{\theta_x}$ induces a bijection t_x, as is easily checked. This takes $x \in \mathrm{SRF}_\tau(G)$ to the basepoint $\delta_{\theta_x} \in \mathrm{SRF}(\delta_\theta, G)$.

We obtain the commutative diagram of (non-pointed) sets:

$$\begin{tikzcd} & & \mathrm{SRF}_\tau(G) \arrow[r] \arrow[d, "t_x"'] & Z^\tau \arrow[d] \\ 1 \arrow[r] & H^1_{\theta_x}(\mathbb{Z}_2, G) \arrow[r] & \mathrm{SRF}(\theta_x, G) \arrow[r] & Z^\tau \end{tikzcd} \tag{7.4}$$

where the final vertical arrow is multiplication by z^{-1}. By the standard twisting argument (or direct calculation) t_x takes the fiber over z to the fiber over 1, which is $H^1_{\theta_x}(\mathbb{Z}_2, G)$. $\square$

Corollary 7.5 *Choose representatives $z_1, \dots, z_n \in Z^\tau$ for the image of* inv : $\mathrm{SRF}_\tau(G) \to Z^\tau$. *For each z_i choose $x_i \in G\delta$ such that* $\mathrm{inv}(x_i) = z_i$. *Then there is a bijection*

$$\mathrm{SRF}_\tau(G) \longleftrightarrow \bigcup_i H^1_{\theta_{x_i}}(\mathbb{Z}_2, G).$$

(Strictly speaking the image of inv is infinite if the center of G contains a compact torus. As in [8] or [4, Section 13] the theory can be modified so this set is finite.)

This gives an interpretation of $\mathrm{SRF}_\tau(G)$ in classical cohomological terms. A similar statement holds in the p-adic case [8].

8 Fibers of the map from $H^1(\Gamma, G) \to H^1(\Gamma, \overline{G})$

Suppose $A \subset Z(G)$, and both G and $\overline{G} = G/A$ are defined over $\mathbb{R}$. It is helpful to analyze the fibers of the map ψ:

$$H^1(\Gamma, A) \to H^1(\Gamma, G) \xrightarrow{\psi} H^1(\Gamma, \overline{G}).$$

The most important special case is when $G = G_{\mathrm{sc}}$ is simply connected, $A = Z_{\mathrm{sc}} = Z(G_{\mathrm{sc}})$ and $\overline{G} = G_{\mathrm{ad}}$. In this case, summing over $H^1(\Gamma, G_{\mathrm{ad}})$, we obtain a description of $H^1(\Gamma, G_{\mathrm{sc}})$, complementary to that of Proposition 6.13.

Suppose σ is a real form of G such that A is σ-invariant. Write $G(\mathbb{R}, \sigma) = G^\sigma$ and $\overline{G}(\mathbb{R}, \sigma) = \overline{G}^\sigma$.

Write p for the projection map $G \to \overline{G}$. This restricts to a map $G(\mathbb{R}, \sigma) \to \overline{G}(\mathbb{R}, \sigma)$, taking the identity component of $G(\mathbb{R}, \sigma)$ to that of $\overline{G}(\mathbb{R}, \sigma)$. Therefore p factors to a map (not necessarily an injection):

(8.1)(a) $$p^* : \pi_0(G(\mathbb{R}, \sigma)) \to \pi_0(\overline{G}(\mathbb{R}, \sigma)).$$

Define

(8.1)(b) $$\pi_0(G, \overline{G}, \sigma) = \pi_0(\overline{G}(\mathbb{R}, \sigma))/p^*(\pi_0(G(\mathbb{R}, \sigma))).$$

There is a natural action of $\overline{G}(\mathbb{R}, \sigma)$ on $H^1(\Gamma, A)$ defined as follows. Suppose $g \in \overline{G}(\mathbb{R}, \sigma)$. Choose $h \in G(\mathbb{C})$ satisfying $p(h) = g$. Then $g : a \to ha\sigma(h^{-1})$ factors to a well defined action of $\overline{G}(\mathbb{R}, \sigma)$ on $H^1(\Gamma, A)$. Furthermore the image of $G(\mathbb{R}, \sigma)$, which includes the identity component, acts trivially, so this factors to an action of $\pi_0(G, \overline{G}, \sigma)$.

Suppose $g \in G$ satisfies $g\sigma(g) \in A$. Then $\sigma' = \mathrm{int}(g) \circ \sigma$ is an involution in the same inner class as σ, normalizing A, and the preceding discussion applies to σ'.

Proposition 8.2 *Fix a real form σ of G which normalizes A, and let $\overline{G} = G/A$. Suppose $\gamma \in H^1(\Gamma, \overline{G})$ is in the image of $\psi : H^1(\Gamma, G) \to H^1(\Gamma, \overline{G})$. Write $\gamma = [g]$ for some $g \in G^{-\sigma}$, and let $\sigma' = \mathrm{int}(g) \circ \sigma$. Then there is a bijection*

$$H^1(\Gamma, G) \supset \psi^{-1}(\gamma) \longleftrightarrow H^1(\Gamma, A)/\pi_0(G, \overline{G}, \sigma').$$

In particular suppose G is semisimple, A is the fundamental group of G, so $G = G_{\mathrm{sc}}/A$, and $\psi : H^1(\Gamma, G_{\mathrm{sc}}) \to H^1(\Gamma, G)$. Then $\psi^{-1}(\gamma) \longleftrightarrow H^1(\Gamma, A)/\pi_0(G(\mathbb{R}, \sigma'))$, and

$$|\psi^{-1}(\gamma)| = |H^1(\Gamma, A)|/|\pi_0(G(\mathbb{R}, \sigma'))|.$$

Proof. First assume γ is trivial, and take $g = 1$. Consider the exact sequence

$$H^0(\Gamma, G) \to H^0(\Gamma, \overline{G}) \to H^1(\Gamma, A) \xrightarrow{\phi} H^1(\Gamma, G) \xrightarrow{\psi} H^1(\Gamma, \overline{G}).$$

This says $\psi^{-1}(\gamma) = \phi(H^1(\Gamma, A))$, i.e. the orbit of the group $H^1(\Gamma, A)$ acting on the identity coset. This is is $H^1(\Gamma, A)$, modulo the action of $H^0(\Gamma, \overline{G})$, and this action factors through the image of $H^0(\Gamma, G)$.

The general case follows from an easy twisting argument, and the final assertion follows after replacing $G, \overline{G}$ with G_{sc}, G, respectively, and using the fact that every real form of G_{sc} is connected.

□

9 Tables

Most of these results can be computed by hand from Theorem 1.2, or using Proposition 8.2 and the classification of real forms (i.e. the adjoint case).

The Atlas of Lie Groups and Representations software can be used to calculate $H^1(\Gamma, G)$ for any real form of a reductive group. This was used to check the tables, and for the Spin groups. See `www.liegroups.org/galois` for more information about using the software to compute Galois cohomology.

9.1 Classical groups

Group	$\vert H^1(\Gamma, G)\vert$	
$SL(n, \mathbb{R}), GL(n, \mathbb{R})$	1	
$SU(p, q)$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$	Hermitian forms of rank $p+q$ and discriminant $(-1)^q$
$SL(n, \mathbb{H})$	2	$\mathbb{R}^*/\mathrm{Nrd}_{\mathbb{H}/\mathbb{R}}(\mathbb{H}^*)$
$Sp(2n, \mathbb{R})$	1	real symplectic forms of rank $2n$
$Sp(p, q)$	$p+q+1$	quaternionic Hermitian forms of rank $p+q$
$SO(p, q)$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$	real symmetric bilinear forms of rank n and discriminant $(-1)^q$
$SO^*(2n)$	2	

Here $\mathbb{H}$ is the quaternions, and Nrd is the reduced norm map from $\mathbb{H}^*$ to $\mathbb{R}^*$ (see [12, Lemma 2.9]). For more information on Galois cohomology of classical groups see [12, Sections 2.3 and s6.6] and [9, Chapter VII].

9.2 Simply connected groups

The only simply connected groups with classical root system, which are not in the table in Section 9.1 are $Spin(p, q)$ and $Spin^*(2n)$.

Define $\delta(p, q)$ by the following table, depending on $p, q \pmod 4$.

	0	1	2	3
0	3	2	2	2
1	2	1	1	0
2	2	1	0	0
3	2	0	0	0

Group	$\lvert H^1(\Gamma, G)\rvert$
$Spin(p, q)$	$\lfloor \frac{p+q}{4} \rfloor + \delta(p, q)$
$Spin^*(2n)$	2

Simply connected exceptional groups					
inner class	group	K	real rank	name	$\lvert H^1(G)\rvert$
compact	E_6	A_5A_1	4	quasisplit' quaternionic	3
	E_6	D_5T	2	Hermitian	3
	E_6	E_6	0	compact	3
split	E_6	C_4	6	split	2
	E_6	F_4	2	quasicompact	2
compact	E_7	A_7	7	split	2
	E_7	D_6A_1	4	quaternionic	4
	E_7	E_6T	3	Hermitian	2
	E_7	E_7	0	compact	4
compact	E_8	D_8	8	split	3
	E_8	E_7A_1	4	quaternionic	3
	E_8	E_8	0	compact	3
compact	F_4	C_3A_1	4	split	3
	F_4	B_4	1		3
	F_4	F_4	0	compact	3
compact	G_2	A_1A_1	2	split	2
	G_2	G_2	0	compact	2

9.3 Adjoint groups

If G is adjoint $|H^1(\Gamma, G)|$ is the number of real forms in the given inner class, which is well known. We also include the component group, which is useful in connection with Proposition 8.2.

One technical point arises in the case of $PSO^*(2n)$. If n is even there are two real forms which are related by an outer, but not an inner, automorphism. See [2, Example 3.3].

Adjoint classical groups		
Group	$\lvert\pi_0(G(\mathbb{R}))\rvert$	$\lvert H^1(\Gamma, G)\rvert$
$PSL(n, \mathbb{R})$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$
$PSL(n, \mathbb{H})$	1	2
$PSU(p, q)$	$\begin{cases} 2 & p = q \\ 1 & \text{otherwise} \end{cases}$	$\lfloor \frac{p+q}{2} \rfloor + 1$
$PSO(p, q)$	$\begin{cases} 1 & pq = 0 \\ 1 & p, q \text{ odd and } p \neq q \\ 4 & p = q \text{ even} \\ 2 & \text{otherwise} \end{cases}$	$\begin{cases} \lfloor \frac{p+q+2}{4} \rfloor & p, q \text{ odd} \\ \frac{p+q}{4} + 3 & p, q \text{ even}, p + q = 0 \pmod 4 \\ \frac{p+q-2}{4} + 2 & p, q \text{ even}, p + q = 2 \pmod 4 \\ \frac{p+q+1}{2} & p, q \text{ opposite parity} \end{cases}$
$PSO^*(2n)$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$	$\begin{cases} \frac{n}{2} + 3 & n \text{ even} \\ \frac{n-1}{2} + 2 & n \text{ odd} \end{cases}$
$PSp(2n, \mathbb{R})$	2	$\lfloor \frac{n}{2} \rfloor + 2$
$PSp(p, q)$	$\begin{cases} 2 & p = q \\ 1 & else \end{cases}$	$\lfloor \frac{p+q}{2} \rfloor + 2$

The groups E_8, F_4 and G_2 are both simply connected and adjoint. Furthermore in type E_6 the center of the simply connected group G_{sc} has order 3, and it follows that $H^1(\Gamma, G_{\text{ad}}) = H^1(\Gamma, G_{\text{sc}})$ in these cases. So the only groups not covered by the table in Section 9.2 are adjoint groups of type E_7.

Adjoint exceptional groups						
inner class	group	K	real rank	name	$\pi_0(G(\mathbb{R}))$	$\lvert H^1(G)\rvert$
compact	E_7	A_7	7	split	2	4
	E_7	D_6A_1	4	quaternionic	1	4
	E_7	E_6T	3	Hermitian	2	4
	E_7	E_7	0	compact	1	4

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