Computing Global Characters Using Atlas

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1 Introduction

Let π be a representation of a real reductive group G. The global character θ_π of π may be considered as a function on the regular semisimple elements of G. It is of great interest to compute this function.

Fix a Cartan subgroup H of G and let D be the Weyl denominator. Let λ be the infinitesimal character of π. The function θ_π restricted to the regular elements H_r of H is roughly of the form

$$\theta_\pi = \frac{\sum_{w\in W} a_w e^{w\lambda}}{D} \tag{1.1}$$

for certain integers a_w. We would like to compute these integers.

There are several methods for doing this in the literature. First of all by the induced character formula [2, Theorem 5.7] one can reduce to the case of a discrete series representation. The most definitive algorithm for this case is due to Rebecca Herb [**?**]. This is combinatorial in nature, and is based on the theory of endoscopy. It proceeds in two steps. First one computes θ_π for a π a stable sum of discrete series representations (this uses the theory of 2-structures). Then one computes θ_π for G in terms of the character of stable sums of discrete series for G and for various endoscopic groups.

Another very different formula is [6]. Some speical cases include [**?**] and unpublished work of Zuckerman.

Properties of the discrete series characters, including the Hecht-Schmid character identities [1], give recursive formulas which determine these constants [4, XIII,§4]. Carrying out this computation would involve running over many subgroups of G of lower semisimple rank. For example see [**?**]. From the point of view of theoretical applications Herb's result is the most useful.

We follow another approach due to David Vogan. The main ideas are in in [**?**] (in particular see [**?**, Theorem 2.3 and Corollary 4.10]); these are combined with the methods of [1], updated to make more direct use of coherent continuation and the cross action, and expressed in terms suitable to `atlas`. In fact the approach here is elementary and self-contained, given the basic facts about characters and coherent continuation. One notable feature is that the use of the ρ-cover of Cartan subgroups makes the formulation considerably simpler.

2 Weyl denominators and related functions

Fix a real reductive group G and a θ-stable Cartan subgroup H. Let $\Delta = \Delta(G,H)$, the roots of G with respect to H. Write $\Delta_r, \Delta_i, \Delta_{cx}, \Delta_{ic}, \Delta_{in}$ for the real, imaginary, complex, imaginary compact and imaginary real roots as usual. If Δ^+ is a set of positive roots write $\Delta_i^+ = \Delta^+ \cap \Delta_i$, etc.

Let Δ^+ be a set of positive roots and let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Also let $\rho_i = \frac{1}{2}\sum_{\alpha\in\Delta_i^+}\alpha$, and ρ_r, ρ_{cx} similarly. Let W_i, W_r be the Weyl groups of Δ_i and Δ_r.

We define the ρ two-fold cover of H as in [1, Section 5]. If we choose another set of positive roots, with corresonding ρ', these covers are canonically isomorphic. This identifies the genuine characters of the ρ-covers of H for all choices of positive roots. We abuse notation a bit and refer to such a collection as as a *genuine character of* $\tilde{H}$. Given such an element Γ and a choice of Δ^+, we obtain a genuine character of the ρ-cover of H.

[1] According to Becky Herb, these should be called the Harish-Chandra character identities. We report, you decide.

Definition 2.1 *Fix a set Δ^+ of of positive roots. Define*

(2.2)(a) $$D^0(g) = \prod_{\alpha\in\Delta^+}(1-e^{-\alpha}(g))$$

(2.2)(b) $$\epsilon_r(g) = sign \prod_{\alpha\in\Delta_r^+}(1-e^{-\alpha}(g))$$

(2.2)(c) $$D^1(g) = \epsilon_r(g)D^0(g)$$

(2.2)(d) $$|D(g)| = |D^0(g)e^\rho|$$

(2.2)(e) $$D(g) = D^0(g)e^\rho$$

The function in (e) is on the ρ-cover of H; its absolute value factors to H, and gives (d).

We write $D^0(\Delta^+, g)$ etc. to indicate the dependence on Δ^+.

The dependence of D on Δ^+ is obvious (modulo chasing the covers a bit): for $w \in W$ we have

(2.3) $$D(\Delta^+, w^{-1}g) = D(w\Delta^+, g) = \mathrm{sgn}(w)D(\Delta^+, g).$$

We will make frequent use of the following condition on Δ^+:

(2.4) $$\alpha > 0, \alpha \text{ complex } \Rightarrow \theta(\alpha) < 0.$$

Write $H = TA$ and let $M = \mathrm{Cent}_G(A)$ as usual. A set of positive roots Δ^+ satisfies (2.4) if the real and imaginary roots are the roots of $\mathfrak{h}$ in $\mathfrak{n}$, where MN is a parabolic subgroup.

Given λ, it is possible to choose Δ^+ making λ dominant for Δ_i^+, Δ_r^+ and satisfying (2.4): define Δ^+ to be the root satisfying

(2.5)(a) $$\langle\lambda|_{\mathfrak{a}}, \alpha^\vee\rangle > 0$$

or

(2.5)(b) $$\langle\lambda|_{\mathfrak{a}}, \alpha^\vee\rangle = 0 \text{ and } \langle\lambda|_{\mathfrak{t}}, \alpha^\vee\rangle > 0.$$

Let W^θ be the elements of W commuting with θ. This acts on H, and contains $W(G,H)$. Note that if Δ^+ satisfies (2.4) then so does $w\Delta^+$ for all $w \in W^\theta$.

Choose Δ^+ satisfying (2.4) and define

(2.6) $$\zeta_{cx}(g) = e^{\rho_{cx}}/|e^{\rho_{cx}}|$$

where $\zeta_{cx} = \frac{1}{2}\sum_{\alpha\in\Delta^+_{cx}}\alpha$. By (2.4) Δ^+_{cx} contains pairs $\alpha, -\theta\alpha$, and from this it is easy to see that $\zeta_{cx}(g) \in \pm 1$ for all g, and is independent of the choice of Δ^+ satisfying (2.4).

Define

$$\gamma(\Delta^+, g) = \frac{e^{\rho_r}(g)}{|e^{\rho_r}(g)|}. \tag{2.7}$$

This is a function on the ρ_r cover of H. Since $e^{2\rho_r}(g) \in \mathbb{R}^\times$,

$$\gamma(\Delta^+_r, g)^4 = 1. \tag{2.8}$$

Note that $\gamma(\Delta^+, g)$ only depends on Δ^+_r.

If $w \in W$ we can write $w = w'w_r$ where $w_r \in W_r$ and

$$(w'\Delta^+)_r = \Delta^+_r \tag{2.9}$$

or equivalently

$$\lambda \text{ is } \Delta^+ - \text{dominant} \Rightarrow w\lambda \text{ is } \Delta^+_r - \text{dominant}. \tag{2.10}$$

For $w \in W$ define

$$\begin{aligned} \tau(\Delta^+, w)(g) &= \gamma(\Delta^+, g)/\gamma(w\Delta^+, g) \\ &= \mathrm{sgn}(e^{\rho_r - w\rho_r}(g)) \end{aligned} \tag{2.11}$$

This factors to a character on H with values in ± 1. If $w \in W^\theta$ then

$$\tau(\Delta^+, w)(g) = \gamma(\Delta^+, g)/\gamma(\Delta^+, wg). \tag{2.12}$$

Fix H and Δ^+. Recall $\Delta^+_i = \Delta^+ \cap \Delta_i$. For $w \in W$ note that $(w\Delta^+)_i = (w\Delta^+) \cap \Delta_i$ is a set of positive roots of Δ_i. For $w \in W$ define $w_i \in W_i$ by: $w_i(\Delta^+_i) = (w\Delta^+)_i$ and

$$\mathrm{sgn}_i(w) = \mathrm{sgn}(w_i) \tag{2.13}$$

Define $\mathrm{sgn}_r(w)$ similarly using the real roots.

Here is another way to think of this. Given Δ^+ we can write $w \in W^\theta$ as $w = w_i w_{cx} w_r$ by [8, ?]. Then $\mathrm{sgn}_i(w) = \mathrm{sgn}(w_i)$ and $\mathrm{sgn}_r(w) = \mathrm{sgn}(w_r)$. Also note that $\mathrm{sgn}(w_{cx}) = 1$ since w_{cx} is a product of terms $s_\alpha s_{\theta(\alpha)}$. In particular note that

$$\mathrm{sgn}(w) = \mathrm{sgn}_i(w)\mathrm{sgn}_r(w). \tag{2.14}$$

Lemma 2.15
(1) $\gamma(\Delta^+,g)=\gamma(\Delta'^+,g)$ *if* $\Delta^+_r=\Delta'^+_r$. *(2) For all* $w\in W^\theta$*:*

$$\tau(\Delta^+,w)=\tau(\Delta^+,w_r) \tag{2.16}$$

(3) For all $x,y\in W$*,*

$$\tau(\Delta^+,xy)=\tau(\Delta^+,y)\tau(y\Delta^+,x) \tag{2.17}$$

(4) For all $x\in W^\theta, y\in W$*,*

$$\tau(\Delta^+,xy)(g)=\tau(\Delta^+,x)(g)\tau(\Delta^+,y)(x^{-1}g) \tag{2.18}$$

or equivalently

$$\tau(\Delta^+,xy)=\tau(\Delta^+,x)(x\cdot\tau(\Delta^+,y)) \tag{2.19}$$

(cf. [1, (8.26)(b)]).

For later use we also determine how $\epsilon_r(\Delta^+,g)$ depends on Δ^+, at least to some extent.

Lemma 2.20 *Suppose* H *is a* θ*-stable Cartan subgroup,* Δ^+ *is a set of positive roots satisfying and* $w\in W^\theta$. *For all* $g\in H$*:*

$$\epsilon_r(\Delta^+,wg)=sgn_r(w)\epsilon_r(\Delta^+,g)\tau(\Delta^+,w^{-1})(g). \tag{2.21}$$

3 Standard Modules

We restrict ourselves to regular integral infinitesimal character.

Definition 3.1 *A regular character for* G *is a pair* (H,Λ) *where* H *is a Cartan subgroup of* G*,* Λ *is a genuine character of* $\widetilde{H}$*, and* $d\Lambda$ *is regular and integral. Write* $\mathcal{P}_0(G)$ *or simpliy* $\mathcal{P}_0$ *for the set of regular characters.*

If $(H,\Lambda)\in\mathcal{P}_0$ *let* $I(G,H,\Lambda)$ *be the standard representation of* G *[8, Definition 8.27]. More precisely* $I(G,H,\Lambda)=I(\Psi,P,\Lambda)$ *where*

$$\mathcal{P}=\{\alpha\in\Delta_i\,|\,\langle\lambda,\alpha^\vee\rangle>0,\quad \Psi=\{\alpha\in\Delta_r\,|\,\langle\lambda,\alpha^\vee\rangle<0\}.$$

The infinitesimal character of $I(G,H,\Lambda)$ *is* $d\Lambda\in\mathfrak{h}^*$.

Let $\pi(G,H,\Lambda)$ *be the unique irreducible submodule of* $I(G,H,\Lambda)$*. Note that* $I(G,H,\Lambda)\simeq I(G,H',\Lambda')$ *if and only if* (G,H,λ) *and* (H,Λ') *are conjugate, and similarly for* $\pi(G,H,\Lambda)$*. Let*

$$\mathcal{P}=\mathcal{P}_0/G. \tag{3.2}$$

We may write $I(H,\Lambda)$ or even $I(\Lambda)$.

Fix H and $\lambda \in \mathfrak{h}^*$, regular and integral. Let

$$\begin{aligned}\mathcal{P}_0(H,\lambda) &= \{(H,\Lambda)\,|\,d\Lambda \in W\lambda\} \\ &= \{(H,\Lambda)\,|\,I(G,H,\Lambda) \text{ has infinitesimal character } \lambda\}\end{aligned} \tag{3.3}$$

and

$$\mathcal{P}(H,\lambda) = \mathcal{P}_0(H,\lambda)/W(G,H). \tag{3.4}$$

The set $\mathcal{P}(H,\lambda)$ parametrizes the irreducible representations of G "coming from" H.

We now describe $I(G,H,\Lambda)$ in terms convenient for our computation.

We now return to the definition of $I(G,H,\Lambda)$. First assume H is relatively compact. Then $I(G,H,\Lambda)$ is a relative discrete series representation. It has Harish-Chandra parameter $d\Lambda$ and central character Λe^{ρ} restricted to the center (note that this is independent of the choice of ρ), and these properties determine it. Alternatively $I(G,H,\Lambda)$ is determined by its character on H, which is given in Proposition 6.2.

Now suppose H is a general θ-stable Cartan subgroup. Let $M = \mathrm{Cent}_G(A)$. The standard module $I(G,H,\Lambda)$ is induced from a relative discrete series representation $I(M,H,\Lambda_M)$, which we now describe.

Definition 3.5 *Let* $\rho_r = \frac{1}{2}\sum_\alpha \alpha$ *where the sum is over* $\{\alpha \in \Delta_r \,|\, \langle\lambda,\alpha^\vee\rangle < 0$. *Define*

$$\Lambda_M(g) = \Lambda(g)\zeta_{cx}(g)\gamma(\Delta_r^+, g) \tag{3.6}$$

Note that $d\Lambda_M = d\Lambda$. This can be viewed as a function on the ρ_i-cover of H, and is therefore the parameter of a (relative) discrete representation

$$\pi_M = I(M,H,\Lambda_M). \tag{3.7}$$

of M. Let

$$I(G,H,\Lambda) = \mathrm{Ind}_{MN}^G(\pi_M). \tag{3.8}$$

4 Character Formulas

Fix a θ-stable Cartan subgroup H, and write H_{reg} for the regular elements of H. Suppose Δ^+ is a set of positive roots. Let

$$H_+ = \{g \in H_{\text{reg}} \,|\, e^\alpha(g) > 1 \text{ for all } \alpha \in \Delta_r^+\}. \tag{4.1}$$

Every element of H_{reg} is conjugate via W_r to an element of H_+.

Proposition 4.2 *Suppose π is an admissible representation, with regular integral infinitesimal character. Fix a θ-stable Cartan subgroup H of G and a set of positive roots Δ^+. Choose $\lambda \in \mathfrak{h}^*$ so that the infinitesimal character of π is (the Weyl group orbit of) λ. Let $q = \frac{1}{2}\dim(G/K)$*

Then θ_π restricted to H_+ may be written

$$\theta_\pi = \frac{\sum_{\Lambda \in \mathcal{P}_0(H,\lambda)} a(\pi, \Delta^+, \Lambda)\Lambda}{D(\Delta^+)} \tag{4.3}$$

where $a(\pi, \Delta^+, \Lambda) \in \mathbb{Z}$ for all π and Λ. The coefficients $a(\Delta^+, \pi, \Lambda)$ are uniquely determined by this equality.

We recall that both the numerator and denominator are functions on H_ρ, and the quotient factors to H.

This almost follows from the Casselman-Osborn conjecture [2], except that the set on which this expansion is valid is larger than that of Casselman and Osborn. Vogan attributes this in [**?**] to Harish-Chandra. I think it follows from the methods in these notes; see Remark 9.14.

We want to compute the integers $a(\pi, \Delta^+, \Lambda)$.

Example 4.4 Suppose $\pi = \mathbb{C}$, the trivial representation. Fix H and Δ^+, and let $\Lambda = e^\rho$, the distinguished genuine character of the ρ-cover of H. Then

$$D(\Delta^+, g) = \sum_{w \in W} (w \times \Lambda)(g). \tag{4.5}$$

Therefore

$$\Theta_{\mathbb{C}} = \frac{\sum_{w \in W} w \times \Lambda}{D(\Delta^+)} \tag{4.6}$$

or in other words

$$a(\mathbb{C}, \Delta^+, w \times \Lambda) = \text{sgn}(w) \text{ for all w.} \tag{4.7}$$

Suppose $\Lambda, \Lambda' \in \mathcal{P}_0$. Then the standard module $I(\Lambda)$ is defined, and we define

$$a(\Lambda, \Delta^+, \Lambda') = a(I(\Lambda), \Delta^+, \Lambda'). \tag{4.8}$$

More precisely we should write $a(\Lambda, H, \Delta^+, \Lambda', H')$ to indicate the Cartans in question; note that Δ^+ is a set of positive roots for H'. The case $H = H'$ is particular important.

By the Kazhdan-Lusztig-Vogan algorithm every irreducible representation can be written as a linear combination of standard modules, so it is (in some sense) enough to compute the $a(\Lambda, \Delta^+, \Lambda')$.

Lemma 4.9 *Fix Δ^+ and $w \in W$. Write $w = xy$ where $x \in W_r$ and $(y\Delta^+)_r = \Delta^+_r$. Then*

$$a(\pi, w\Delta^+, \Lambda) = sgn(y) a(\pi, \Delta^+, x^{-1}\Lambda). \tag{4.10}$$

In particular

$$\begin{aligned} &a(\pi, w\Delta^+, \Lambda) = a(\pi, \Delta^+, w^{-1}\Lambda) \quad (w \in W_r) \\ &a(\pi, w\Delta^+, \Lambda) = sgn(w) a(\pi, \Delta^+, \Lambda) \quad (w\Delta^+)_r = \Delta^+_r. \end{aligned} \tag{4.11}$$

Suppose $w \in W^\theta$, and write $w = w_i w_{cx} w_r$ as in Section 2. Then

$$a(\pi, w\Delta^+, \Lambda) = sgn(w_i) a(\pi, \Delta^+, w_r^{-1}\Lambda) \tag{4.12}$$

In Section 7 we will see how $a(\pi, \Delta^+, \Lambda)$ behaves under the coherent continuation action of W on π.

5 Cross Action

We use the cross action to write (4.3) in a more convenient form.

Suppose Λ is a character of H and $d\Lambda$ is integral. For $w \in W$ write $wd\Lambda = d\Lambda + \sum n_\alpha \alpha$ (sum over the roots) and define

$$w \times \Lambda = \Lambda + \sum n_\alpha \alpha. \tag{5.1}$$

This is the *cross action*. It satisfies $d(w \times \Lambda) = wd\Lambda$. Note that if $w \in W(G, H)$ then $w\Lambda$ is defined, but it is *not* necessarily the case that $w \times \Lambda = w\Lambda$.

The same definition holds for Λ a genuine character of $\tilde{H}$.

Note that this definition of the cross action is simpler than that of [7, Definition 8.3.1], thanks to the fact that we are using a more natural parametrization of representations.

Fix a set Δ^+ of positive roots, and choose λ dominant with respect to Δ^+ so that π has infinitesimal character λ. If $\Lambda \in \mathcal{P}(H, \lambda)$ then $w \times \Lambda$ has differential λ for some $w \in W$.

Lemma 5.2 *Let $\Lambda_1, \dots, \Lambda_r$ be the elements of $\mathcal{P}_0(H, \lambda)$ with differential λ. With notation as in the preceding Lemma,*

$$\theta_\pi = \frac{\sum_{i=1}^{r} \sum_W a(\Delta^+, \pi, w \times \Lambda_i) w \times \Lambda_i}{D} \tag{5.3}$$

6 Character formula for standard modules

Suppose H is a θ-stable Cartan subgroup and $(H, \Lambda) \in \mathcal{P}_0$. We give the formula for $\Theta_{I(G,H,\Lambda)}$ on H. This combines the character formula for discrete series on the compact Cartan with the induced character formula. We first state these.

Given G, let K be a maximal compact subgroup, and let G_d, K_d be the derived groups. Define

$$q_G = \frac{1}{2} \dim(G_d/K_d). \tag{6.1}$$

Proposition 6.2 (Harish-Chandra) *Suppose H is relatively compact and $(H, \Lambda) \in \mathcal{P}(H)$. Then $\pi = \pi(H, \Lambda) = I(H, \Lambda)$ is a relative discrete series representation. Let $\epsilon(\Delta^+, \Lambda) = sgn(w)$ where $\langle wd\Lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in \Delta^+$. Then for $g \in H_{reg}$ we have:*

$$\Theta_\pi(g) = (-1)^{q_G} \epsilon(\Delta^+, \Lambda) D(\Delta^+, g)^{-1} \sum_{w \in W(G,H)} sgn(w)(w\Lambda)(g). \tag{6.3}$$

The next Proposition is known as the induced character formula. See [2, Theorem 5.7], where it is attributed to Hirai and Wolf.

Proposition 6.4 *Suppose H is a θ-stable Cartan subgroup, write $H = TA$ and let $M = Cent_G(A)$. Let $P = MN$ be a parabolic subgroup, and suppose* *σ is an admissible representation of M. Let* $\pi = Ind_{MN}^G(\sigma\otimes 1)$ *(normalized induction). Then for* $g \in H_{reg}$*:*

$$\Theta_\pi(g) = |D(\Delta^+, g)|^{-1} \sum_{w\in W(M,H)\backslash W(G,H)} |D(\Delta_i^+, wg)|\Theta_\sigma(wg) \tag{6.5}$$

Here is the main result of this section.

Proposition 6.6 *Given* Λ*, choose* Δ^+ *satisfying* (2.4) *and so that* λ *is dominant for* Δ_i^+ *and* Δ_r^+ *(this is always possible, cf.* (2.5)*). Then*

$$\Theta_\pi(g) = \frac{(-1)^{q_M}\epsilon_r(\Delta^+, g)}{D(\Delta^+, g)} \sum_{w\in W_G} sgn_i(w)\tau(\Delta^+, w)(g)(w^{-1}\Lambda)(g) \tag{6.7(a)}$$

Furthermore assume $g \in H_+$*. Then*

$$\Theta_\pi(g) = \frac{(-1)^{q_M}}{D(\Delta^+, g)} \sum_{w\in W_G} sgn_i(w)\tau(\Delta^+, w)(g)(w^{-1}\Lambda)(g). \tag{6.7(b)}$$

In other words

$$a(\pi, \Delta^+, \tau(\Delta^+, w)(w\Lambda)) = sgn_i(w)(-1)^{q_M}. \tag{6.7(c)}$$

Then coefficients for other Δ^+ *can be computed from this and Lemma 4.9.*

An important special case is:

Lemma 6.8 *Suppose* $\Lambda \in \mathcal{P}_0(H)$ *and* Δ^+ *is a set of positive roots. Assume* Δ^+ *satisfies* (2.4) *and* $d\Lambda$ *is dominant for* Δ_i^+, Δ_r^+*. Then*

$$a(I(\Lambda), \Delta^+, \Lambda) = (-1)^{q_M} \tag{6.9}$$

Here is a formula for any Δ^+.

Proposition 6.10 *Fix* $\Lambda \in \mathcal{P}_0(H)$ *and let* $\lambda = d\Lambda$*. Let* $\pi = I(G, H, \Lambda)$*. Let* Δ^+ *be any set of positive roots. Choose* $u \in W_r$ *so that*

$$u(\Delta_r^+) = \{\alpha \in \Delta_r \,|\, \langle\lambda, \alpha^\vee\rangle > 0\} \tag{6.11}$$

and let

$$n = |\{\alpha \in \Delta^+ cx \,|\, \theta(\alpha) > 0\}|. \tag{6.12}$$

Then for $g \in H_{reg}$ we have

(6.13)

$$\Theta_\pi(g) = \frac{(-1)^{q_M}(-1)^n\epsilon(\Delta_i^+,\Lambda)\epsilon_r(\Delta^+,g)}{D(\Delta^+,g)} \sum_{w\in W_G} sgn_i(w)\tau(\Delta^+,w)(g)(wu^{-1}\Lambda)(g)$$

In other words

$$a(\pi,\Delta^+,\tau(\Delta^+,w)(wu^{-1}\Lambda)) = (-1)^{q_M+n}\epsilon(\Delta_i^+,\Lambda)\epsilon_r(\Delta^+,g)sgn_i(w). \tag{6.14}$$

Here is an elementary proof of the Proposition. A much shorter proof, using coherent continuation, is at the end of Section 9. This approach has the advantage that it makes the domain of validity of the formula clear.

Proof of Proposition 6.10. Write $W_G = W(G,H)$ and $W_M = W(M,H)$. Recall $I(G,H,\Lambda)$ is induced from $I(M,H,\Lambda_M)$ where Λ_M is given by (3.6). Let $\pi = I(G,H,\Lambda)$. By the preceding propositions:

$$\begin{aligned}
\Theta_\pi(g) &= |D(\Delta^+,g)|^{-1}\sum_{w\in W_M\backslash W_G}|D(\Delta_i^+,wg)|\Theta_{I(M,H,\Lambda_M)}(wg)\\
&= |D(\Delta^+,g)|^{-1}\sum_{w\in W_M\backslash W_G}|D(\Delta_i^+,wg)|(-1)^{q_M}\epsilon(\Delta_i^+,\Lambda_M)\times\\
&\qquad D(\Delta_i^+,wg)^{-1}\sum_{y\in W_M}\mathrm{sgn}(y)(y\Lambda_M)(wg).
\end{aligned} \tag{6.15}$$

Here we have used the fact that $\frac{1}{2}\dim(M/M\cap K) = \frac{1}{2}\dim(G/K) \pmod 2$ [5]. Regrouping terms gives

$$\begin{aligned}
&(-1)^{q_M}\epsilon(\Delta_i^+,\Lambda_M)D(\Delta^+,g)^{-1}\sum_{w\in W_M\backslash W_G}\frac{D(\Delta^+,g)}{|D(\Delta^+,g)|}\frac{|D(\Delta_i^+,wg)|}{D(\Delta_i^+,wg)}\times\\
&\qquad \sum_{y\in W_M}\mathrm{sgn}(y)\Lambda_M(y^{-1}wg)
\end{aligned} \tag{6.16}$$

Recall $D(\Delta^+,wg) = \mathrm{sgn}(w)D(\Delta^+,g)$ for all $w \in W(G,H)$. Therefore

(6.17)

$$\begin{aligned}
\frac{D(\Delta^+,g)}{|D(\Delta^+,g)|}\frac{|D(\Delta_i^+,wg)|}{D(\Delta_i^+,wg)} &= \mathrm{sgn}(w)\frac{D(\Delta^+,wg)}{|D(\Delta^+,wg)|}\frac{|D(\Delta_i^+,wg)|}{D(\Delta_i^+,wg)}\\
&= \mathrm{sgn}(w)\frac{\prod_{\alpha\in\Delta_r^+}(1-e^{-\alpha}(wg))}{\prod_{\alpha\in\Delta_r^+}|(1-e^{-\alpha}(wg))|}\frac{\prod_{\alpha\in\Delta_{cx}^+}(1-e^{-\alpha}(wg))}{\prod_{\alpha\in\Delta_{cx}^+}|(1-e^{-\alpha}(wg))|}\frac{e^{\rho-\rho_i}(wg)}{|e^{\rho-\rho_i}(wg)|}
\end{aligned}$$

Write $\rho - \rho_i = \rho_{cx} + \rho_r$, and regroup terms:

(6.18)

$$= \operatorname{sgn}(w) \frac{\prod_{\alpha\in\Delta_r^+}(1-e^{-\alpha}(wg))e^{\rho_{cx}}(wg)}{\prod_{\alpha\in\Delta_r^+}|(1-e^{-\alpha}(wg))e^{\rho_{cx}}(wg)|} \frac{\prod_{\alpha\in\Delta_{cx}^+}(1-e^{-\alpha}(wg))}{\prod_{\alpha\in\Delta_{cx}^+}|(1-e^{-\alpha}(wg))|} \frac{e^{\rho_r}(wg)}{|e^{\rho_r}(wg)|}$$

If Δ^+ satisfies (2.4) then the term involving complex roots equals $\zeta_{cx}(wg)$. It is straightforward to see that in general this terms equals $(-1)^n\zeta_{cx}(wg)$ where n is the number of pairs of positive complex roots $\alpha, \theta\alpha$.

By the definition of ϵ_r and γ we obtain

$$\operatorname{sgn}(w)\zeta_{cx}(wg)(-1)^n\epsilon_r(\Delta^+, wg)\gamma(\Delta^+, wg). \tag{6.19}$$

Recall by Lemma (2.20) we have

$$\epsilon_r(\Delta^+, wg) = \operatorname{sgn}_r(w)\epsilon_r(\Delta^+, g)\tau(\Delta^+, w^{-1})(g). \tag{6.20}$$

which gives

$$\operatorname{sgn}(w)\operatorname{sgn}_r(w)\zeta_{cx}(wg)(-1)^n\epsilon_r(\Delta^+, g)\tau(\Delta^+, w^{-1})(g)\gamma(\Delta^+, wg). \tag{6.21}$$

Letting $\mu = (-1)^{q_M+n}\epsilon(\Delta_i^+, \Lambda_M)\epsilon_r(\Delta^+, g)D(\Delta^+, g)^{-1}$ and plugging this in gives

$$\begin{aligned}\Theta_\pi(g) = \mu \sum_{w\in W_M\backslash W_G}\sum_{y\in W_M} &\operatorname{sgn}(w)\operatorname{sgn}_r(w)\operatorname{sgn}(y)\times \\ &\zeta_{cx}(wg)\tau(\Delta^+, w^{-1})(g)\gamma(\Delta^+, wg)\Lambda_M(y^{-1}wg)\end{aligned} \tag{6.22}$$

Let $\Delta^{'+} = \{\alpha \,|\, \langle\lambda, \alpha^\vee\rangle < 0\}$. Recall

$$\Lambda_M(g) = \gamma(\Delta^{'+}, wg)\zeta_{cx}(g)\Lambda(wg) \tag{6.23}$$

Therefore

$$\begin{aligned}&\zeta_{cx}(wg)\tau(\Delta^+, w^{-1})(g)\gamma(\Delta^+, wg)\Lambda_M(y^{-1}wg) = \\ &\zeta_{cx}(wg)\tau(\Delta^+, w^{-1})(g)\gamma(\Delta^+, wg)\gamma(\Delta^{'+}, y^{-1}wg)\zeta_{cx}(y^{-1}wg)\Lambda(y^{-1}wg)\end{aligned} \tag{6.24}$$

Since $y \in W_i$ $\zeta_{cx}(y^{-1}wg) = \zeta_{cx}(wg)$; since $\zeta_{cx}(wg) = \pm 1$ these terms cancel. Also $\gamma(\Delta^+, y^{-1}wg) = \gamma(\Delta^+, wg)$, This gives

(6.25) $$\tau(\Delta^+,w^{-1})(g)\gamma(\Delta^+,wg)\gamma(\Delta'^+,wg)\Lambda(y^{-1}wg)$$

Choose $u\in W_r$ satisfying

(6.26) $$u(\Delta_r^+)=-\Delta_r'^+=\{\alpha\in\Delta_r\mid\langle\lambda,\alpha^\vee\rangle>0\}$$

(note the minus sign). Then

(6.27) $$\begin{aligned}\gamma(\Delta^+,wg)\gamma(\Delta'^+,wg)&=\gamma(\Delta^+,wg)\gamma(-u\Delta^+,wg)\\&=\gamma(\Delta^+,wg)/\gamma(u\Delta^+,wg)\\&=\mathrm{sgn}(e^{\rho_r-u\rho_r}(wg))\\&=\tau(\Delta^+,u)(wg).\end{aligned}$$

and

(6.28) $$\begin{aligned}\tau(\Delta^+,w^{-1})\gamma(\Delta^+,wg)\gamma(\Delta'^+,wg)&=\tau(\Delta^+,w^{-1})\tau(\Delta^+,u)(wg)\\&=\tau(\Delta^+,w^{-1}u)(g).\end{aligned}$$

Therefore
(6.29)
$$\Theta_\pi(g)=\mu\sum_{w\in W_M\backslash W_G}\sum_{y\in W_M}\mathrm{sgn}(w)\mathrm{sgn}_r(w)\mathrm{sgn}(y)\tau(\Delta^+,w^{-1}u)(g)\Lambda(y^{-1}wg)$$

Let $v=y^{-1}w,w=yv$. Since $y\in W_i$,

(6.30) $$\begin{aligned}\mathrm{sgn}(w)\mathrm{sgn}_r(w)\mathrm{sgn}(y)&=\mathrm{sgn}(yv)\mathrm{sgn}_r(yv)\mathrm{sgn}(y)\\&=\mathrm{sgn}(y)\mathrm{sgn}(v)\mathrm{sgn}_r(v)\mathrm{sgn}(y)\\&=\mathrm{sgn}(v)\mathrm{sgn}_r(v).\end{aligned}$$

Also note that $w^{-1}u=(yv)^{-1}u=v^{-1}y^{-1}u$. Since $y\in W_i,u\in W_r$ they commute, so this equals $v^{-1}uy^{-1}$. Then $\tau(\Delta^+,w^{-1}u)=\tau(\Delta^+,v^{-1}uy^{-1})$. Since $y\in W_i$, $\tau(\Delta^+,v^{-1}uy^{-1})=\tau(\Delta^+,v^{-1}uy)$. This gives

(6.31) $$\Theta_\pi(g)=\mu\sum_{v\in W}\mathrm{sgn}(v)\mathrm{sgn}_r(v)\tau(\Delta^+,v^{-1}u)(g)\Lambda(vg).$$

Now let $w=u^{-1}v$, so $v=uw$ and $v^{-1}u=w^{-1}$:

(6.32) $$\Theta_\pi(g)=\mu\sum_{w\in W}\mathrm{sgn}(uw)\mathrm{sgn}_r(uw)\tau(\Delta^+,w^{-1})(g)\Lambda(uwg).$$

It is easy to see $(uw)_r = w_r u^{-1}$, so $\mathrm{sgn}_r(uw) = \mathrm{sgn}((uw)_r) = \mathrm{sgn}(w_r u^{-1}) = \mathrm{sgn}(w_r)\mathrm{sgn}(u) = \mathrm{sgn}_r(w)\mathrm{sgn}(u)$. Therefore

$$\mathrm{sgn}(uw)\mathrm{sgn}_r(uw) = \mathrm{sgn}(u)\mathrm{sgn}(w)\mathrm{sgn}_r(w)\mathrm{sgn}(u) = \mathrm{sgn}(w)\mathrm{sgn}_r(w). \tag{6.33}$$

This gives

$$\Theta_\pi(g) = \mu \sum_{w\in W} \mathrm{sgn}(w)\mathrm{sgn}_r(w)\tau(\Delta^+, w^{-1})(g)\Lambda(uwg). \tag{6.34}$$

By (2.14) $\mathrm{sgn}(w)\mathrm{sgn}_r(w) = \mathrm{sgn}_i(w)$. Writing $\Lambda(uwg) = (w^{-1}u^{-1})\Lambda(g)$, and making one last change $w \to w^{-1}$ gives

$$\Theta_\pi(g) = \mu \sum_{w\in W} \mathrm{sgn}_i(w)\tau(\Delta^+, w)(g)(wu^{-1}\Lambda)(g). \tag{6.35}$$

Finally plugging back in μ gives

$$\Theta_\pi(g) = \frac{(-1)^{q_M}(-1)^n\epsilon(\Delta_i^+,\Lambda)\epsilon_r(\Delta^+,g)}{D(\Delta^+,g)} \sum_{w\in W_G} \mathrm{sgn}_i(w)\tau(\Delta^+, w)(g)(wu^{-1}\Lambda)(g) \tag{6.36}$$

□

We may write any virtual representation with infinitesimal character λ as a formal sum of standard modules:

$$\pi = \sum_{\Lambda\in\mathcal{P}(\lambda)} M(\Lambda,\pi)I(\Lambda) \tag{6.37}$$

This is given by the Kazhdan-Lusztig polynomials, which are available via the `klbasis` command of the `atlas` software. See Section 10. We may therefore compute θ_π by (6.37) together with (6.13). For this reason (6.37) is sometimes referred to as a character formula.

Example 6.38 Let $G = SL(2,\mathbb{R})$, H be the split Cartan subgroup. Write $x = \mathrm{diag}(x, x^{-1}) \in H$, and choose $\Lambda(x) = |x|^\nu$ or $|x|^\nu\mathrm{sgn}(x)$ for $\nu \in \mathbb{Z}$. Assume $\nu < 0$ so we can take Δ^+ to be the usual root $\alpha(x) = x^2$. Then

$$\begin{aligned}\Theta_\pi(x) &= \frac{\mathrm{sgn}(1-\frac{1}{x^2})}{x-\frac{1}{x}}(\Lambda(x)+\Lambda(x^{-1}))\\ &= \frac{\mathrm{sgn}(x-\frac{1}{x})}{x-\frac{1}{x}}\mathrm{sgn}(x)(\Lambda(x)+\Lambda(x^{-1}))\\ &= \frac{(\Lambda\mathrm{sgn})(x)+(\Lambda\mathrm{sgn})(x^{-1})}{|x-\frac{1}{x}|}\end{aligned} \tag{6.39}$$

This is the familiar character formula for $\mathrm{Ind}_B^G(\Lambda\otimes\mathrm{sgn})$. Note that $\mathrm{sgn}(e^{\rho_r}(x)) = \mathrm{sgn}(x)$, so $\Lambda\otimes\mathrm{sgn} = \Lambda_M$. For example $\Lambda(x)=x$ (i.e. $|x|\mathrm{sgn}(x)$) gives

$$\Theta_\pi(x) = \frac{x+\frac{1}{x}}{|x-\frac{1}{x}|}; \tag{6.40}$$

this is the character of the principal series containing the trivial representation.

Note that $H_+ = \{x \,|\, |x|>1\}$ and

$$a(I(\Lambda),\Delta^+,\Lambda) = a(I(\Lambda),\Delta^+,\Lambda^{-1}) = 1. \tag{6.41}$$

Example 6.42 Principal series for $PGL(2,\mathbb{R}) = SO(2,1)$ are somewhat interesting. Let H be the split Cartan, and write $x = \mathrm{diag}(x,\frac{1}{x},1)\in SO(2,1)$. Take $\Delta^+ = \{\alpha\}$ where $\alpha(x)=x$, and drop it from the notation. We need the ρ-cover. Thus $H_\rho = \{(x,z)\,|\,z^2=x\}$, and

That is

$$\begin{aligned}\rho(x,\epsilon\sqrt{|x|}) &= \epsilon\sqrt{|x|} \quad (x>0)\\ \rho(x,i\epsilon\sqrt{|x|}) &= i\epsilon\sqrt{|x|} \quad (x<0).\end{aligned} \tag{6.43}$$

We compute

$$\tau(\Delta^+,s_\alpha)(x) = e^{\rho-s_\alpha\rho}(x)/|e^{\rho-s_\alpha\rho}(x)| = \mathrm{sgn}(x). \tag{6.44}$$

and

$$\Delta(x,z) = (1-\frac{1}{x})z. \tag{6.45}$$

Also note that $s_\alpha(x,z) = (\frac{1}{x},\frac{z}{x})$.

Let $\Lambda(x,z) = \mu(x)z$ where μ is a charater of $\mathbb{R}^\times$. This has differential $2d\mu+\rho$. If $\mu(x)=|x|^\nu$ or $|x|^\nu\mathrm{sgn}(x)$ then $d\Lambda = \nu+\frac{1}{2}$. In particular if μ is trivial then $\Lambda(x)=z$.

We use Proposition 6.6. For this we need $\langle\lambda,\alpha^\vee\rangle\ge 0$, i.e. $\nu\ge 0$. Then

$$
\begin{aligned}
\Theta_\pi(x) &= \Theta_\pi(x,z)\\
&= \frac{\mathrm{sgn}(1-\frac{1}{x})}{(1-\frac{1}{x})z}(\mu(x)z+\mu(\frac{1}{x})\frac{z}{x}\mathrm{sgn}(x))\\
&= \frac{1}{|1-\frac{1}{x}|}(\mu(x)+\mu(\frac{1}{x})|x|^{-1})\\
&= \frac{1}{|1-\frac{1}{x}||x|^{\frac12}}(\mu(x)|x|^{\frac12}+\mu(\frac{1}{x})|x|^{-\frac12})\\
&= \frac{|x|^{\nu+\frac12}+|x|^{-\nu-\frac12}}{||x|^{\frac12}-|x|^{-\frac12}|} \text{ or } \frac{|x|^{\nu+\frac12}\mathrm{sgn}(x)+|x|^{-\nu-\frac12}\mathrm{sgn}(x)}{||x|^{\frac12}-|x|^{-\frac12}|}
\end{aligned}
\tag{6.46}
$$

There is a dangerous bend here. By Definition 3.5 we have $\Lambda_M(g)=\Lambda(g)\otimes\gamma(-\Delta^+,g)$; note that we have to use $-\Delta^+$ here. This is a function on the $-\rho$-cover of H, while Λ is defined on the ρ-cover. With the obvious notation the isomorphism between the $-\rho$ and ρ covers is

$$(x,z)_\rho\to(x,z/x)_{-\rho}. \tag{6.47}$$

Note that $z^2=x$ implies $(z/x)^2=\frac{1}{x}$. Then $\gamma(x,u)_{-\rho}=u/|u|$, carried over to the ρ-cover gives $\gamma(x,z)_\rho=(z/x)/|z/x|=\mathrm{sgn}(x)z/|z|$. Therefore

$$\Lambda_M(x)=\mu(x)z\mathrm{sgn}(x)z/|z|=\mu(x)|x|^{\frac12}. \tag{6.48}$$

We are therfore getting the usual character formula for $\mathrm{Ind}_B^G(\Lambda_M)$, as desired.

In particular note that $\Lambda=e^\rho$, i.e. $\nu=\frac12$, $\Lambda(x,z)=z$, gives the spherical principal series, containing the trivial representation:

$$\frac{|x|+|x|^{-1}}{||x|^{\frac12}-|x|^{-\frac12}|} \tag{6.49}$$

In this case $H_+=\{x\,|\,1-\frac{1}{x}>0\}=\{x\,|\,x>1 \text{ or } x<0\}$ and

$$a(I(\Lambda),\Delta^+,\Lambda)=a(I(\Lambda),\Delta^+,\Lambda^{-1}\mathrm{sgn})=1. \tag{6.50}$$

7 Coherent Continuation

Fix H, Δ^+ and an element $\mathfrak{h}^*$ which is dominant for Δ^+, regular, and integral. We consider the space $\mathcal{M}(\lambda)$ of virtual characters with infinitesimal character

λ. The Weyl group $W = W(G(\mathbb{C}), H(\mathbb{C}))$ acts on $\mathcal{M}(\lambda)$ by the *coherent continuation* action [7, Definition 7.28]. That is, suppose $\pi \in \mathcal{M}(\lambda)$, and let Φ be a coherent family [7, Definition 7.25] such that $\Phi(\lambda) = \pi$. Then we define $w \cdot \pi = \Phi(w^{-1}\lambda)$.

Remark 7.1 There is an important subtle point here. This action of W depends on the choice of Δ^+. Define $w * \pi$ to be the coherent continuation action with respect to $y\Delta^+$ for some $y \in W$. Then

$$w * \pi = y^{-1}wy \cdot \pi \tag{7.2}$$

In the setting of [7] the coherent continuation action of the *abstract Weyl group* is independent of any such choice. For our explicit application it is better to use this version.

Proposition 7.3 *Suppose π is a virtual character with regular integral infinitesimal character. Fix H and Δ^+, and suppose the infinitesimal character of π is $\lambda \in \mathfrak{h}^*$ where λ is Δ^+-dominant. Let $\Lambda_1, \dots, \Lambda_r$ be the characters of H with $d\Lambda = \lambda$. Write θ_π restricted to H_+ as in* (5.3)*:*

$$\theta_\pi = \frac{\sum_{i=1}^r \sum_W a(\pi, \Delta^+, w \times \Lambda_i) w \times \Lambda_i}{D} \tag{7.4}$$

Define the coherent continuation action with respect to Δ^+. For $y \in W$ we have

$$\theta_{y\cdot\pi} = \frac{\sum_{i=1}^r \sum_W a(\pi, \Delta^+, w \times \Lambda_i) wy^{-1} \times \Lambda_i}{D} \tag{7.5}$$

A version of this result using (4.3) instead of (5.3) is slightly messier:

Proposition 7.6 *Let H, Δ^+, π and λ be as in the previous Proposition,*

$$\theta_\pi = \frac{\sum_{\Lambda \in \mathcal{P}_0(H)} a(\pi, \Delta^+, \Lambda)\Lambda}{D} \tag{7.7}$$

For each $\Lambda \in \mathcal{P}_0(H, \lambda)$ choose w_Λ so that $d(w_\Lambda^{-1}\Lambda) = \lambda$. Then

$$\theta_{y\cdot\pi} = \frac{\sum_{\Lambda \in \mathcal{P}_0(H,\lambda)} a(\pi, \Delta^+, \Lambda)(w_\Lambda y^{-1} w_\Lambda^{-1} \times \Lambda)}{D} \tag{7.8}$$

We state these results in terms of the coefficients $a(\pi, \Delta^+, \Lambda)$ (see (4.3)).

Corollary 7.9 *Assume $d\Lambda$ is Δ^+-dominant, and define coherent continuation with respect to Δ^+. Then for all $w, y \in W$,*

$$a(y \cdot \pi, \Delta^+, w \times \Lambda) = a(\pi, \Delta^+, wy \times \Lambda) \tag{7.10)(a}$$

or equivalently

$$a(\pi, \Delta^+, w \times \Lambda) = a(y \cdot \pi, \Delta^+, wy^{-1} \times \Lambda) \tag{7.10)(b}$$

Alternatively, using Proposition 7.6 instead of 7.3, or from (7.10)(b) we have:

Corollary 7.11 *Given $\Lambda \in \mathcal{P}_0(H, \lambda)$ choose w so that $w^{-1}d\Lambda = \lambda$. Then, with coherent continuation defined with respect to Δ^+, we have*

$$a(y \cdot \pi, \Delta^+, \Lambda) = a(\pi, \Delta^+, wyw^{-1} \times \Lambda) \tag{7.12)(a}$$

or equivalently

$$a(\pi, \Delta^+, \Lambda) = a(w^{-1}y^{-1}w \cdot \pi, \Delta^+, y \times \Lambda) \tag{7.12)(b}$$

8 Computation of $a(\pi, \Delta^+, \Lambda)$ for leading terms

Definition 8.1 *Suppose $H = TA$ is a θ-stable Cartan subgroup and Δ^+ is a set of positive roots satisfying (2.4). Let $\Phi^+ = \{\alpha|_{\mathfrak{a}} \,|\, \alpha \in \Delta^+\}$. Suppose $\lambda \in \mathfrak{h}^*$. We say $\lambda \in \mathfrak{h}^*$ is a leading term with respect to Δ^+ if*

$$\langle \lambda|_{\mathfrak{a}}, \alpha^\vee \rangle \geq \langle w\lambda|_{\mathfrak{a}}, \alpha^\vee \rangle \tag{8.2}$$

for all $w \in W$ and $\alpha \in \Phi^+$. We say a genuine character Λ of $\widetilde{H}$ is a leading term if this holds for $d\Lambda$.

The `atlas` software doesn't deal directly with restricted roots. I think the following is sufficient for our purposes.

Lemma 8.3 *Suppose H is a θ-stable Cartan subgroup, Δ^+ is a set of positive roots satisfying (2.4), and $\lambda \in \mathfrak{h}^*$ is integral. Then λ is a leading term with respect to Δ^+ if $w\lambda$ is dominant for Δ^+ for some $w \in W_i$.*

Sketch of proof. Clearly being leading is invariant under the action of W_i, so assume λ is dominant. We want to show

$$\langle \lambda - w\lambda|_{\mathfrak{a}}, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+. \tag{8.4}$$

Equivalently

$$\langle \lambda - w\lambda, \beta^\vee - \theta\beta^\vee \rangle \geq 0 \text{ for all } \beta \in \Delta^+. \tag{8.5}$$

For β real or complex β and $-\theta\beta$ are positive, so this holds since λ is dominant. It is obvious if β is imaginary. □

Recall (Lemma 6.8) $a(I(\Lambda), \Delta^+, \Lambda) = (-1)^{q_M}$ if Δ^+ satisfies (2.4) and $d\Lambda$ is dominant for Δ_i^+, Δ_r^+. By the theory of leading exponents [2, Section 3] no other standard module $I(H', \Lambda')$ contains the term Λ in its character expansion:

Proposition 8.6 *Fix H and $\Lambda \in \mathcal{P}_0(H)$. Suppose $wd\Lambda$ is Δ^+-dominant for some $w \in W_i$. Let $I(H', \Lambda')$ be any standard module. Then*

(8.7)

$$a(I(H', \Lambda'), \Delta^+, \Lambda) = \begin{cases} sgn(w)(-1)^{q_M} & (H', \Lambda') \text{ is conjugate to } (H, \Lambda) \\ 0 & \text{otherwise.} \end{cases}$$

Recall we write

$$\pi = \sum_{\Lambda \in \mathcal{P}(\lambda)} M(\Lambda, \pi) I(\Lambda) \tag{8.8}$$

using the Kazhdan-Lusztig algorithm.

Corollary 8.9 *Assume Δ^+ satisfies* (2.4) *and $wd\Lambda$ is dominant for some Δ_i^+ and Δ_r^+ for $w \in W_i$. Then*

$$a(\pi, \Delta^+, \Lambda) = sgn(w)(-1)^{q_M} M(\Lambda, \pi) \tag{8.10}$$

We obtain the general coefficient $a(\pi, \Delta^+, \Lambda)$ from these via coherent continuation.

9 Computation of general $a(\pi, \Delta^+, \Lambda)$

We now apply this to our computation of $a(\pi, \Delta^+, \Lambda)$ for any Λ. If $d\Lambda$ is dominant for Δ^+ then $a(\pi, \Delta^+, \Lambda)$ is given by Corollary regc:worst. Otherwise we use this together with Corollary 9.8.

Proposition 9.1 *Fix H, Δ^+ satisfying* (2.4)*, and π with regular integral infinitesimal character $\lambda \in \mathfrak{h}^*$, chosen to be dominant for Δ^+. Suppose $\Lambda \in \mathcal{P}_0(H)$ and $d\Lambda \in W\lambda$.*

Define the coherent continuation action with respect to Δ^+. Choose $w \in W$ so that $w\lambda = d\Lambda$. Then

$$\begin{aligned} a(\pi, \Delta^+, \Lambda) &= a(w \cdot \pi, \Delta^+, w^{-1} \times \Lambda) \\ &= (-1)^{q_M} M(w^{-1} \times \Lambda, w \cdot \pi) \end{aligned} \tag{9.2}$$

Proof. Write

$$a(\pi, \Delta^+, \Lambda) = a(\pi, \Delta^+, w \times (w^{-1} \times \Lambda)). \tag{9.3}$$

By (7.10)(b) for all $y \in W$ the right hand side equals

$$a(y \cdot \pi, \Delta^+, wy^{-1} \times (w^{-1} \times \Lambda). \tag{9.4}$$

Choose $y = w$ to conclude

$$a(\pi, \Delta^+, \Lambda) = a(w \cdot \pi, \Delta^+, w^{-1} \times \Lambda). \tag{9.5}$$

By Corollary 8.9 this equals $(-1)^{q_M} M(w^{-1} \times \Lambda, w \cdot \pi)$. □

Corollary 9.6 *Suppose Δ^+ satisfies* (2.4) *and $d\Lambda$ is dominant for Δ^+. Define the coherent continuation action with respect to Δ^+. Suppose π has infinitesimal character $d\Lambda$. Then for all $w \in W$*

$$a(\pi, \Delta^+, w \times \Lambda) = (-1)^{q_M} M(\Lambda, w \cdot \pi). \tag{9.7}$$

The following version is the one we will use in Section 10.

Corollary 9.8 *Let H, Δ^+, π and λ be as in the Proposition. Write $\Lambda_1, \dots, \Lambda_r$ for elements of $\mathcal{P}_0(H)$ satisfying $d\Lambda_i = \lambda$. For any $w \in W$:*

$$a(\pi, \Delta^+, w \times \Lambda_i) = (-1)^{q_M} M(\Lambda_i, w \cdot \pi). \tag{9.9}$$

Here is another proof of Proposition 6.6.

Proof. Given Λ, assume Δ^+ satisfies (2.4) and $d\Lambda$ is dominant for Δ^+_i, Δ^+_r. For $w \in W(G,H)$ we show $a(I(\Lambda), \Delta^+, \tau(\Delta^+, w)(w\Lambda)) = \mathrm{sgn}_i(w)(1)^{q_M}$, i.e. (6.7)(c).

Suppose $d\Gamma = d\Lambda$. By the corollary

$$a(I(\Lambda), \Delta^+, w \times \Gamma) = (-1)^{q_M} M(\Gamma, w \cdot I(\Lambda)). \tag{9.10}$$

We need:

Lemma 9.11 $w \cdot I(\Lambda) = sgn(w_i) I((w \times \Lambda)\tau(\Delta^+, w)) + \dots$

where the other terms all come from other (less compact) Cartan subgroups.

Note that $I((w \times \Lambda)\tau(\Delta^+, w)) = I(w^{-1}(w \times \Lambda)\tau(\Delta^+, w))$

Take $\Gamma = w^{-1}(w \times \Lambda)\tau(\Delta^+, \Lambda)$. Then $M(\Gamma, w \cdot I(\Lambda)) = 1$, and therefore

$$a(I(\Lambda), \Delta^+, w \times w^{-1}(w \times \Lambda)\tau(\Delta^+, w)) = (-1)^{q_M} \mathrm{sgn}_i(w). \tag{9.12}$$

The result follows from the fact that

$$w \times (w^{-1}(w \times \Lambda)) = w\Lambda. \tag{9.13}$$

□

Remark 9.14 I think Proposition 4.2 follows from this.

10 Computing characters using the atlas software

The calculations in the setting of the atlas software are fairly straightforward. The hard part is passing back and forth between classical language and the setting of the atlas software. In this regard the Dictionary of Representations will be very useful.

We are given *basic data* (G, γ), consisting of a complex group and an involution in $\mathrm{Out}(G)$. This determines and inner class of real forms of G. We are given a (complex) Cartan subgroup H and set of positive roots Δ^+, which are fixed once and for all.

Suppose (x, y) is a parameter. This means that x is an H-conjugacy class of strong involutions of G^Γ If ξ is a strong involution in this conjugacy class we can talk about $(\mathfrak{g}, K_\xi)$-modules.

We work in the setting of a block of parameters, in the sense of atlas. The `block` command lists these parameters, together with some extra information. We fix an infinitesimal character for G associated to this block/ The parameters are labelled as $0, \dots, n$. Suppose parameter i is (x, y). Associated to x is a well defined involution θ_x of H (although not of G), and associated to i is a one-dimensional $(\mathfrak{h}, H_\rho^{\theta_x})$ module $\Lambda(x, y)$. If ξ is a strong involution mapping to x then associated to $\Lambda(x, y)$ is a standard $(\mathfrak{g}, K_\xi)$-module $I(i)$, and and irreducible module $\pi(i)$. See Annegret's Paul's notes for more information.

Now fix a conjugacy class of twisted involutions in the Weyl group. This corresponds to a choice of conjugacy class of real Cartan subgroup $H(\mathbb{R})$ in the quasisplit form of G, as well as some other real forms. Fix a representative τ of this conjugacy class.

Suppose $0 \le i \le n$ is a parameter. This maps to a twisted involution given by the last column in the output of `block`. It isn't elementary to see when two such twisted involutions are conjugate, but two parameters i, j lie map to the same conjugacy class of twisted involution if and only if you can get from i to j by a series of cross actions.

Given τ, we need to choose a parameter x, mapping to the conjugacy class of τ, such that Δ^+ satisfies (2.4) with respect to θ_x. Here is how to do this:

Lemma 10.1 *Fix a parameter (x, y). If there is a simple root of type C^+, apply a cross action in this root. Continue until there are no simple roots of type C^+. This gives a parameter (x', y'), mapping to the same conjugacy class of twisted involutions, such that Δ^+ satisfies* (2.4) *with respect to $\theta_{x'}$.*

Given this parameter (x, y), let $H(\mathbb{R})$ be the real group corresonding to (H, θ_x), with two-fold cover $H(\mathbb{R})_\rho$. Let S be the set of parameters (x, y') (with the same x) in the block. There correspond to characters Λ of $H(\mathbb{R})_\rho$, all with the same differential. Recall $\lambda = d\Lambda$ is automatically dominant for Δ^+.

Now all characters of $H(\mathbb{R})_\rho$ occuring for this block are of the form $w \times \Lambda(i)$ for some $w \in W, i \in S$.

The analogue of Corollary 9.8 in this setting is:

Lemma 10.2 *For all $0 \le j \le n$ and $i \in S$ we have:*

$$a(I(j), \Delta^+, w \times \Lambda(i)) = (-1)^{q_M} M(\Lambda(i), w \cdot I(j)). \tag{10.3}$$

.

This reduces us to the computation of $M(\Lambda(i), w \cdot I(j))$ for all $i \in S$ and w. We now sketch how to compute this. We first define some integers coming from atlas output.

For irreducible representations $\pi(i)$ we have

$$w \cdot \pi(i) = \sum_{j=0}^{n} b(w,i,j)\pi(j) \tag{10.4)(a}$$

where the integers $b(w,i,j)$ may be computed from the output of the `wgraph` command. The helper application `coherentContinuation` gives these integers.

On the other hand for standard modules we have

$$w \cdot I(i) = \sum_{j=0}^{n} c(w,i,j)I(j) \tag{10.4)(b}$$

and $c(w,i,j)$ is obtained from the `block` command. The helper application `coherentContinuation` also gives these integers.

The Kazhdan-Lusztig character identities are

$$\pi(i) = \sum_{j=0}^{n} M(i,j)I(j) \tag{10.4)(c}$$

and $M(i,j)$ is obtained from the output of `klbasis`. The `klbasis` command outputs integers $P(i,j)$, and $M(i,j) = (-1)^{\ell(i)-\ell(j)}P(i,j)$. The helper application `kl` gives the integers $M(i,j)$, with the correct signs. These are the integers $M(\Gamma(i), \pi)$ used earlier.

Finally the inverse identities are

$$I(i) = \sum_{j=0}^{n} m(i,j)\pi(j) \tag{10.4)(d}$$

These are obtained by taking the inverse of the $M(i,j)$ matrix. This is given by `kl` (for small rank; for large rank it is better to use mathematica or another similar program to compute the inverse).

We return to our computation of $M(\Gamma(j), w\cdot\pi)$. How we proceed will depend on π. If $\pi = I(j)$ is a standard module things then we simply have

$$M(\Gamma(j), w\times\pi(i)) = c(w,i,j). \tag{10.5}$$

On the other hand suppose $\pi = \pi(i)$ is an irreducible representation. From (10.4)(a) and then (d) we conclude:

$$M(\Gamma(j), w\cdot\pi(i)) = \sum_k b(w,i,k)M(k,j) \tag{10.6}$$

Alternatively we could apply (10.4) (e) and then (c) to conclude

$$M(\Gamma(j), w\cdot\pi(i)) = \sum_k M(i,k)c(w,k,j) \tag{10.7}$$

In any event we summarize this discussion and give the main result:

Proposition 10.8 *Fix a block, with parameters* $,0,\dots,n$*, and fix a twisted involution* τ*. Fix a parameter* (x,y) *with* x *lying over the conjugacy class of* τ *in the fiber over* τ*, and let* S *be the set of parameters* (x,y') *for which no simple roots are of type* C^+*. After a series of cross actions applied to* (x,y) *we may assume assume* S *is non-empty.*

For $i\in S_\tau$ *let* $\Lambda(i)$ *be the corresponding character. Let* $I(i)$ *and* $\pi(i)$ *be the corresponding standard and irreducible modules.*

Let π *be any representation in the block. Then for all* $w\in W, i\in S_\tau$*,*

$$a(\pi,\Delta^+, w\times\Lambda(i)) = M(I(i), w\cdot\pi). \tag{10.9}$$

The integer $M(I(i), w\cdot\pi)$ *is the multiplicity of* $I(i)$ *in the expression of* $w\cdot\pi$ *as a sum of standard modules. It may be computed using the output of the* `block`*,* `wgraph` *and* `klbasis` *commands.*

The question about how to relate this to actual characters of a given real group G is subtle; see Annegret Paul's talk on the Dictionary. Roughly speaking here is the situation.

Fix your own personal real group $G(\mathbb{R})$, a θ-stable Cartan subgroup $H(\mathbb{R})$ of it, and a choice of positive roots Δ_0^+. Suppose you are interested in computing $a(\pi,\Delta^+,\Lambda)$ where Λ is a genuine one-dimensional $H(\mathbb{R})_\rho$-module and π is an admissible representation with our given infinitesimal character character.

11 Examples

All of the data from this section is available from the `block`, `wgraph` and `klbasis` commands of `atlas`. The help applications `coherentContinuation` and `kl` (`www.liegroups.org/software/helper`) are quite useful, but not essential (these applications do calculations based only on the output of the aforementioned atlas commands).

Throughout this section we use some fairly standard notation and conventions, which shoulbe be self-explanatory. The translation to actual representations of real groups is fraught with peril; we won't try to be precise here.

The output of `block` is a list of parameters numbered $0, \dots, n$. Each parameter i is a pair (x_i, y_i).

11.1 The Trivial Representation

Fix any Cartan subgroup H, a positive system Δ^+, and let $\Lambda = e^\rho$, the distinguished character of the ρ-cover of H. Let $\pi = \mathbb{C}$, the trivial representation. Assume Δ^+ satisfies (2.4). Then by Corollary 9.6

$$a(\mathbb{C}, \Delta^+, w \times \Lambda) = (-1)^{q_M} M(\Lambda, w \cdot \mathbb{C}). \tag{11.1}$$

It is easy to see that $w \cdot \mathbb{C} = \mathrm{sgn}(w)\mathbb{C}$ for all $w \in W$, so

$$a(\mathbb{C}, \Delta^+, w \times \Lambda) = (-1)^{q_M} \mathrm{sgn}(w) M(\Lambda, \mathbb{C}). \tag{11.2}$$

Recall (4.7) $a(\mathbb{C}, \Delta^+, w\Lambda) = \mathrm{sgn}(w)$. It is easy $w \times \Lambda = w\Lambda$ for all w (Λ is a holomorphic character of $H(\mathbb{C})$, restricted to H). This proves:

$$M(w\Lambda, \mathbb{C}) = (-1)^{q_M} \quad \text{for all } w \in W. \tag{11.3}$$

Zuckerman's Theorem [7, Proposition 2.2.2] gives a formula for $M(\Lambda, \mathbb{C})$ in different coordinates. After some ρ-shifts it says

$$M(\Lambda, \mathbb{C}) = (-1)^{\ell_0 - \ell(\Lambda)} \tag{11.4}$$

where $\ell(\Lambda)$ is the length of the parameter Λ, and ℓ_0 is the length of the trivial parameter.

Therefore either

$$(-1)^{q_M} = (-1)^{\ell_0 - \ell(\Lambda)} \tag{11.5}$$

and we have re-proved Zuckerman's theorem, or I have a sign error somewhere.

11.2 $SL(2,\mathbb{R})$ and $PGL(2,\mathbb{R})$

We work on the block of the trivial representation. Here is the output of the `block` command of `atlas` (the last two columns have been added):

```
0(0,1):  1    (2,*)    [i1]  0      T   discrete series
1(1,1):  0    (2,*)    [i1]  0      T   discrete series
2(2,0):  2    (*,*)    [r1]  1  1   A   trivial representation
```

We follow Proposition 10.8. There are two Cartan subgroups and three twisted involutions. Here are the Cartan subgroups, with choices of twisted involution τ, x and S as in Proposition 10.8.

H	τ	x	S
T	id	0	0
A	s_α	2	2

Here is a table of the coherent continuation action on standard modules. This is easy to read off from the output of `block`, or use `coherentConinuation`. The third column gives $w \cdot \pi$ as a virtual of standard modules.

π	w	$w \cdot \pi$
$I(0)$	id	0
$I(0)$	1	-1+2
$I(1)$	id	1
$I(1)$	1	-0+2
$I(2)$	id	2
$I(2)$	1	2

From this we can compute the character of each standard representation $I(j)$. Here is the discrete series representation $I(0)$:

π	H	w	$w \cdot \pi$	j	$a(\pi, \Delta^+, w \times \Lambda(j) = (-1)^{q_M} M(\Lambda(j), w \cdot \pi)$
$I(0)$	T	id	0	0	$-1 \cdot 1 = -1$
$I(0)$	T	1	-1+2	0	$-1 \cdot 0 = 0$
$I(0)$	H	id	0	2	$1 \cdot 0 = 0$
$I(0)$	H	1	-1+2	2	$1 \cdot 1 = 1$

With some "obvious" choices $\Lambda(0) = e^{i\theta}$ and $\Lambda(2) = x$ (see the remark below). Note that $s_\alpha\Lambda(2) = x^{-1}$. So on T this is just Harish-Chandra's character formula

(11.6) $$\theta_{\pi(0)} = \frac{-e^{i\theta}}{e^{i\theta} - e^{-i\theta}}$$

and on $A_+ = \{x \mid |x| > 1\}$ this is also well known:

(11.7) $$\theta_{\pi(0)} = \frac{x^{-1}}{x - x^{-1}}$$

Remark 11.8 *The choices aren't so obvious; you could have chosen* $\Lambda(0) = e^{-i\theta}$ *and* $\Lambda(x) = \frac{1}{x}$*. Note that there is no intrinsic way to specify a discrete series representation of* $SL(2, \mathbb{R})$ *to call holomorphic; the two discrete series are interchanged by an outer automorphism of* $SL(2, \mathbb{R})$*. This has to do with the translation from the* `atlas` *language, as discussed in the Dictionary.*

The other discrete series representation $I(1)$ is similar (and the same on A). Here is the spherical principal series representation $I(2)$:

π	H	w	$w\cdot\pi$	j	$a(\pi,\Delta^+,w\times\Lambda(j)=(-1)^{q_M}M(\Lambda(j),w\cdot\pi)$
$I(2)$	T	id	2	0	$(-1)\cdot 0=0$
$I(2)$	T	1	2	0	$(-1)\cdot 0=0$
$I(2)$	A	id	2	2	$1\cdot 1=1$
$I(2)$	A	1	2	2	$1\cdot 1=1$

π	H	$u^{-1}w$	$u^{-1}w\cdot\pi$	j	$i=u\times j$	w	$(-1)^qM(j,u^{-1}w\cdot\pi)(w\times\Gamma(i))$
$I(2)$	T	id	2	0	0	id	
$I(2)$	T	1	2	0	0	1	
$I(2)$	A	id	2	2	2	id	$1\cdot 1\cdot x=x$
$I(2)$	A	1	2	2	2	1	$1\cdot 1\cdot s_1(x)=x^{-1}$

Of course this is just the induced character formula: $\theta_{I(2)}$ vanishes on T, and on A_+ we have

(11.9) $$\theta_{I(2)}=\frac{x+x^{-1}}{x-x^{-1}}\quad(|x|>1).$$

It is worthwhile repeating part of the computation in the case of $PGL(2,\mathbb{R})$. Here is the output of `block`:

```
0(0,2):  0    (1,2)    [i2]  0      T  Discrete series
1(1,0):  2    (*,*)    [r2]  1   1  A  Principal series
2(1,1):  1    (*,*)    [r2]  1   1  A  Principal seris
```

In this case we have:

H	τ	x	S
T	id	0	0
A	1	1	1,2

Here is the coherent continuation action:

$I(j)$	w	$w \cdot I(j)$
0	id	0
0	1	-0 1 2
1	id	1
1	1	2
2	id	2
2	1	1

Here is the character of the discrete series representation $I(0)$:

$I(j)$	H	w	$w \cdot I(j)$	j	$a(\pi, \Delta^+, w \times \Lambda(j) = (-1)^{q_M} M(\Lambda(j), w \cdot \pi)$
0	T	id	0	0	$-1 \cdot 1 = -1$
0	T	1	-0+1+2	0	$-1 \cdot (-1) = 1$
0	A	id	0	1,2	$1 \cdot 0 = 0$ $(j = 1, 2)$
0	A	1	-0+1+2	1,2	$1 \cdot 1 = 1$ $(j = 1, 2)$

With the usual notation $\Lambda(0) = e^{i\theta/2}$; this is a genuine character of the ρ-cover of T. Then

$$\Theta_{I(0)} = \frac{-e^{i\theta/2} + e^{-i\theta/2}}{(1 - e^{-i\theta})e^{i\theta/2}} = -1 \tag{11.10}$$

(This is true: the character of the discrete series representation of $SO(2,1)$ is equal to the negative of the character of the trivial representation on T.)

Note that $A_+ = \{x \mid 1 - \frac{1}{x} > 0\}$, i.e. $x > 1$ or $x < 0$. Then on A_+ we have

$$\Theta_{I(0)} = \frac{\Lambda(1) + \Lambda(2)}{D}. \tag{11.11}$$

Here is a slightly dangerous bend. With notation as in Example 6.42 we have $\Lambda(1)(x,z) = z$ and $\Lambda(2)(x,z) = \text{sgn}(x)z$, *or vice versa*. For this computation it won't matter which is which. Taking $s_\alpha\Lambda(i)$ $(i = 1, 2)$ we get the two characters $\Lambda(x,z) = z^{-1}$ and $\Lambda(x,z) = \text{sgn}(x)z^{-1}$. On A_+ we then have

$$\begin{aligned} \Theta_{I(0)}(x,z) &= \frac{\frac{1}{z} + \text{sgn}(x)\frac{1}{z}}{(1 - \frac{1}{x})z} \\ &= \frac{|x|^{-\frac{1}{2}} + |x|^{-\frac{1}{2}}\text{sgn}(x)}{(1 - \frac{1}{x})|x|^{\frac{1}{2}}} \end{aligned} \tag{11.12}$$

Note that $\Theta_{I(0)}(x) = 0$ for $x < 0$; the usual way to write this character, which requires more cases, is

$$\Theta_{I(0)}(x) = \begin{cases} \frac{2|x|^{-\frac{1}{2}}}{(1-\frac{1}{x})|x|^{\frac{1}{2}}} & x > 1 \\ 0 & x < 0 \end{cases} \tag{11.13}$$

Here is the character formula for $I(1)$:

$I(j)$	H	w	$w \cdot I(j)$	j	$a(\pi, \Delta^+, w \times \Lambda(j) = (-1)^{q_M} M(\Lambda(j), w \cdot \pi)$
1	T	id	1	0	$-1 \cdot 0 = 0$
1	T	1	2	0	$-1 \cdot 0 = 0$
1	A	id	1	1	$1 \cdot 1 = 1$
1	A	id	1	2	$1 \cdot 0 = 0$
1	A	1	2	1	$1 \cdot 0 = 0$
1	A	1	2	2	$1 \cdot 1 = 1$

Here is the character formula for $I(1)$ in more familiar terms. Here we have a slightly more dangerous bend. Suppose we decree that

$$\begin{aligned} \Lambda(1)(x,z) &= z \\ \Lambda(2)(x,z) &= \operatorname{sgn}(x)z \end{aligned} \tag{11.14}$$

Then $\Lambda(2)(x,z) = \operatorname{sgn}(x)z$. Note that $\Lambda(1)$ and $s_\alpha\Lambda(2)$ occur in the character formula. Note that $s_\alpha\Lambda(2)(x,z) = x\operatorname{sgn}(x)/z$. A short calculation gives

$$\Theta_{I(1)}(x) = \frac{|x|^{\frac{1}{2}} + |x|^{-\frac{1}{2}}\operatorname{sgn}(x)}{(1-\frac{1}{x})|x|^{\frac{1}{2}}} \quad (x \in A_+). \tag{11.15}$$

This is the character of the principal series representation containing the trivial representation. See Example 6.42.

However we could also have switched parameters 1 and 2: from the output of `block`, you can't tell which one is the trivial representation, and which is the *sgn*. (This is dual to the fact that we can't tell canonically tell which discrete series representation to call holomorphic, and which to call anti-holomorphic.) So, suppose we take:

$$\begin{aligned} \Lambda(2)(x,z) &= z \\ \Lambda(1)(x,z) &= \operatorname{sgn}(x)z \end{aligned} \tag{11.16}$$

Then we get

$$\Theta_{I(1)}(x) = \frac{|x|^{\frac{1}{2}}\mathrm{sgn}(x) + |x|^{-\frac{1}{2}}}{(1-\frac{1}{x})|x|^{\frac{1}{2}}} \quad (x \in A_+). \tag{11.17}$$

This is the principal series representation containing sgn.

For completeness here is the character formula for $I(2)$:

$I(j)$	H	w	$w \cdot I(j)$	j	$a(\pi, \Delta^+, w \times \Lambda(j) = (-1)^{q_M} M(\Lambda(j), w \cdot \pi)$
2	T	id	2	0	$-1 \cdot 0 = 0$
2	T	1	1	0	$-1 \cdot 0 = 0$
2	A	id	2	1	$1 \cdot 0 = 0$
2	A	id	2	2	$1 \cdot 1 = 1$
2	A	1	1	1	$1 \cdot 1 = 1$
2	A	1	1	2	$1 \cdot 0 = 0$

11.3 $Sp(4,\mathbb{R})$

We work on the block of the trivial representation of $Sp(4,\mathbb{R})$:

```
 0( 0,6): 1  2 ( 6, *)  ( 4, *) [i1,i1] 0      T      large discrete series
 1( 1,6): 0  3 ( 6, *)  ( 5, *) [i1,i1] 0      T      large discrete series
 2( 2,6): 2  0 ( *, *)  ( 4, *) [ic,i1] 0      T      discrete series
 3( 3,6): 3  1 ( *, *)  ( 5, *) [ic,i1] 0      T 1    discrete series
 4( 4,4): 8  4 ( *, *)  ( *, *) [C+,r1] 1 2    S^1R^* A_q(lambda)
 5( 5,4): 9  5 ( *, *)  ( *, *) [C+,r1] 1 2    S^1R^* A_q(lambda)
 6( 6,5): 6  7 ( *, *)  ( *, *) [r1,C+] 1 1    C*     A_q(lambda)
 7( 7,2): 7  6 (10,11)  ( *, *) [i2,C-] 2 212  C*
 8( 8,3): 4  9 ( *, *)  (10, *) [C-,i1] 2 121  S^1R^*
 9( 9,3): 5  8 ( *, *)  (10, *) [C-,i1] 2 121  S^1R^*
10(10,0): 11 10( *, *)  ( *, *) [r2,r1] 3 1212 A      trivial representation
11(10,1): 10 11 ( *, *) ( *, *) [r2,rn] 3 1212 A      PS
```

There are 4 conjugacy classes of twisted involutions, corresponding to the four conjugacy classes of Cartan subgroups of $Sp(4,\mathbb{R})$ appearing in the table above.

Here are choices of H, τ, x and S:

H	τ	x	S
T	id	0	0
$S^1\mathbb{R}^\times$	121	8	8
$\mathbb{C}^\times$	212	7	7
A	1212	10	10,11

Let's compute the character of the large discrete series representation $I(0)=\pi(0)$. Here is the coherent continuation action on $\pi(0)=I(0)$, computed using `block`. The third column indicates a sum of standard modules.

π	w	$w \cdot \pi$
$I(0)$	id	0
$I(0)$	1	-1+6
$I(0)$	2	-2+4
$I(0)$	12	2+8
$I(0)$	21	3-5+7
$I(0)$	121	-3 -7 -9 10 11
$I(0)$	212	-0 4 -9 10
$I(0)$	1212	1 -5 -6 8 11

Now we compute $\theta_{\pi(0)}$ on T.

π	H	w	$w \cdot \pi$	j	$a(\pi, \Delta^+, w \times \Lambda(j) = (-1)^{q_M} M(\Lambda(j), w \cdot \pi)$
$I(0)$	T	id	0	0	$-1 \cdot 1 = -1$
$I(0)$	T	1	-0+4-9+10	0	$-1(-1) = 1$

Let $\Lambda = \Lambda(0)$. Then

$$\theta_{\pi(0)} = \frac{\Lambda + (s_1 s_2 s_1 \times \Lambda)}{D(\Delta^+)}. \tag{11.18}$$

Note that $s_1 s_2 s_1$ is reflection in the non-simple short root. For the large discrete series representation this is the compact root. (In the usual coordinates $\Lambda = (2, -1)$ and $s_1 s_2 s_1 \Lambda = (-1, 2)$).

Next we compute $\pi(0)$ on A. Here $S = \{10, 11\}$.

π	H	w	$w \cdot \pi$	j	$a(\pi, \Delta^+, w \times \Lambda(j) = (-1)^{q_M} M(\Lambda(j), w \cdot \pi)$
$I(0)$	A	121	-3 -7 -9 10 11	10	$1 \cdot 1 = 1$
$I(0)$	A	121	-3 -7 -9 10 11	11	$1 \cdot 1 = 1$
$I(0)$	A	212	-0 4 -9 10	10	$1 \cdot 1 = 1$
$I(0)$	A	212	-0 4 -9 10	11	$1 \cdot 0 = 0$
$I(0)$	A	1212	1 -5 -6 8 11	10	$1 \cdot 0 = 0$
$I(0)$	A	1212	1 -5 -6 8 11	11	$1 \cdot 1 = 1$

Write $A = \{(x, y) = \mathrm{diag}(x, y, 1/x, 1/y)\}$, and choose Δ^+ so that $e^\rho(x, y) = x^2 y$. Write (a, b) for the character $(x, y) = x^a y^b$. Then $e^\rho = (2, 1)$, and this is $\Lambda(10)$. We also have

$$\begin{aligned} s_1 s_2 s_1 \times (2, 1) &= (-2, 1) \\ s_2 s_1 s_2 \times (2, 1) &= (-1, -2) \\ s_1 s_2 s_1 s_2 \times (2, 1) &= (-2, -1). \end{aligned} \tag{11.19}$$

Also $\Lambda(11)(x, y) = x^2 y \mathrm{sgn}(xy)$, and the preceding formulas hold, with $\Lambda(11)$ on the left hand side, and an additional $\mathrm{sgn}(xy)$ term on the right.

With these coordinates we have on A_+:

$$D\theta_{\pi(0)}(x, y) = x^{-2} y + \mathrm{sgn}(xy) x^{-2} y + x^{-1} y^{-2} + \mathrm{sgn}(xy) x^{-2} y^{-1} \tag{11.20}$$

A more familiar way to write this is as follows. Let $t = (\epsilon_1, \epsilon_2) = \mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2) \in A$ with $\epsilon_1, \epsilon_2 = \pm 1$. Suppose $X \in \mathfrak{a}_0$ satisfies $\alpha(X) > 0$ for all $\alpha \in \Delta^+$. Then if $\epsilon_1 \epsilon_2 = 1$ we have

$$D\theta_{\pi(0)}(t \exp(X)) = 2e^{(-2,1)(X)} + e^{(-1,-2)(X)} + e^{(-2,-1)(X)}. \tag{11.21}$$

and if $\epsilon_1 \epsilon_2 = -1$ we have

$$D\theta_{\pi(0)}(t \exp(X)) = e^{(-1,-2)(X)} - e^{(-2,-1)(X)} \tag{11.22}$$

These agree with [3, page 249]. Note that the single formula (11.20) holds on all of A_+.

Finally we compute $\theta_{\pi(0)}$ on the intermediate Cartans.

π	H	w	$w \cdot \pi$	j	$a(\pi, \Delta^+, w \times \Lambda(j) = (-1)^{q_M} M(\Lambda(j), w \cdot \pi)$
$I(0)$	$S^1\mathbb{R}^\times$	12	2 8	8	$-1 \cdot 1 = -1$
$I(0)$	$S^1\mathbb{R}^\times$	1212	1 -5 -6 8 11	8	$-1 \cdot 1 = -1$
$I(0)$	$\mathbb{C}^\times$	21	3 -5 7	7	$-1 \cdot 1 = -1$
$I(0)$	$\mathbb{C}^\times$	121	-3 -7 -9 10 11	7	$-1 \cdot -1 = 1$

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