Formula for y

November 10, 2009

More details can be found in *Some Notes on Parametrizing Representations* on the atlas web site. In particular this has formulas for computing Cayley transforms and cross actions in various coordinates.

The main point is the formula for y. See Marc's email in the Appendix.

We have fixed $G, G^\vee$, and $T, T^\vee$ as usual. Let $X^* = X^*(T)$, which is canonically identified with $X_*(T^\vee)$.

0.1 Maps of the Weil group

Give (y, λ) satisfying:

1. $\lambda \in X^* \otimes \mathbb{C}$,
2. $y \in \mathrm{Norm}_{G^{\vee\Gamma} - G^\vee}(T^\vee)$,
3. $y^2 = \exp(2\pi i \lambda)$

Define ϕ by

1. $\phi(z) = z^\lambda \overline{z}^{y\lambda}$
2. $\phi(j) = \exp(-\pi i \lambda) y$.

The first line is shorthand for $\phi(e^w) = \exp(w\lambda + y\overline{w}\lambda)$.

0.2 Character of covers of $T(\mathbb{R})$

Fix $\gamma \in \frac{1}{2}Z^*$. Then the two-fold cover T_γ of the complex torus T is defined. It is an abelian algebraic group; its identity component is a torus of index 1 or 2. In our applications we often have $\gamma = \rho$.

If θ is a Cartan involution of T satisfying $\theta\gamma - \gamma \in X^*$ then the inverse image $T(\mathbb{R})_\gamma$ of the real form $T(\mathbb{R})$ of T defined by θ is defined.

Fix θ and (λ, κ) satisfying:

1. $\lambda \in X^* \otimes \mathbb{C}$,
2. $\kappa \in (\gamma + X^*)/(1-\theta)X^*$,
3. $\lambda + \theta\lambda = \kappa + \theta\kappa$

In (2) we identify κ with a representative in $\gamma + X^*$; any formula involving $\kappa + \theta\kappa$ is well defined.

Associated to (λ, κ) is a genuine character $\Lambda(\lambda, \kappa)$ of $T(\mathbb{R})_\gamma$. See [2, Proposition 5.8].

0.3 Characters of tori defined by (y, λ)

Fix $\gamma \in \frac{1}{2}X^*$ as in Section 0.2. Suppose we are given (y, λ, y_0) with (y, λ) as in Section 0.1, y_0 in the same *fiber* as y, and $y_0^2 = \exp(2\pi i\gamma)$. That is:

1. $y, y_0 \in \mathrm{Norm}_{G^\vee\Gamma - G^\vee}(T^\vee)$,
2. $yy_0^{-1} \in T^\vee$,
3. $\lambda \in X^* \otimes \mathbb{C}$
4. $y^2 = \exp(2\pi i\lambda)$,
5. $y_0^2 = \exp(2\pi i\gamma)$.

Assume θ is a Cartan involution of T satisfying $\theta^t = -\mathrm{Ad}(y)$. This defines a character $\Lambda[y, \lambda, y_0]$ of $T_\gamma(\mathbb{R})$ as follows. By (2) choose $\tau \in X^* \otimes \mathbb{C}$ satisfying $y = \exp(2\pi i\tau)y_0$, and let

$$\kappa = \lambda - (\tau - \theta\tau). \tag{1}$$

We check that conditions (1-3) in Section 0.2 hold. It is obvious that another choice of τ modifies κ by an element of $(1-\theta)X^*$, and $\lambda+\theta\lambda=\kappa+\theta\kappa$. Also

$$\begin{aligned}
\exp(2\pi i\kappa) &= \exp(2\pi i(\lambda-\tau+\theta\tau) \\
&= y^2\exp(2\pi i(-\tau+\theta\tau)) \\
&= \exp(2\pi i(\tau+\theta^\vee\tau))y_0^2\exp(2\pi i(-\tau+\theta\tau)) \\
&= y_0^2 = \exp(2\pi i\gamma).
\end{aligned} \tag{2}$$

We can therefore define

$$\Lambda[y,\lambda,y_0]=\Lambda(\lambda,\kappa) \tag{3}$$

0.4 Formula for y

The main point is now a formula inverting (3): given (λ,κ) and y_0, give a formula for y, so that $\Lambda[y,\lambda,y_0]=\Lambda(\lambda,\kappa)$.

To spell things out, suppose we are given $(\theta,\gamma,\lambda,\kappa)$ satisfying

1. $\gamma\in\frac{1}{2}X^*$,
2. $\gamma-\theta\gamma\in X^*$,
3. $\lambda\in X^*\otimes\mathbb{C}$,
4. $\kappa\in\gamma+X^*/(1-\theta)X^*$
5. $\lambda+\theta\lambda=\kappa+\theta\kappa$.

Then the genuine character $\Lambda(\lambda,\kappa)$ of $T_\gamma(\mathbb{R})$ is defined as in Section 0.2. Fix $y_0\in\mathrm{Norm}_{G^\vee\Gamma-G^\vee}(T^\vee)$ so that $\theta^t=-Ad(y)$ satisfying

$$y_0^2=\exp(2\pi i\gamma). \tag{4}$$

Then there exists y satisfying

1. $y\in\mathrm{Norm}_{G^\vee\Gamma-G^\vee}(T^\vee)$,
2. $y^2=\exp(2\pi i\lambda)$,
3. $yy_0^{-1}\in T^\vee$

so that $\Lambda[y, \lambda, y_0]$ is defined, and

$$\Lambda[y, \lambda, y_0] = \Lambda(\lambda, \kappa). \tag{5}$$

We want a formula for y.

Set $y = \exp(2\pi i\mu)y_0$, and solve for μ. By (1) we need to choose μ so that

$$\kappa = \lambda - (\mu - \theta\mu). \tag{6}$$

This is equivalent to $\kappa - \lambda = \mu - \theta\mu$, and this is possible since by (5)

$$\lambda - \kappa = -\theta(\lambda - \kappa). \tag{7}$$

In fact we can set

$$\begin{aligned} \mu &= \frac{1}{4}((\lambda - \kappa) - \theta(\lambda - \kappa)) \\ &= \frac{1}{2}(\lambda - \kappa). \end{aligned} \tag{8}$$

We check this:

$$\begin{aligned} \lambda - (\mu - \theta\mu) &= \lambda - \frac{1}{2}((\lambda - \kappa) - \theta(\lambda - \kappa)) \\ &= \lambda - \frac{1}{2}(2(\lambda - \kappa)) \text{ by (7)} \\ &= \lambda - (\lambda - \kappa) = \kappa \end{aligned} \tag{9}$$

as desired.

This proves:

Lemma 10 *Fix θ, γ, and suppose (λ, κ) are given, so $\Lambda(\lambda, \kappa)$ is a genuine character of $T_\gamma(\mathbb{R})$. Fix y_0 satisfying $y_0^2 = \exp(2\pi i\gamma)$ and $\theta^t = Ad(y)$. Let*

$$y = \exp(\pi i(\lambda - \kappa))y_0. \tag{11}$$

Then $\Lambda(\lambda, \kappa) = \Lambda[y, \lambda, y_0]$.

In particular:

Lemma 12 *Suppose we are in the usual atlas setting, with integral infinitesimal character. Fix $(x, y) \in \mathcal{Z}$. Let $\tau^\vee = p(y)$, the twisted involution defined by y. Fix $y[\tau^\vee]$ also lying over $\tau^\vee$, satisfying $y[\tau^\vee]^2 = \exp(2\pi i\rho)$.*

Let $y = \exp(\pi i(\lambda - \kappa))y[\tau^\vee]$. Then

$$\Lambda(\lambda, \kappa) = \Lambda(y, \lambda, y_0). \tag{13}$$

Note this holds for any $y[\tau^\vee]$ with the correct square. While this gives a well defined genuine character of $T_\gamma(\mathbb{R})$, we need to choose $y[\tau^\vee]$ more carefully; this is the basepoint issue.

In this setting λ is always dominant, and this formulation gives a dominant character of $T_\gamma(\mathbb{R})$. This may get moved by an element of the Weyl group when making an actual standard representation.

I've used my terminology to avoid going crazy. Marc uses different convention:

my (λ, κ) are Marc's (γ, λ), and he chooses $y[\tau^\vee] = \sigma_w \delta_1$, so in his terms we have

$$y = \exp(\pi i(\gamma - \lambda))\sigma_w \delta_1$$

which agrees with what he wrote in his email (see the appendix).

0.5 Basepoint

Fix basic data (G, γ) (γ is an involution in $\mathrm{Out}(G)$), and then define G^Γ and $\mathcal{X}$ as usual. Recall $\mathcal{I}_W$ is the involutions in W^Γ. For $\tau \in \mathcal{I}_W$ $\mathcal{X}_\tau$ denotes the fiber of τ. The *basepoint issue* is: for each τ we need to canonically choose an element $x[\tau] \in \mathcal{X}_\tau$. This must satisfy certain properties, ultimately coming down to [2, Proposition 6.30].

We first show there is a canonical way to pick $\{x[\tau]\}$, and then show these satisfy the required properties.

Recall we have fixed a pinning datum $(B, T, \{X_\alpha\}$. For α a simple root let σ_α denote the corresponding element of the Tits group $\widetilde{W}$. If $w \in W$ let σ_w be its canonical lift to the Tits group.

Remark 14 There is a dangerous bend here. For α simple we have defined σ_α; we have also defined σ_{s_α}; and these are equal. If α is not simple then σ_{s_α} is defined, by σ_α is not, and writing σ_α for a non-simple root can cause confusion or even (gasp!) mistakes.

Lemma 15 *Fix τ and any element $x \in \mathcal{X}_\tau$. If α is a simple τ-complex root define $x[s_\alpha \tau s_\alpha^{-1}] = s_\alpha \times x$. Repeating this process gives a well-defined element in each fiber $\mathcal{X}_{\tau'}$ for τ' in the conjugacy class of τ.*

Recall $s_\alpha \times x = \sigma_\alpha x \sigma_\alpha^{-1}$. Also recall the conjugacy class of τ corresponds to a conjugacy class of Cartan subgroups of the quasisplit form.

Proof. Suppose $\alpha_1, \ldots, \alpha_n$ is a sequence of simple roots, such that α_1 is τ-complex, α_2 is $s_{\alpha_1}\tau$-complex, and so on. Also suppose $w = s_{\alpha_1} \ldots s_{\alpha_n}$ satisfies $w\tau w^{-1} = \tau$, i.e. $w \in W^\tau$. It is enough to show $wxw^{-1} = x$.

Let $\rho_i(\tau), \rho_r(\tau)$ be the half sums of the τ-imaginary and τ-real roots, respectively. Let $W_C(\tau) = \mathrm{Stab}_W(\rho_i(\tau), \rho_r(\tau))$.

Let $\alpha = \alpha_1$. If β is τ-imaginary then $(s_\alpha \tau s_\alpha)(s_\alpha \beta) = s_\alpha \tau \beta = s_\alpha \beta$, so $s_\alpha \beta$ is $s_\alpha \tau s_\alpha$-imaginary. Since α is simple and complex $\beta > 0$ implies $s_\alpha \beta > 0$. Similarly for real roots, so

$$s_\alpha(\rho_i(\tau)) = \rho_i(s_\alpha \tau s_\alpha), s_\alpha(\rho_r(\tau)) = \rho_i(s_\alpha \tau s_\alpha) \tag{16}$$

Applying this repeatedly we conclude $w\rho_i(\tau) = \rho_i(w\tau w^{-1}) = \rho_i(\tau)$ since $w \in W^\tau$. Similarly for $\rho_r(\tau)$, so $w \in W_C(\tau)$. Also $w \in W^\tau$, so $w \in W_C(\tau)^\tau$. By [1, Proposition 12.16] $W_C(\tau)^\tau$ acts trivially on $\mathcal{X}_\tau$, proving the result. □

Remark 17 This gives an algorithm to choose of $x[\tau']$ for all τ' in a conjugacy class once we've picked a single $x[\tau]$.

Cayley transforms require just a little more thought. We start on the fundamental fiber.

Definition 18 *Recall δ is a fixed element of G^Γ, and we use the same notation for the corresonding elements of $\mathcal{X}$ and of $\mathcal{I}_W$.*

Define

$$m_{\rho^\vee} = \exp(\pi i \rho^\vee), \quad z_\rho = m_{\rho^\vee}^2 = \exp(2\pi i \rho^\vee) \in Z(G). \tag{19(a)}$$

Define the basepoint in $\mathcal{X}_\delta$ to be:

$$x[\delta] = \exp(\pi i \rho)\delta. \tag{19(b)}$$

Since we make repeated use of this element we define

$$\delta_1 = \exp(\pi i \rho)\delta. \tag{19(c)}$$

Recall $\exp(\pi i \rho)\delta$ *is a particularly important element; see [2, 9.7(b)]. In particular every simple δ-imaginary root is non-compact with respect to $x[\delta]$ (the corresonding Borel is* large*) and $x[\delta]^2 = z_\rho$.*

By the Lemma this defines $x[\tau]$ for all τ conjugate to δ (this is a singleton in the equal rank case). Marc suggest defining $x[\tau]$ in general as follows.

Definition 20 *Given $\tau \in \mathcal{I}_W$, write $\tau = w\delta \in \mathcal{I}_W = \langle W, \delta\rangle$ (recall $w\theta_\delta(w) = 1$). Let σ_w be the canonical lift of w to the Tits group. Let $x[\tau] = \sigma_w\delta_1$.*

This definition has several remarkable properties. First of all it is consistent with Lemma 12.

Lemma 21 *If α is a simple τ-complex root then $s_\alpha \times x[\tau] = x[s_\alpha\tau s_\alpha]$. In particular this definition of $x[\tau]$ is consistent with the algorithm of Remark 17.*

Proof. Write $\tau = w\delta$, and let $\beta = \tau(\alpha)$, so $s_\alpha(w\delta)s_\alpha = s_\alpha w s_\beta\tau$. Without loss of generality we may assume (after switching w and $s_\alpha w s_\beta$ if necessary) that $\ell(s_\alpha w s_\beta) > \ell(w)$, so $s_\alpha w s_\beta$ is a reduced expression. We have to show

$$\sigma_\alpha(\sigma_w m_{\rho^\vee}\delta)\sigma_\alpha^{-1} = (\sigma_\alpha w \sigma_\beta)m_{\rho^\vee}\delta \tag{22}$$

(careful with the inverses here, they are correct). Recall $\sigma_\alpha^2 = \exp(\pi i\alpha^\vee)$, and we denote the latter element m_α. Then $\sigma_\alpha^{-1} = \sigma_\alpha m_\alpha$, and the left hand side is

$$\begin{aligned}\sigma_\alpha(\sigma_w m_{\rho^\vee}\delta)\sigma_\alpha^{-1} &= \sigma_\alpha\sigma_w m_{\rho^\vee}\delta\sigma_\alpha m_\alpha\\ &= \sigma_\alpha\sigma_w m_{\rho^\vee}\sigma_\beta m_\beta\delta\\ &= \sigma_\alpha\sigma_w\sigma_\beta(\sigma_\beta^{-1}m_{\rho^\vee}\sigma_\beta)m_\beta\delta\end{aligned} \tag{23}$$

Now $(\sigma_\beta^{-1}m_{\rho^\vee}\sigma_\beta)m_\beta = \exp(\pi i(\rho^\vee - \beta^\vee))\exp(\pi i\beta^\vee) = m_{\rho^\vee}$. Plugging this in gives the right hand side. $\square$

Let α be τ-imaginary root, and assume it is non-compact with respect to $x[\tau]$. Then the Cayley transform $c^\alpha x[\tau]$ is defined. See [1, Definition 14.1]. In fact

$$c^\alpha x[\tau] = \sigma_\alpha x[\tau]. \tag{24}$$

We are being a little sloppy here. Recall $\widetilde{\mathcal{X}}$ is a subset of G^Γ, and $\mathcal{X}$ is the quotient of $\widetilde{\mathcal{X}}$ by conjugation by H. See [1, Section 9]. We are identifying $\mathcal{X}$ with a subset of representatives in $\widetilde{\mathcal{X}}$. This is legitimate: if $\xi \in \widetilde{\mathcal{X}}$ represents $x \in \mathcal{X}$, and α is x-non-compact-imaginary, then $\sigma_\alpha\xi$ represents $c_\alpha x$ [1, Lemma 14.2]. Note that this is *not* necessarily true for real (inverse) Cayley transforms.

In any event we have:

Lemma 25 *Suppose α is a τ-imaginary root, which is non-compact imaginary with respect to $x[\tau]$. Then*

$$x[s_\alpha \tau] = c_\alpha x[\tau]. \tag{26}$$

Proof. Suppose $\tau = w\delta$. We have to show $x[s_\alpha w\tau] = \sigma_\alpha x[w\tau]$, or $\sigma_{s_\alpha w}\delta_1 = \sigma_\alpha\sigma_w\delta_1$, i.e. $\sigma_{s_\alpha w} = \sigma_\alpha\sigma_w$. This comes down to the fact that $\ell(s_\alpha w) > \ell(s_\alpha)$ since α is simple and τ-imaginary (right, Marc?). $\square$

This gives a strong property of the basepoints. Start at $x[\delta]$, and apply *any* sequence of simple complex cross actions and non-compact imaginary Cayley transforms. The result will be a basepoint.

What is left is to confirm this satisfies the requirements of [2, Proposition 6.30]. To be continued...

0.6 Appendix

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Subject:    Re: minor matters
From:     "Marc van Leeuwen" <Marc.van-Leeuwen@math.univ-poitiers.fr>
Date:     Thu, November 5, 2009 7:51 am

...

First the identification of the parameters (what is a bit confusing is that all
of them have several "components", some of which overlap with those of others).
There is a strong involution (actually given as KGB element) x, which has as
component an involution theta of the root datum (the other part, the torus part
of x, will play no role for y). Then there is the discrete parameter lambda,
which is a character of the rho-cover of the real Cartan at theta (please
correct me if the wording is wrong) and lies in rho+X^* and is defined modulo
(1-theta)X^*; one can distinguish the free part of lambda (determined by
(1+theta)lambda) and the torsion part of lambda, although the decomposition may
not be mathematically very meaningful. Finally there is the parameter gamma,
taken as a rational weight, which is a representative of the infinitesimal
character; it has a discrete part (projection on +1 eigenspace of theta) that
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is bound to the free part of lambda by the relation (1+theta)(gamma-lambda)=0, and a continuous part (projection on -1 eigenspace of theta) that is also known as nu. I suppose that the Weyl group can act on everything to give an equivalent set of parameters, but this aspect confuses me because (1) the formulas seem to depend on choosing definite representatives and (2) if we are going to determine a block element (x,y) then certainly we cannot operate by cross actions on x (and y) and hope to get an equivalent block element; therefore I will ignore any W action and suppose (x,lambda,gamma) is fixed.

What I need is a formula to find y associated to these parameters, in the form y = t.sigma_w.delta where t is a dual torus element whose square is central in G^\vee(\gamma), the dual group whose root system is the subsystem of the coroot system of G integral on gamma (this means that all coroots integral on gamma should have values plus or minus 1 on t); sigma_w is the canonical lift of a twisted involution in the dual Tits group (in fact w is determined by theta), and \delta is some fixed basic involution of G^\vee. In order to have a formula with some chance of being valid I take \delta_1=exp(i\pi\rho)\delta_0 with delta_0 the pinning-preserving involution of G^\vee (never mind whether that makes sense). This choice makes sigma_w.delta_1 the base-point for the dual fiber. The formula I propose is

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   y = exp(i\pi(gamma-lambda)) . sigma_w.delta_1
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There are a number of sanity checks that this formula seems to pass. The square of y should be exp(2i\pi(\gamma)); I think I checked that by computation in the Tits group of G^\vee (more or less). Then y should encompass the torsion information of lambda (because that is what lives in the dual fiber group that atlas uses to compute with y); note that taking the difference gamma-lambda precisely ignores the free part of lambda, and the discrete part of gamma. The value of y is only meaningful modulo the image of the +1 \Q-eigenspace of theta^t on \Q\tensor X_* (or the -1 eigenspace of -theta^t if you prefer); changing just the free part of lambda gives a change precisely in that direction (this is not really a check, but I find it reassuring nonetheless).

Then we should have coincidence with the formula (4.3) in the sp4forms paper given at theta=-1 (which implies w=e); that formula has an unspecified binary parameter epsilon, and setting epsilon = rho-lambda (mod 2X^*) gives my formula above (this was indeed the starting point for my formula). Finally y should transform correctly under cross actions and Cayley transforms; since I do not know how to do these in general at the level of (x,lambda,gamma) I could only

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check that for complex cross actions, where things seem to work well. So I
cross my fingers and hope this is the right formula. (By the way, I must have
mentioned this formula at least twice in previous emails, but with perhaps not
enough explanation).

-- Marc
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References

[1] Jeffrey Adams and Fokko du Cloux. Algorithms for representation theory of real reductive groups. *J. Inst. Math. Jussieu*, 8(2):209–259, 2009.

[2] Jeffrey Adams and David A. Vogan, Jr. L-groups, projective representations, and the Langlands classification. *Amer. J. Math.*, 114(1):45–138, 1992.