Affine Weyl group and coherent families

David Vogan (Notes by Siddhartha Sahi)

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1 Weyl groups

Let G be a complex reductive group, with dual group $G^\vee$,
Weyl group W, character lattice X^*, and root lattice R
The affine Weyl group is $W^{aff} := W \ltimes R$
The extended affine Weyl group is $W^{ext} := W \ltimes X^*$
They act on X^*, $\mathfrak{h}^*_{\mathbb{R}} := X^* \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathfrak{h}^* := X^* \otimes_{\mathbb{Z}} \mathbb{C}$

$W^{aff}/W^{ext} \approx X^*/R \approx$ algebraic characters of $Z(G)$
$\mathfrak{h}^*_{\mathbb{R}}/W^{aff}$ is the fundamental alcove
$\mathfrak{h}^*/W^{ext} \leftrightarrow$ conjugacy classes of semisimple elements in $G^\vee$
$\mathfrak{h}^*/W \leftrightarrow$ characters of $\mathcal{Z}(\mathfrak{g})$ (Harish-Chandra isomorphism)

2 Coherent family

Let Ξ be an X^*-coset in $\mathfrak{h}^*/X^*$.
A *coherent family* for $G_{\mathbb{R}}$ based on Ξ is a map
$\Theta : \Xi \to \{\text{virtual representations of } G_{\mathbb{R}}\}$ such that

1. $\Theta(\xi)$ has infinitesimal character ξ, for all ξ in Ξ
2. for any finite dimensional representation F of G we have
   $F \otimes \Theta(\xi) = \sum_\mu m_F(\mu)\,\Theta(\xi+\mu)$
   where $m_F(\mu)$ is the multiplicity of the weight μ in F.

The set of all such Θ forms a $\mathbb{Z}$-module $CF(\Xi)$.
[More generally we can consider coherent families
with values in other categories of representations]

Problem 1 *Define a W^{aff} representation on $CF(\Xi)$*

3 W action

Given $w \in W$ and Θ in $CF(\Xi)$, define
$(w\Theta)(\xi) := \Theta\left(w^{-1}\xi\right)$ for $\xi \in w\Xi$
Then $w\Theta$ is a coherent family based on $w\Xi$
This defines a W representation on the *sum* of the various $CF(w\Xi)$ as w ranges over W

However, consider the following subgroups of W :
$W_{[\Xi]} := \{w \in W : w(\Xi) = \Xi\}$
$W_{\Xi} := \{w \in W : w(\xi) - \xi \in R$ for some (hence all) ξ in $\Xi\}$
Then these act naturally on $CF(\Xi)$ itself.
[W_{Ξ} is a parabolic subgroup of W^{aff}, though not of W]

Let $s = s(\Xi)$ be the semisimple element in $H^{\vee} \subset G^{\vee}$ corresponding to the natural map $\mathfrak{h}^*/X^* \to \mathfrak{h}^*/W^{ext}$.
Its centralizer $G_s^{\vee}$ is reductive (perhaps disconnected).
Then $W_{[\Xi]}$ is the Weyl group $W(G_s^{\vee}, H^{\vee})$ and we have $W_{[\Xi]}/W_{\Xi} \approx G_s^{\vee}/\{$identity component$\}$

4 Associated variety and cycle

Suppose $\xi_0 \in \Xi$ is regular. The map $\Theta \to \Theta(\xi_0)$ gives a bijection:
$CF(\Xi) \to \{$Virtual representations with infinitesimal character $\xi_0\}$
Given Θ in $CF(\Xi)$, write $\Theta(\xi_0) = \sum m_i X_i$, where X_i are irreducible
The *associated variety* of X_i is a union of closures of nilpotent K orbits
Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be the maximal such orbits (obtained as i varies).
This collection is independent of the regular $\xi_0 \in \Xi$
Write $\mathcal{O}_j \approx K/K_j$ where K_j is the stabilizer of some point in $\mathcal{O}_j$.

The *associated cycle* of Θ gives for each ξ in Ξ, a virtual representation $\tau_{ij}(\xi)$ of each K_j.
We define $\tau_j(\xi) = \sum_i m_i \tau_{ij}(\xi)$, then we have
$\tau_j(\xi) \otimes F = \sum_\mu m_F(\mu)\ \tau_j(\xi + \mu)$

We want to compute $\tau_j(\xi)$; the following result should be useful:

Proposition 2 *If $\phi : X^* \to \mathbb{C}$ is a function satisfying*

$$\dim(F)\,\phi(\lambda) = \sum_\mu m_F(\mu)\,\phi(\lambda + \mu)$$

Then ϕ is a harmonic polynomial in $S(\mathfrak{h})$.