# BALA-CARTER TYPE CLASSIFICATION OF NILPOTENT ORBITS OF REAL LIE GROUPS

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Abstract. We present a new classification of nilpotent orbits of real reductive Lie algebras under the action of their adjoint group. This classification generalizes the one given by P. Bala and R. Carter in 1976, for complex semisimple Lie algebras.**[1]**,**[2]**

## 1. Introduction

Let $\mathfrak{g}$ be a real reductive Lie algebra with adjoint group $G$ and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Also let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. Finally, let $\theta$ be a Cartan involution of $\mathfrak{g}$ and $\sigma$ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with regard to $\mathfrak{g}$. Then, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Denote by $K_{\mathbb{C}}$ the connected subgroup of the adjoint group $G_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$. We prove that the orbits $K_{\mathbb{C}}.e$ are in one-to-one correspondence with the triples of the form $(l, q_l, \mathfrak{w})$, where $e$ is a non zero nilpotent in $\mathfrak{p}_{\mathbb{C}}$, $l$ is a minimal $(\theta, \sigma)$-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing $e$, $q_l$ is a $\theta$ stable parabolic subalgebra of $[l, l]$ and $\mathfrak{w}$ is a certain $L \cap K_{\mathbb{C}}$ prehomogeneous subspace of $q_l \cap \mathfrak{p}_{\mathbb{C}}$ containing $e$. $L$ is the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $l$. Thus, we obtain a classification for real nilpotents $G$-orbits in $\mathfrak{g}$ via the Kostant-Sekiguchi correspondence which establishes a one to one correspondence between $G$-orbits in $\mathfrak{g}$ and $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ **[7]**. The Bala-Carter classification for complex nilpotent orbits was covered in Monty Mc Govern's talk.

## 2. The Classification

Let $\mathfrak{g}_0 = \mathfrak{k} \oplus i\mathfrak{p}$. Then, $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. Let $\kappa$ be the Killing form on $\mathfrak{g}_{\mathbb{C}}$ and $\tau$, the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}_0$. For $X$ and $Y$ in $\mathfrak{g}_{\mathbb{C}}$ define $\kappa'(X, Y) = -\kappa(X, \tau(Y))$. Then $\kappa'$ is a hermitian form on $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{q} = l \oplus u$ be a $\theta$-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{m}$ be the orhogonal complement of $[u \cap \mathfrak{k}_{\mathbb{C}}, [u \cap \mathfrak{k}_{\mathbb{C}}, u \cap \mathfrak{p}_{\mathbb{C}}]]$ relative to $\kappa'$ inside $u \cap \mathfrak{p}_{\mathbb{C}}$. Define $\mathfrak{w}$ to be an $L \cap K_{\mathbb{C}}$ module in $\mathfrak{m}$. Finally, let $\hat{\mathfrak{w}} = \mathfrak{w} \oplus [l \cap \mathfrak{p}_{\mathbb{C}}, \mathfrak{w}]$ be an $L$-module. Clearly, $\hat{\mathfrak{w}}$ is $\theta$-stable.

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**Definition**. Define $\mathfrak{L}$ to be the set of triples $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ such that:

1. $\mathfrak{w}$ has a dense $L \cap K_{\mathbb{C}}$ orbit.
2. dim $l \cap \mathfrak{k}_{\mathbb{C}} =$ dim $\mathfrak{w}$.
3. $L$ has an open dense orbit on $\hat{\mathfrak{w}}$ and that dense orbit comes from an element $\hat{e} \in \mathfrak{w}$ such that $(L \cap K_{\mathbb{C}}).\hat{e}$ is dense in $\mathfrak{w}$.
4. $\hat{\mathfrak{w}} \perp [u, [u, u]]$
5.$[u, \hat{\mathfrak{w}}] \cap \hat{\mathfrak{w}} = \{0\}$

Let $e$ be a non-zero nilpotent element of $\mathfrak{p}_{\mathbb{C}}$. It is known that $e$ can be embedded in a normal triple $(x, e, f)$ in Kostant-Rallis's sense that is $[x, e] = 2e$, $[x, f] = -2f$ and $[e, f] = x$, with $x \in \mathfrak{k}_{\mathbb{C}}$, $e$ and $f \in \mathfrak{p}_{\mathbb{C}}$ **[4]**. Since we are interested in $K_{\mathbb{C}}$-conjugacy classes of nilpotents in $\mathfrak{p}_{\mathbb{C}}$ we can assume that $\sigma(e) = f$.**[7]**

From the representation theory of $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{g}_{\mathbb{C}}$ has the following eigenspace decomposition:

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(j)} \text{ where } \mathfrak{g}_{\mathbb{C}}^{(j)} = \{z \in \mathfrak{g}_{\mathbb{C}} | [x, z] = jz\}.$$

The $\theta$-stable subalgebra $\mathfrak{q}_x = \bigoplus_{j\epsilon\mathbb{N}} \mathfrak{g}_{\mathbb{C}}^{(j)}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with a Levi part $l = \mathfrak{g}_{\mathbb{C}}^{(0)}$ and nilradical $u = \bigoplus_{j\in\mathbb{N}^*} \mathfrak{g}_{\mathbb{C}}^{(j)}$. Call $\mathfrak{q}_x$ the Jacobson-Morosov parabolic subalgebra of $e$ relative to the triple $(x, e, f)$.

**Definition.** A nilpotent element $e$ in $\mathfrak{p}_{\mathbb{C}}$ (or its $K_{\mathbb{C}}$-orbit) is *noticed* if the only $(\sigma, \theta)$-stable Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing $e$ (or equivalently meeting $K_{\mathbb{C}}.e$) is $\mathfrak{g}_{\mathbb{C}}$ itself.

A Levi subalgebra $l$ contains $e$ if and only if $[l, l]$ does. Thus if $e$ is noticed in $l$ it is actually noticed in the semi-simple subalgebra $[l, l]$. Finally if $l$ is $(\theta, \sigma)$-stable then so is $[l, l]$, and any nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ is noticed in any minimal $(\theta, \sigma)$-stable Levi subalgebra $l$ containing it. If $e$ is noticed then the normal triple $(x, e, f)$ is said to be noticed.

Let $\mathfrak{S}$ be the set of noticed normal triples $(x, e, f)$ of $\mathfrak{g}_{\mathbb{C}}$.

We have a map $\mathfrak{F}$ from $\mathfrak{S}$ to $\mathfrak{L}$ which associates a triple $(x, e, f)$ of $\mathfrak{S}$ to an element $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}_x, \mathfrak{w})$ of $\mathfrak{L}$ where $\mathfrak{q}_x$ is the $\theta$-stable parabolic subalgebra relative to $(x, e, f)$ and $\mathfrak{w} = \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}$. From Kostant and Rallis **[4]** we know that $L.e$ (respectively $L \cap K_{\mathbb{C}}$.e) is dense on $\mathfrak{g}_{\mathbb{C}}^{(2)}$ (respectively $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$). Also dim $\mathfrak{g}_{\mathbb{C}}^{(0)} \cap \mathfrak{k}_{\mathbb{C}} =$ dim $\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$ because $e$ is noticed. Since $[\mathfrak{p}_{\mathbb{C}}^x, \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}] = \mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{k}_{\mathbb{C}}$ **[6]**, it is clear that $\mathfrak{F}$ is well defined.

From a theorem of Kostant & Rallis **[4]**, there is a bijection between the non-zero nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ and the $K_{\mathbb{C}}$-conjugacy classes of normal triples. Two normal noticed triples $(x, e, f)$ and $(x', e', f')$ are $K_{\mathbb{C}}$ conjugate if and only if their corresponding triples $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w})$ and $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}', \mathfrak{w}')$ are $K_{\mathbb{C}}$ conjugate. Hence, $\mathfrak{F}$ induces a one to one map from $K_{\mathbb{C}}$-orbits of $\mathfrak{S}$ and the $K_{\mathbb{C}}$-conjugacy classes of the triples of $\mathfrak{L}$. The following theorem tells us that such a map is also surjective.

**Theorem 1.** *For any triple* $\{\mathfrak{g}_{\mathbb{C}}, \mathfrak{q}, \mathfrak{w}\}$ *of* $\mathfrak{L}$ *there exists a normal triple* $(x, e, f)$ *in* $\mathfrak{S}$ *such that* $\mathfrak{q}$ *is the Jacobson-Morosov parabolic subalgebra for* $(x, e, f)$ *and* $\mathfrak{w} =$

$\mathfrak{g}_{\mathbb{C}}^{(2)} \cap \mathfrak{p}_{\mathbb{C}}$.

*Proof.* See **[6]**.

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Let $l$ be a $(\theta, \sigma)$-stable Levi Subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Define the set of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$ to have the same properties as the triples of $\mathfrak{L}$, replacing $\mathfrak{g}_{\mathbb{C}}$ by $l$. Here $\mathfrak{q}_l$ is a $\theta$-stable parabolic subalgebra of $[l, l]$. Then we have:

**Theorem 2.** *There is a one-to-one correspondance between nilpotent $K_{\mathbb{C}}$-orbits on $\mathfrak{p}_{\mathbb{C}}$ and $K_{\mathbb{C}}$-conjugacy classes of triples $(l, \mathfrak{q}_l, \mathfrak{w}_l)$.*

*Proof.* We noted before that a Levi subalgebra $l$ contains a nilpotent element $e \in \mathfrak{p}_{\mathbb{C}}$ if and only if $[l, l]$ does. Two Levi subalgebras are $K_{\mathbb{C}}$-conjugate if and only if their derived subalgebras are. Each nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$ can be put in a normal triple $(x, e, f)$ inside of the minimal Levi subalgebra $l$ containing $e$. Any two minimal Levi subalgebras containing $e$ are conjugate under $K_{\mathbb{C}}^e$ **[6]**. Hence the theorem follows from theorem 1.

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## 3. Example

Let $\mathfrak{g}$ be $\mathfrak{sl}(3, \mathbb{R})$ , the set of $3 \times 3$ real matrices of trace 0, Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(3, \mathbb{C})$, and $\mathfrak{p}_{\mathbb{C}}$ is the space of $3 \times 3$ complex symmetric matrices. The Cartan involution $\theta$ is defined as $\theta(X) = -X^T$ for $X \in \mathfrak{g}$. Denote by $\bar{Y}$, the complex conjugate of a matrix $Y \in \mathfrak{g}_{\mathbb{C}}$.

The set of orthogonal matrices $(K_{\mathbb{C}})$ preserves the set of symmetric matrices $(\mathfrak{p}_{\mathbb{C}})$ under conjugation. The nilpotent orbits of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ are parametrized by the partitions of 3. Therefore, there are exactly two non zero nilpotent classes since the zero nilpotent class corresponds to the partition $[1, 1, 1]$. A computation shows that the following matrices

$$H_1 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$E_1 = \frac{1}{2} \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & -1 \\ i & 1 & 0 \end{pmatrix}$$

generate the only $\theta$-stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$. Let

$$\mathfrak{b} = \mathbb{C}H_1 \oplus \mathbb{C}E_3 \oplus \mathbb{C}H_2 \oplus \mathbb{C}E_1 \oplus \mathbb{C}E_2.$$

Of course $\mathfrak{b}$ is conjugate to the set of upper triangular matrices of $\mathfrak{sl}(3,\mathbb{C})$.

We see that $l \cap \mathfrak{k}_{\mathbb{C}} = \mathbb{C}H_1$ and $\mathfrak{m} = \mathbb{C}E_1 \oplus \mathbb{C}E_2$. Let $\mathfrak{w}_1 = \mathbb{C}E_1$ and $\mathfrak{w}_2 = \mathbb{C}E_2$. Clearly, $L \cap K_{\mathbb{C}}$ has a dense orbit on $\mathfrak{w}_1$ and $\mathfrak{w}_2$ respectively for $[H_1, E_1] = 2E_1$ and $[H_1, E_2] = E_2$. Also

$$\dim\, l \cap \mathfrak{k}_{\mathbb{C}} = \dim\, \mathfrak{w}_1 = \dim\, \mathfrak{w}_2 = 1.$$

For each $\mathfrak{w}_i$ one verifies easily that the triple ,$(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_i)$, satisfies all the requirements specified above.

Thus we obtain the following correspondence:

$$(H_1, E_1, \bar{E}_1) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_1)$$

$$(2H_1, E_2, \bar{E}_2) \longleftrightarrow (\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}, \mathfrak{w}_2)$$

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