How to attach a representation to a pair (x,y)
Atlas of Lie Groups Workshop

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1 Setup

Let (G,τ) be given. Here G is an algebraic group and τ an involution in $Out(G)$ determining an inner class of real forms. Fix a Cartan subgroup and a Borel $H \subseteq B \subseteq G$ and a set of root vectors $\{X_\alpha\}$. This determines a unique distinguished involution γ of G mapping to τ and fixing the pinning $(H, B, \{X_\alpha\})$. Form the extended group $G^\Gamma = G \rtimes \Gamma$, where $\Gamma = \{1, \sigma\}$ is the Galois group of $\mathbb{C}$ over $\mathbb{R}$, and σ acts on G by the involution γ. Write $\delta = 1 \times \sigma$, so $G^\Gamma = G \sqcup G\delta$. Let $(G^\vee, \tau^\vee)$ be the dual group and the involution dual to τ, and choose the corresponding objects $\delta^\vee$, $H^\vee$, $B^\vee$, and $(G^\vee)^\Gamma$. We have also made various identifications $\mathfrak{h}^* \simeq \mathfrak{h}^\vee$, $X^*(H) = X_*(H^\vee)$, etc.

2 What are the Pairs (x,y)?

Translation families of irreducible admissible representations with regular integral infinitesimal character of real forms of G in the given inner class are parametrized by certain sextuples of data (integral L-data). They are defined up to conjugation by $G \times G^\vee$. By fixing (H, B, $H^\vee$, $B^\vee$), these sets of data can be reduced to pairs (x, y), which are now $H \times H^\vee$ conjugacy classes of things.

Definition 1 *A strong involution for (G,τ) is an element $\xi \in G^\Gamma \backslash G = G\delta$ such that $\xi^2 \in Z(G)$. Two strong involutions are equivalent if they are conjugate by an element of G. If ξ is a strong involution then we let $\theta_\xi = int(\xi)$, and $K_\xi = Stab_G(\xi)$.*

Recall that a strong involution defines a Cartan involution of G, and hence a real form $G(\mathbb{R})$, with complexified maximal compact subgroup K_ξ.

Definition 2 *The set $\widetilde{\mathcal{X}}$ is the set of strong involutions of G normalizing H. The group H acts on this set by conjugation. The set $\mathcal{X}$ is the set of H conjugacy classes in $\widetilde{\mathcal{X}}$.*

$$\widetilde{\mathcal{X}} = \{\xi \text{ strong involution of } G : \theta_\xi(H) = H\} \tag{1}$$
$$\mathcal{X} = \widetilde{\mathcal{X}}/H. \tag{2}$$

Clearly
$$\widetilde{\mathcal{X}} \subseteq N^\Gamma = Norm_{G^\Gamma}(H). \tag{3}$$
Let
$$W^\Gamma = N^\Gamma/H. \tag{4}$$
Then the Weyl group $W \subseteq W^\Gamma$, and it acts on $\widetilde{\mathcal{X}}$, and $\mathcal{X}$ by conjugation.

The canonical map
$$\widetilde{p} : \widetilde{\mathcal{X}} \longrightarrow W^\Gamma \tag{5}$$
descends to a map
$$p : \mathcal{X} \longrightarrow W^\Gamma, \tag{6}$$
so that we have a well-defined action of $\mathcal{X}$ on H: if $x \in \mathcal{X}$ and $h \in H$ we can define
$$\theta_x(h) = \theta_\xi(h), \tag{7}$$
where ξ is any strong involution mapping to x. The image of p is the set of twisted involutions in W^Γ
$$\mathcal{I}_W = \left\{\tau \in W^\Gamma \backslash W : \tau^2 = 1\right\}. \tag{8}$$
If $\tau \in \mathcal{I}_W$, we denote the inverse image of τ in $\mathcal{X}$ by $\mathcal{X}_\tau$.

Since we have fixed H and B, there is a fixed choice of positive roots Ψ^+. An element $x \in \mathcal{X}$ determines a real form of G, a real form of H, and sets of real, imaginary and complex roots $\Psi_{R,x}$, $\Psi_{im,x}$, $\Psi_{cx,x}$ (these depend on the twisted involution $\tau = p(x)$ only), as well as a grading of the imaginary roots according to compact/noncompact.

In addition to the partition of $\mathcal{X}$ into fibers $\mathcal{X}_\tau$, we also have a partition according to equivalence classes of strong real forms. For $x_0 \in \mathcal{X}$ and $\tau \in \mathcal{I}_W$, we define

$$\mathcal{X}[x_0] = \{x \in \mathcal{X} : x \text{ is conjugate to } x_0 \text{ by } G\}, \text{ and} \tag{9}$$
$$\mathcal{X}_\tau[x_0] = \mathcal{X}[x_0] \cap \mathcal{X}_\tau. \tag{10}$$

Let $\mathcal{Y}$ be the corresponding space for $G^\vee$. The Weyl group of $G^\vee$ may be identified with W via $s_\alpha \longleftrightarrow s_{\alpha^\vee}$.

Definition 3 *The two-sided parameter space $\mathcal{Z}$ is defined to be*

$$\mathcal{Z} = \left\{(x,y) \in \mathcal{X} \times \mathcal{Y} : \theta_y = -(\theta_x)^t \text{ as involutions of } H \text{ and } H^\vee\right\}. \tag{11}$$

This means we are pairing fibers $\mathcal{X}_\tau \times \mathcal{Y}_{\tau^\vee}$, where τ and $\tau^\vee$ are related by $\theta_y = -(\theta_x)^t$.

The set $\mathcal{Z}$ parametrizes the set of translation families of irreducible admissible representations with regular integral infinitesimal character of real forms of G in the given inner class.

3 Cross Actions, Cayley Transforms and Blocks

The Weyl group W acts on each $\mathcal{X}[x_0]$, with W_{im} (the Weyl group of $\Psi_{im,\tau}$) preserving and acting transitively on $\mathcal{X}_\tau[x_0]$, and the Weyl group W_R acting trivially. If $w \in W$ then $w \cdot \mathcal{X}_\tau[x_0] \in \mathcal{X}_{w \cdot \tau}[x_0]$. Here $w \cdot \tau$ denotes the conjugation action of w in W^Γ. This is the cross action.

Definition 4 *If $x \in \mathcal{X}$ is given and α is a simple noncompact imaginary root with respect to x, let ξ be an element of $\widetilde{\mathcal{X}}$ mapping to x. Let σ_α be an element of $Norm_{G_\alpha}(H_\alpha)$ representing s_α.The Cayley transform of x through α is the image $\sigma_\alpha x$ of $\sigma_\alpha \xi$ in $\mathcal{X}$*

This is well defined, and if $p(x) = \tau$ then $p(\sigma_\alpha x) = s_\alpha \tau$ (multiplication in W^Γ). Also, Cayley transforms preserve the real form. The Cayley transform is 1-1 or 2-1 depending on whether α is type II or type I.

Starting from an element $x \in \mathcal{X}_\delta$, we can get to any other element of $\mathcal{X}_\delta[x]$ by a sequence of cross actions and Cayley transforms through simple roots. The atlas command "kgb" gives $\mathcal{X}[x]$ for any (weak) real form of G, along with information about where cross actions and Cayley transforms take you.

We have cross actions and Cayley transforms defined on elements of $\mathcal{Z}$ as well. If α is a simple root then the cross action by s_α on (x, y) is

$$s_\alpha \cdot (x,y) = (s_\alpha \cdot x, s_\alpha \cdot y). \tag{12}$$

Here we are using the same notation for α and $\alpha^\vee$.

If α is simple imaginary noncompact for x then the Cayley transform through α is

$$c^\alpha(x,y) = (\sigma_\alpha x, \sigma^\alpha y), \tag{13}$$

where $\sigma^\alpha y$ denotes the possibly double valued inverse Cayley transform of y through the real (for y) root $\alpha^\vee$,

$$\sigma^\alpha y = \{y' : \sigma_\alpha y' = y\}. \tag{14}$$

Cross actions and Cayley transforms preserve blocks of parameters.

If $(x,y)\in\mathcal{Z}$ consider the collection of parameters

$$\mathcal{B}_{x,y}=\{(x',y')\in\mathcal{Z}:x'\in\mathcal{X}[x],y'\in\mathcal{Y}[y]\}\,. \tag{15}$$

So we are pairing x's and y's corresponding to a fixed pair of real forms of G and $G^\vee$. The corresponding representations form a block in the sense of [8]. The atlas command "block" gives any block of parameters. Of course, not every pair of (weak) real forms will give a block of representations; a compact group, e. g., must be paired with a split one since the torus dual to a compact one is split.

There is a path between any two parameters in a block consisting of simple cross actions and Cayley transforms, and we have the following "chain condition":

Chain Condition 5 *Fix a block $\mathcal{B}_{x,y}$. Choose $(x_{\min},y_{\min})$ and $(x_{\max},y_{\max})$ with $x_{\min}$ corresponding to the most compact Cartan and $x_{\max}$ corresponding to the most split Cartan. Choose a chain C of simple cross actions and Cayley transforms through non-compact simple roots taking $(x_{\min},y_{\min})$ to $(x_{\max},y_{\max})$. Then this determines every intermediate parameter (x,y) uniquely.*

4 Other Parametrizations of Representations

We will use two parametrizations for representations, which we call the Λ-parameters and the Γ-parameters. The Λ-parameters associated to an irreducible admissible Harish-Chandra module consist of a triple $(H(\mathbb{R}),\Psi_R^+,\Lambda)$ (modulo some equivalence relation) of a Cartan subgroup $H(\mathbb{R})$, a choice of positive real roots Ψ_R^+, and a genuine character Λ of the ρ-cover of $H(\mathbb{R})$ (see [4]). The differential λ of Λ corresponds to the infinitesimal character of the representation. In general, one also needs to choose a system of positive imaginary roots; in the regular case, however, this system is uniquely determined, so we omit it.

The Γ-parameters may be more familiar (see [7]); they consist of a triple $(H(\mathbb{R}),\Gamma,\overline{\gamma})$ of a Cartan subgroup, a character Γ of $H(\mathbb{R})$, and an element $\overline{\gamma}\in\mathfrak{h}^*$ representing the infinitesimal character such that

$$d\Gamma=\overline{\gamma}+\rho_i-2\rho_{i,c}, \tag{16}$$

where ρ_i and $\rho_{i,c}$ are one half the sums of the positive imaginary and compact imaginary roots determined by $\overline{\gamma}$.

We may identify $\overline{\gamma}$ with λ. We want to assign to each triple (x,y,λ) satisfying $(x,y)\in\mathcal{Z}$ and $y^2=\exp(2\pi i\lambda)$ Λ-parameters or Γ-parameters. The parameter x determines both a real form of G and a Cartan subgroup, so we will write (x,Ψ_R^+,Λ) and (x,Γ,λ).

We will specify Γ by a pair of parameters; its differential, and an algebraic character $\nu \in X^*(H)$ which agrees with Γ on $H^\theta = T$. Similarly for Λ; here the second parameter will be an element of $\rho + X^*(H)$. So

$$\Lambda = \Lambda(\lambda, \kappa) = \Lambda(x, \lambda, \kappa); \tag{17}$$

$$\Gamma = \Gamma(\gamma, \nu) = \Gamma(x, \gamma, \nu). \tag{18}$$

We can convert between Λ-parameters and Γ-parameters as follows:

$$(x, \Psi_R^+, \Lambda(\lambda, \kappa)) \longleftrightarrow (x, \Gamma(\lambda + \rho_i - 2\rho_{i,c}, \kappa + \rho_i - 2\rho_{i,c} + \rho_R + \rho_{cx}^\circ)), \tag{19}$$

where ρ_R corresponds to Ψ_R^+, and ρ_{cx}° corresponds to a specially chosen system of complex roots. It is not the system determined by λ.

We are looking for the correct correspondence

$$(x, y, \lambda) \longleftrightarrow \left(x, \Psi_R^+, \Lambda(\lambda, \kappa)\right). \tag{20}$$

Notice that if $H(\mathbb{R})$ is connected, then Λ is uniquely determined, and there is no dependence on Ψ_R^+, so we are done (with this part). The difficult part of the assignment is the correct choice of the pair (Ψ_R^+, κ) when the Cartan has $\mathbb{Z}/2\mathbb{Z}$ factors.

5 The Parameter y and L-parameters

Given a pair (y, λ) with $y^2 = \exp(2\pi i \lambda)$, we get by [4] an L-parameter

$$\phi : W_{\mathbb{R}} \longrightarrow (G^\vee)^\Gamma \text{ (modulo conjugation by } G^\vee). \tag{21}$$

The corresponding L-packet should contain the representations parametrized by the pairs (x, y) with this fixed y. The image of ϕ may be conjugated to lie inside the subgroup of $(G^\vee)^\Gamma$ generated by $H^\vee$ and y, an E-group of H. Such maps parametrize genuine characters of the λ-cover of the real form of H determined by x. This parametrization depends on choosing a fixed y_0 in the fiber of y (i. e., $p(y_0) = p(y)$). Then $y \in \langle H^\vee, y_0 \rangle$, and we can write

$$y = hy_0 = \exp(2\pi i \tau) y_0 \tag{22}$$

for some $h \in H^\vee = \exp(\mathfrak{h}^\vee)$. Then the character associated to y (and also depending on y_0 and λ) is

Definition 6

$$\Lambda[x, y, \lambda, y_0] = \Lambda(x, \lambda, \lambda - (\tau - \theta_x \tau)) = \Lambda(x, \lambda, \lambda - (\tau + \theta_y^\vee \tau)). \tag{23}$$

Strictly speaking, we need to choose elements of $\widetilde{\mathcal{Y}}$ mapping to y and y_0; however, the resulting character is independent of these choices. Notice that

$$\Lambda[x, y_0, \lambda, y_0] = \Lambda(x, \lambda, \lambda). \tag{24}$$

The element y_0 must be such that $B^\vee$ is large with respect to y_0, i. e., the real form must be quasisplit, and the simple imaginary roots must be noncompact. (These are the real roots for x.) There may be more than one such y_0 per fiber (these correspond to large fundamental series on the dual side). Choosing a Λ-parameter amounts to choosing such a basepoint, and at the same time choosing the correct system of real roots. These choices are not independent and quite subtle, and we must be sure to make the choices carefully. One requirement is, of course, that the resulting assignment commute with Cayley transforms and cross actions. Because of the Chain Condition, up to a choice each in the most compact and the most split fiber, there is at most one matching of parameters for a given block which commutes with cross actions and Cayley transforms.

This is what we (some of us) refer to as the *Basepoint Issue*.

6 The Trivial Representation

A place to start is the trivial representation, at infinitesimal character ρ. Suppose G is split. Then the trivial representation of G has Λ-parameter

$$\left(x_s, \Psi_R^-, \Lambda(\rho, \rho)\right), \tag{25}$$

where x_s is the unique element of $\mathcal{X}$ giving the split Cartan, and Ψ_R^- is the set of real roots α such that $\langle \rho, \alpha^\vee \rangle < 0$. The y-parameter of the trivial representation of the quasisplit form will be our fundamental basepoint; it corresponds to a large discrete series for the quasisplit form of the dual group. A difficulty is that there may be more than one candidate for this parameter; if $G(\mathbb{R})$ is disconnected, it has more than one one-dimensional character, which atlas sees, but we don't know how to decide which of them is the trivial representation. For example, the parameters for the trivial and the sign representation of $SO(3,2)$ are indistinguishable. (This corresponds to the dual group $Sp(4)$ having nontrivial center and hence two large discrete series.) So we must make a choice. Any alternative choice will result in an assignment that differs by tensoring every representation by a one-dimensional character.

Recall that we can write the trivial representation as a formal sum of standard modules. It turns out that the standard modules occurring in the character formula of the trivial representation provide us with a consistent way of choosing our desired basepoints! Of course, the different sets of parameters for irreducible representations also parametrize standard modules.

Proposition 7 *Fix an equivalence class of strong real forms. A standard module occurs in the character formula of the trivial representation if and only if its* Λ*-parameter is of the form* $\big(x, \Psi_R^-, \Lambda(\rho,\rho)\big)$*, where* Ψ_R^- *is the system of real roots* α *such that* $\langle \rho, \alpha^\vee \rangle < 0$.

This follows by an easy calculation as a corollary of the following result.

Theorem 8 *(Zuckerman) A standard module occurs in the character formula of the trivial representation if and only if the corresponding character of the component group of* $H(\mathbb{R})$ *is trivial.*

The character χ of the component group may be computed fairly easily from the Γ-parameters.

To see that these standard modules provide us with a point in each fiber, we have

Proposition 9 *The set of standard modules which appear in the character formula of the trivial representation is closed under cross actions by imaginary and simple complex roots, and by Cayley transforms through simple real roots (on the corresponding parameters).*

On the level of the y's, i. e., on the dual side, these operations are cross actions through real roots (no effect), complex roots, and Cayley transforms through simple noncompact imaginary roots; and we know that we can reach each fiber by a sequence of such operations. Now if (x_1, y_1) and (x_2, y_2) both parametrize standard modules occurring in the character formula of the trivial representation and both are in the same fiber, then $(x_{1,}y_2)$ has the same property since we are allowed to move parameters by imaginary cross actions. Since the y parameter directly affects the character of the component group, we must have $y_1 = y_2$. So we get precisely one element in each fiber. Now our fundamental basepoint y is large by choice. This property is easily seen to be preserved by simple complex cross action, and it takes more work (actually: David's help) to show this for simple noncompact Cayley transforms. So we have

7 Our Basepoints

Proposition 10 *For the quasisplit form of* G*, choose a large KGB orbit* x_δ *(the fundamental base point). Then the set of orbits obtained by applying sequences of simple complex cross actions and simple noncompact Cayley transforms gives a set of basepoints for* G*, i. e., one point* x_τ *in each fiber making* B *large.*

This gives us now a consistent assignment of parameters for all representations of the quasisplit form whose standard module occur in the character formula of the trivial representation:

$$(x, y_\tau, \rho) \longmapsto \big(x, \Psi_R^-, \Lambda(\rho, \rho)\big) = \big(x, \Psi_R^-, \Lambda[x, y_\tau, \rho, y_\tau]\big) \tag{26}$$

This motivates the following definition.

Definition 11 *If $(x, y) \in \mathcal{Z}$ and $y^2 = \exp(2\pi i\lambda)$, we define the Λ-parameter assigned to (x, y, λ) by*

$$\Phi(x, y, \lambda) = \big(x, \Psi_R^-, \Lambda[x, y, \lambda, y_\tau]\big) = (x, \Lambda[x, y, \lambda, y_\tau])\,. \tag{27}$$

Here Ψ_R^- is the system of real roots α such that $\langle \lambda, \alpha^\vee \rangle < 0$, which we drop from the notation, and y_τ is the basepoint in the fiber of y given by Proposition 10.

Proposition 12 *The map Φ given in (27) is well-defined and 1-1 and commutes with Cayley transforms and cross actions.*

Since all parameters in the large block may be obtained from those occurring in the character formula of the trivial representation by real cross actions, this gives the unique matching of parameters for this block which assigns the trivial representation to our chosen base point, and commutes with cross actions and Cayley transforms.

If we move to the large block of other real forms of G, the trivial representation has to be given by a set of parameters (x, y, ρ) with x giving a maximally split Cartan in this real form, and y large. If there is only one such y, it will be the basepoint y_τ in this fiber. If there is more than one, the basepoint will give us one of the choices, just as for the quasisplit from. Propositions 7 and 9 hold for all real forms, so our assignment is consistent for the large block of any real form.

8 Unraveling x

What is the representation parametrized by a Λ-parameter (x, Λ)? Much of the data is encoded in the parameter x.If $\xi \in \widetilde{\mathcal{X}}$ maps to x then (x, Λ) determines a $(\mathfrak{g}, K_\xi)$-module (a different choice for ξ yields an H-conjugate). As x changes over $\mathcal{X}[x]$, so does $G(\mathbb{R})$ inside G, keeping H fixed, but varying the real form $H(\mathbb{R})$. We will want to conjugate these parameters around so that for each weak real form and each conjugacy class of real Cartans, we fix θ (and hence $G(\mathbb{R})$) and $H(\mathbb{R})$, and move Λ (and Γ) instead. This will require a second choice in each block, this time on the side of the x's. For example, for $SL(2, \mathbb{R})$, there are two discrete series which look indistinguishable; one will be called holomorphic, the other antiholomorphic.

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