A Practical View of $\widehat{W}$: A Guide to Working with Weyl Group Representations, With Special Emphasis on Branching Rules

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Contents

These notes provide some practical tools for doing explicit calculations with representations of (complex) Weyl groups. We place special emphasis on branching rules and permutation characters induced from reflection subgroups, since this kind of information may prove to be useful for understanding the Weyl group action on characters (of real Lie group representations) by coherent continuation.

1. Type A

The irreducible representations of S_n are indexed by partitions of n, or equivalently, Young diagrams of size n. Being deliberately vague, we will write

$$\widehat{S}_n = \{\chi^\lambda : |\lambda| = n\}$$

as a notation for either the set of irreducible representations or their characters.

There are various ways to nail down which representation corresponds to which partition, but for now, let us merely take note that

$$\chi^{(n)} = \text{trivial}; \quad \chi^{(1^n)} = \text{sign}; \quad \chi^\lambda = \text{sign} \otimes \chi^{\lambda'},$$

where λ' denotes conjugation (transposition of Young diagrams).

A. The induction ring.

There is an obvious embedding of $S_m \times S_n$ in S_{m+n}, so the "induction product"

$$\chi \cdot \theta := (\chi \times \theta) \uparrow_{S_m \times S_n}^{S_{m+n}}$$

provides a graded, associative, commutative ring structure for

$$R = R^A := \bigoplus_{n \geqslant 0} \mathbb{Z}\widehat{S}_n.$$

Note that this product is *NOT* the tensor product that $\mathbb{Z}\widehat{S}_n$ enjoys by itself as a Grothendieck ring. We will prevent confusion about this by consistently using three different multiplication symbols: $\cdot$ (for induction products), $\otimes$ (for tensor products of representations of a single group), and $\times$ (for outer tensor products of representations of two groups).

It is not hard to show that the trivial and sign representations $\{\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \dots\}$ and $\{\chi^{(1)}, \chi^{(11)}, \chi^{(111)}, \dots\}$ both provide algebraically independent generators for R^A; i.e.,

$$R^A = \mathbb{Z}[\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \dots] = \mathbb{Z}[\chi^{(1)}, \chi^{(11)}, \chi^{(111)}, \dots].$$

It follows that the monomials

$$\pi^\mu := \chi^{(\mu_1)} \cdot \chi^{(\mu_2)} \cdots = 1 \uparrow_{S_{\mu_1} \times S_{\mu_2} \times \cdots}^{S_n},$$
$$\varepsilon^\mu := \chi^{1^{\mu_1}} \cdot \chi^{1^{\mu_2}} \cdots = \mathrm{sgn} \uparrow_{S_{\mu_1} \times S_{\mu_2} \times \cdots}^{S_n},$$

as μ varies over partitions of n, both form nice $\mathbb{Z}$-bases for $\widehat{S}_n$.

Rephrasing for emphasis, the above remarks indicate that every representation of S_n is (uniquely) expressible as a polynomial in the trivial representations of various symmetric groups, so many calculations in $\widehat{S}_n$ reduce to polynomial arithmetic.

B. Schur-Weyl duality.

It is useful to keep in mind that for all $r \geqslant 1$, the Grothendieck ring of polynomial representations of gl_r is a quotient of the induction ring R^A. More explicitly, there is a ring morphism $R^A \to \mathbb{Z}\widehat{gl_r}$ in which

$$\chi^\lambda \mapsto \begin{cases} V_\lambda & \text{if } \ell(\lambda) \leqslant r, \\ 0 & \text{if } \ell(\lambda) > r, \end{cases}$$

where V_λ denotes the irreducible representation of gl_r with highest weight λ, and $\ell(\lambda)$ denotes the number of parts in λ (the number of rows in the Young diagram).

In particular, the images of the trivial and sign representations are symmetric and exterior powers; i.e., $\chi^{(n)} \mapsto S^n(\mathbb{C}^r)$ and $\chi^{(1^n)} \mapsto \Lambda^n(\mathbb{C}^r)$.

C. The Pieri rule.

To decompose π^μ or ε^μ into irreducibles, it suffices to know the "Pieri rules"

$$\chi^{(n)} \cdot \chi^\mu = \sum_{\lambda \in H_n(\mu)} \chi^\lambda,$$

$$\chi^{(1^n)} \cdot \chi^\mu = \sum_{\lambda' \in H_n(\mu')} \chi^\lambda,$$

where $H_n(\mu)$ denotes the set of partitions that can be obtained from μ by adding a "horizontal n-strip"; i.e., adding n nodes to the Young diagram of μ so that there is at most one new node per column. Iterating these rules, one obtains

$$\pi^\mu = \sum K_{\lambda,\mu} \chi^\lambda, \tag{1}$$

where $K_{\lambda,\mu}$ is the number of semistandard Young tableaux of shape λ and weight μ; i.e., the number of increasing chains of Young diagrams from 0 to λ so that the ith member of the chain is obtained by adding a horizontal μ_i-strip to the previous member.

It is a remarkable miracle that $K_{\lambda,\mu}$ is also the dimension of the μ-weight space of V_λ. Note that this miracle is manifested entirely within the gl_r-world (and hence is not simply a warmed-over instance of Schur-Weyl duality): the dimension of the μ-weight space of V_λ is also the multiplicity of V_λ in $S^{\mu_1}(\mathbb{C}^r) \otimes S^{\mu_2}(\mathbb{C}^r) \otimes \cdots$.

It is easy to show that (1) $K_{\lambda,\lambda} = 1$, and (2) $K_{\lambda,\mu} > 0$ only if $\lambda \geqslant \mu$, where '$\geqslant$' denotes the usual "dominance" partial order (i.e., λ can be obtained from μ by a series of operations that involve shrinking smaller parts and increasing larger parts). Conjugation is order-reversing, so

$$\pi^\mu = \chi^\mu + \text{ later terms}, \tag{2}$$

$$\varepsilon^\mu = \chi^{\mu'} + \text{ earlier terms}. \tag{3}$$

This provides probably the least encumbered way to nail down which irreducible representation of S_n is which: χ^λ is the unique constituent common to both π^λ and $\varepsilon^{\lambda'}$.

D. The Jacobi-Trudi identity.

We can invert (1), expressing χ^λ as a polynomial in trivial or sign representations:

$$\chi^\lambda = \det\left[\chi^{(\lambda_i - i + j)}\right]_{1 \leqslant i,j \leqslant \ell(\lambda)},$$

$$\chi^{\lambda'} = \det\left[\chi^{1^{\lambda_i - i + j}}\right]_{1 \leqslant i,j \leqslant \ell(\lambda)},$$

following the convention that $\chi^{(-r)} = \chi^{1^{-r}} = 0$ for $r > 0$. Of course these determinants are to be evaluated in R^A.

This provides a very efficient algorithm for decomposing a polynomial in R^A into irreducibles. Bearing in mind that $\chi^\lambda = \pi^\lambda + \text{later } \pi^\mu\text{'s}$ (see (2)), it follows that if $c_\mu \pi^\mu$ is the leading term of some polynomial π in the $\chi^{(r)}$'s, then c_μ is multiplicity of χ^μ in π. Evaluating the above determinant for χ^μ allows one to replace $\pi \leftarrow \pi - c_\mu \chi^\mu$ and iterate.

E. The Littlewood-Richardson rule.

This is a rule for decomposing the induction product of two irreducibles; i.e.,

$$\chi^\mu \cdot \chi^\nu = \sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda,$$

or dually (by Frobenius reciprocity),

$$\chi^\lambda \downarrow^{S_n}_{S_k \times S_{n-k}} = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, \chi^\mu \times \chi^\nu.$$

These decompositions may be described in many different combinatorial terms. One description with relatively little baggage is as follows:

$$\chi^\mu \cdot \chi^\nu = \sum_T \chi^{\mu + \mathrm{wt}(T)},$$

where the sum ranges over all semistandard tableaux T of shape ν such that $\mu + \mathrm{wt}(T_{\geqslant j})$ is dominant for all $j \geqslant 1$, where $T_{\geqslant j}$ is the subtableau of T formed by columns $j, j+1, \ldots$.

In other words, $c^\lambda_{\mu,\nu}$ is the number of semistandard tableaux of shape ν and weight $\lambda - \mu$ such that $\mu + \mathrm{wt}(T_{\geqslant j})$ is dominant for all $j \geqslant 1$.

Note that it is clear from the definition (but not so clear from the rule) that

$$c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu} = c^{\lambda'}_{\mu'\nu'}.$$

On the other hand, it is clear from the rule (but not so clear from the definition) that $c^\lambda_{\mu\nu} = 0$ unless λ contains the diagram of μ (and hence also ν, by symmetry).

As we shall see, the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ occur in many contexts beyond the world of type A.

2. Type B

The irreducible representations of the Weyl group B_n are indexed by ordered pairs of partitions of total size n; say,

$$\widehat{B}_n := \{\chi^{(\mu,\nu)} : |\mu| + |\nu| = n\}.$$

If we regard B_n as the group of $n \times n$ signed permutation matrices, then there is a natural "ignore signs" epimorphism $B_n \to S_n$, and each irreducible representation χ^λ of S_n lifts back through this morphism to an irreducible representation of B_n. By convention, these are the irreducible representations labeled $\chi^{(\lambda,0)}$.

Let $\delta = \delta_n$ denote the one-dimensional B_n-representation that tracks the parity of signs in each signed permutation matrix (equivalently, this is the parity of "short" reflections). By convention, the irreducible representation $\delta_n \otimes \chi^{(\lambda,0)}$ is the one labeled $\chi^{(0,\lambda)}$.

In these terms, the one-dimensional representations of B_n are as follows: $\chi^{(n,0)}$ (trivial), $\chi^{(0,n)} = \delta_n$, $\chi^{(1^n,0)}$ (the parity of "long" reflections), and $\chi^{(0,1^n)}$ (the sign representation).

A. The induction ring.

There is an obvious embedding of $B_m \times B_n$ in B_{m+n}, so the induction product

$$\chi \cdot \theta := (\chi \times \theta)\uparrow^{B_{m+n}}_{B_m \times B_n}$$

provides a graded, associative, commutative ring structure for

$$R^B := \bigoplus_{n \geqslant 0} \mathbb{Z}\widehat{B}_n.$$

Furthermore, the remaining irreducible representations are products in this ring:

$$\chi^{(\mu,\nu)} = \chi^{(\mu,0)} \cdot \chi^{(0,\nu)}.$$

The restriction of δ_{m+n} to $B_m \times B_n$ is $\delta_m \times \delta_n$, so tensoring by δ is a ring automorphism of R^B and

$$\delta \otimes \chi^{(\mu,\nu)} = \chi^{(\nu,\mu)}.$$

It follows that the trivial representations $\chi^{(n,0)}$, and their δ-twists $\chi^{(0,n)}$, both freely generate subrings of R^B isomorphic to R^A, and together freely generate R^B; i.e.,

$$R^B \;=\; \mathbb{Z}[\chi^{(n,0)}, \chi^{(0,n)} : n \geqslant 1] \;\cong\; R^A \otimes R^A.$$

In particular, every representation of B_n is (uniquely) expressible as a polynomial in the trivial and δ-representations of B_k for $k \leqslant n$.

B. Branching from B_n to $B_k \times B_{n-k}$.

The induction product in B_n is completely determined by the Littlewood-Richardson Rule (i.e., the type A induction product):

$$\begin{aligned}\chi^{(\mu,\nu)} \cdot \chi^{(\alpha,\beta)} &= (\chi^{(\mu,0)} \cdot \chi^{(\alpha,0)}) \cdot (\chi^{(0,\nu)} \cdot \chi^{(0,\beta)}) \\ &= \Big(\sum_\theta c^\theta_{\mu,\alpha}\, \chi^{(\theta,0)}\Big) \cdot \Big(\sum_\psi c^\psi_{\nu,\beta}\, \chi^{(0,\psi)}\Big) = \sum_{\theta,\psi} c^\theta_{\mu,\alpha} c^\psi_{\nu,\beta}\, \chi^{(\theta,\psi)}.\end{aligned}$$

Applying Frobenius reciprocity, this yields a branching rule:

$$\chi^{(\theta,\psi)}\downarrow^{B_n}_{B_k \times B_{n-k}} = \sum_{\mu,\nu,\alpha,\beta} c^\theta_{\mu,\alpha} c^\psi_{\nu,\beta}\, \chi^{(\mu,\nu)} \times \chi^{(\alpha,\beta)}.$$

C. Branching from B_n to S_n.

Littlewood-Richardson coefficients also show up when you branch from B_n to S_n:

$$\begin{aligned}\chi^\lambda\uparrow^{B_n}_{S_n} &= \sum_{\mu,\nu} c^\lambda_{\mu,\nu} \chi^{(\mu,\nu)}, \\ \chi^{(\mu,\nu)}\downarrow^{B_n}_{S_n} &= \sum_\lambda c^\lambda_{\mu,\nu} \chi^\lambda.\end{aligned}$$

Special cases include $1\uparrow^{B_n}_{S_n} = \sum_{k=0}^n \chi^{(k,n-k)}$ and $(\mathrm{sgn})\uparrow^{B_n}_{S_n} = \sum_{k=0}^n \chi^{(1^k,1^{n-k})}$.

D. Branching from S_{2n} to B_n.

Note that B_n embeds in S_{2n} as the centralizer of a fixed-point free involution (e.g., the longest element). While this is not a reflection embedding, the fact that it occurs as the centralizer of an involution indicates that induction/restriction between this pair of groups is significant in the context of real Weyl groups.

If χ and θ are elements of R^B of degrees m and n, then

$$(\chi \cdot \theta)\uparrow_{B_{m+n}}^{S_{2m+2n}} = \left(\chi\uparrow_{B_m}^{S_{2m}} \times \theta\uparrow_{B_n}^{S_{2n}}\right)\uparrow_{S_{2m}\times S_{2n}}^{S_{2m+2n}} = \chi\uparrow_{B_m}^{S_{2m}} \cdot \theta\uparrow_{B_n}^{S_{2n}}.$$

It follows that the maps $\chi \mapsto \chi\uparrow_{B_n}^{S_{2n}}$ define a degree-doubling ring morphism $R^B \to R^A$.

Since $\chi^{(\mu,\nu)} = \chi^{(\mu,0)} \cdot \chi^{(0,\nu)}$, and we already know how to compute induction products in R^A (e.g., using the Littlewood-Richardson Rule), it follows that we can reduce the problem of $B_n \to S_{2n}$ induction to the special cases $\chi^{(\mu,0)}\uparrow_{B_n}^{S_{2n}}$ and $\chi^{(0,\mu)}\uparrow_{B_n}^{S_{2n}}$. However, as far as we know, there are no general combinatorial rules known for the expansions

$$\chi^{(\mu,0)}\uparrow_{B_n}^{S_{2n}} = \sum_\lambda S^{(2)}_{\mu\lambda}\,\chi^\lambda, \quad \chi^{(0,\mu)}\uparrow_{B_n}^{S_{2n}} = \sum_\lambda \Lambda^{(2)}_{\mu\lambda}\,\chi^\lambda.$$

We chose the peculiar coefficient names $S^{(2)}_{\mu\lambda}$ and $\Lambda^{(2)}_{\mu\lambda}$ in order to emphasize that they also occur in a natural (but equally unsolved) problem over in the gl-world: $S^{(2)}_{\mu\lambda}$ is the multiplicity of V_λ in the gl_r-representation obtained by restricting the $gl_{r(r+1)/2}$-representation of highest weight μ to gl_r (embedded via the second symmetric power), and there is a similar description involving the second exterior power for $\Lambda^{(2)}_{\lambda\mu}$. Writing this in shorthand,

$$S^{(2)}_{\mu\lambda} = \langle V_\mu(S^2(\mathbb{C}^r)), V_\lambda\rangle_{gl}, \quad \Lambda^{(2)}_{\mu\lambda} = \langle V_\mu(\Lambda^2(\mathbb{C}^r)), V_\lambda\rangle_{gl}. \tag{4}$$

To be a bit more careful, one should add the qualification that all of the above requires r to be sufficiently large (in fact $r = 2n$ is sufficiently large). If r is not sufficiently large, then the only failure of (4) is that some left-hand sides may involve partitions λ that don't correspond to highest weights for gl_r (i.e., they have too many parts).

A nice reduction of the problem may be obtained by noticing that the sign representation of S_{2n} restricts to δ_n, which implies that our $R^B \to R^A$ morphism intertwines the actions of these two characters on R^B and R^A; i.e.,

$$\left(\delta_n \otimes \chi\right)\uparrow_{B_n}^{S_{2n}} = \operatorname{sgn} \otimes \left(\chi\uparrow_{B_n}^{S_{2n}}\right).$$

Thus the decompositions of $\chi^{(\mu,0)}\uparrow_{B_n}^{S_{2n}}$ and $\chi^{(0,\mu)}\uparrow_{B_n}^{S_{2n}}$ are sign twists of each other. This has the amusing corollary that

$$S^{(2)}_{\mu\lambda} = \Lambda^{(2)}_{\mu\lambda'},$$

which is not at all obvious if you are stuck inside the gl-world.

We should also mention that there are a few important special cases for which the expansions in (4) are known explicitly. These two correspond to the gl_r-decompositions of $S^n(S^2(\mathbb{C}^r))$ and $S^n(\Lambda^2(\mathbb{C}^r))$:

$$\begin{aligned} 1\uparrow_{B_n}^{S_{2n}} &= \sum_{\lambda} \chi^{2\lambda}, \\ \delta_n\uparrow_{B_n}^{S_{2n}} &= \sum_{\lambda} \chi^{(2\lambda)'}, \end{aligned}$$

and these two correspond to $\Lambda^n(S^2(\mathbb{C}^r))$ and $\Lambda^n(\Lambda^2(\mathbb{C}^r))$:

$$\begin{aligned} \chi^{(1^n,0)}\uparrow_{B_n}^{S_{2n}} &= \sum_{\sigma \text{ strict}} \chi^{\sigma_*}, \\ \chi^{(0,1^n)}\uparrow_{B_n}^{S_{2n}} &= \sum_{\sigma \text{ strict}} \chi^{\sigma^*}, \end{aligned}$$

where σ ranges over partitions of n into *distinct* parts, and $\sigma \mapsto \sigma_* = (\sigma^*)'$ denotes the following doubling operation, illustrated in the case $\sigma = 6421$:

$$\begin{array}{cccccc} x & x & x & x & x & x \\ & x & x & x & x & \\ & & x & x & & \\ & & & x & & \end{array} \quad \longmapsto \quad \begin{array}{ccccccc} y & x & x & x & x & x & x \\ y & y & x & x & x & x & \\ y & y & y & x & x & & \\ y & y & y & y & x & & \\ y & y & & & & & \\ y & & & & & & \end{array}$$

All four of these expansions are due to Littlewood.

E. Permutation representations induced by reflection subgroups.

The standard realizations of S_n, B_n ($n \geqslant 1$) and D_n ($n \geqslant 2$) involve reflections that permute (up to sign) n coordinates. Even though there may be isomorphisms among these groups (e.g., $D_3 \cong S_4$, $B_1 \cong S_2$), their standard realizations are all different (e.g., they involve permuting different numbers of coordinates). In this way, given any triple of partitions $(\alpha;\beta;\gamma)$ of total size n (but with no 1's in β), we can understand

$$W = S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times D_{\beta_1} \times D_{\beta_2} \times \cdots \times B_{\gamma_1} \times B_{\gamma_2} \times \cdots$$

as specifying a reflection subgroup of B_n in which the n coordinates have been partitioned into disjoint blocks of sizes α_i, β_i and γ_i, and then standard realizations of the appropriate reflection (sub)groups of type S, D, or B have been selected for each block. In particular, choosing $\alpha_i = 1$ amounts to selecting a coordinate that will be fixed by every reflection.

It is not hard to show that the type vector $(\alpha;\beta;\gamma)$ classifies the conjugacy classes of reflection subgroups of B_n; i.e., every reflection subgroup of B_n is conjugate to a reflection subgroup of exactly one of the types $(\alpha;\beta;\gamma)$.

If W is of type $(\alpha;\beta;\gamma)$, then the action of B_n on B_n/W is an induction product involving terms of the form $1\uparrow_{S_m}^{B_m} = \sum_k \chi^{(k,m-k)}$ (see §2C), $1\uparrow_{D_m}^{B_m} = \chi^{(m,0)} + \chi^{(0,m)}$, and $1\uparrow_{B_m}^{B_m} = \chi^{(m,0)}$ for various $m \geqslant 1$. It follows that to decompose any such product into irreducibles, all one needs to know are the multiplication rules

$$\chi^{(k,0)} \cdot \chi^{(\mu,\nu)} = \sum_{\lambda \in H_k(\mu)} \chi^{(\lambda,\nu)},$$
$$\chi^{(0,k)} \cdot \chi^{(\mu,\nu)} = \sum_{\lambda \in H_k(\nu)} \chi^{(\mu,\lambda)}.$$

These are corollaries of the Pieri Rule in §1C.

3. Type D

Notice that D_n is the kernel of δ_n, so $\chi^{(\mu,\nu)}$ and $\chi^{(\nu,\mu)} = \delta_n \otimes \chi^{(\mu,\nu)}$ both restrict to the same representation of D_n. Thus it is reasonable to define a family of D_n-representations indexed by *unordered* pairs of partitions of total size n:

$$\chi^{\{\mu,\nu\}} := \chi^{(\mu,\nu)}\downarrow_{D_n}^{B_n} = \chi^{(\nu,\mu)}\downarrow_{D_n}^{B_n}.$$

It follows from standard facts about subgroups of index 2 that (1) if $\mu \neq \nu$ (i.e., $\chi^{(\mu,\nu)} \ncong \delta_n \otimes \chi^{(\mu,\nu)}$), then $\chi^{\{\mu,\nu\}}$ is irreducible, and (2) if $\mu = \nu$, then $\chi^{\{\mu,\nu\}}$ decomposes into two irreducible, nonisomorphic pieces, say

$$\chi^{\{\mu,\mu\}} = \chi_+^{\{\mu,\mu\}} + \chi_-^{\{\mu,\mu\}},$$

and the outer automorphism of D_n provided by B_n interchanges these two pieces. Thus

$$\widehat{D}_n = \bigl\{\chi^{\{\mu,\nu\}} : |\mu| + |\nu| = n,\ \mu \neq \nu\bigr\} \cup \bigl\{\chi_\pm^{\{\mu,\mu\}} : |\mu| = n/2\bigr\}.$$

Of course the split pieces do not exist when n is odd.

The only non-trivial D_n-representation of degree one is the sign representation $\chi^{\{1^n,0\}}$. It acts on $\widehat{D}_n$ according to the rules $\mathrm{sgn} \otimes \chi^{\{\mu,\nu\}} = \chi^{\{\mu',\nu'\}}$ and

$$\mathrm{sgn} \otimes \chi_\pm^{\{\mu,\mu\}} = \begin{cases} \chi_\pm^{\{\mu',\mu'\}} & \text{if } n/2 \text{ is even,} \\ \chi_\mp^{\{\mu',\mu'\}} & \text{if } n/2 \text{ is odd.} \end{cases}$$

We will see that there is a quasi-natural way to nail down how the '+' and '−' labels should be assigned to the two irreducible constituents of $\chi^{\{\mu,\mu\}}$. This may sound paradoxical at first, given that there is an automorphism that interchanges them, but the point is that once we think of D_n as being a group of signed permutation matrices, then we have distinguished a copy of S_n: the one formed by the subgroup of true permutation matrices. Distinguishing this copy is equivalent to deciding which of the two nodes at the forked end of the Dynkin diagram of D_n to throw away in order to generate S_n.

A. Unsplit branching.

Branching rules for $\chi^{\{\mu,\nu\}}$ are just warmed-over rules for B_n; e.g.,

$$\chi^{\{\mu,\nu\}}\downarrow^{D_n}_{D_k\times D_{n-k}} = \sum_{\alpha,\beta,\psi,\theta} c^{\mu}_{\alpha,\psi} c^{\nu}_{\beta,\theta}\, \chi^{\{\alpha,\beta\}} \times \chi^{\{\psi,\theta\}},$$

$$\chi^{\{\mu,\nu\}}\downarrow^{D_n}_{S_n} = \sum_{\lambda} c^{\lambda}_{\mu\nu}\, \chi^{\lambda}.$$

B. Reflection subgroups.

Before tackling split branching, we need to consider the reflection subgroups of D_n. Certainly these are also reflection subgroups of B_n, so (following §2E) each of these subgroups is conjugate (in B_n) to exactly one reflection subgroup of type $(\alpha;\beta;0)$; i.e.,

$$S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times D_{\beta_1} \times D_{\beta_2} \times \cdots .$$

However, two reflection subgroups conjugate in B_n need not be conjugate in D_n. It is not hard to see that there is no element of $B_n - D_n$ that centralizes a reflection subgroup of type $(\alpha;\beta;0)$ if and only if $\beta = 0$ and every part of α is even. (In particular, n must be even.) In such cases, reflection subgroups of this type split into two conjugacy classes. In the particular case of S_n (i.e., $(\alpha;\beta;0) = (n;0;0)$), the parabolic subgroups obtained by deleting either of the nodes at the forked end of the Dynkin diagram are representatives of the two classes.

To remind us of the fact that there are ambiguities of conjugacy, let S^+_n denote the subgroup of permutation matrices in D_n.

C. Split branching from D_{2n} to S^+_{2n}.

It is somewhat unexpected (and definitely not well known) that there do exist combinatorial rules for describing the coefficients in the expansions

$$\chi^{\{\mu,\mu\}}_{\pm}\downarrow^{D_{2n}}_{S^+_{2n}} = \sum_{\lambda} c^{\lambda}_{\mu}(\pm)\chi^{\lambda}.$$

More precisely, over in the gl-world, there is a combinatorial rule due to Carré and Leclerc [**CL**] for describing the irreducible decomposition of the exterior square and symmetric square of any irreducible representation of gl_r. It turns out that these are exactly the same multiplicities that occur above; i.e.,

$$c^{\lambda}_{\mu}(+) = \langle S^2(V_\mu), V_\lambda\rangle_{gl}, \quad c^{\lambda}_{\mu}(-) = \langle \Lambda^2(V_\mu), V_\lambda\rangle_{gl}.$$

We have never seen this fact in print, although it can be deduced from Theorem 7.5 in [**S**].

This also justifies our previous remark that there is a "correct" way to label the two constituents of $\chi^{\{\mu,\mu\}}$.

Dualizing, we get the following induction rule:

$$\chi^\lambda \uparrow_{S_{2n}^+}^{D_{2n}} = \sum_{\mu \neq \nu} c_{\mu\nu}^\lambda \, \chi^{\{\mu,\nu\}} + \sum_\mu c_\mu^\lambda(+)\chi_+^{\{\mu,\mu\}} + \sum_\mu c_\mu^\lambda(-)\chi_-^{\{\mu,\mu\}}.$$

In particular,

$$\begin{aligned} 1\uparrow_{S_{2n}^+}^{D_{2n}} &= \sum_{k<n} \chi^{\{k,2n-k\}} + \chi_+^{\{n,n\}}, \\ \operatorname{sgn}\uparrow_{S_{2n}^+}^{D_{2n}} &= \sum_{k<n} \chi^{\{1^k,1^{2n-k}\}} + \chi_\varepsilon^{\{1^n,1^n\}}, \end{aligned}$$

where $\varepsilon = (-1)^n$. Notice that the action of sgn on $\widehat{D}_{2n}$ and $\widehat{S}_{2n}$ implies the following relations among the multiplicities in exterior and symmetric squares:

$$c_\mu^\lambda(\pm) = c_{\mu'}^{\lambda'}(\pm) \text{ if } n \text{ is even}, \quad c_\mu^\lambda(\pm) = c_{\mu'}^{\lambda'}(\mp) \text{ if } n \text{ is odd}.$$

Once again, this is not at all obvious if you are stuck inside the gl-world.

D. Split branching from D_{2n} to $D_k \times D_{2n-k}$.

If k is odd, then $\chi_+^{\{\mu,\mu\}}$ and $\chi_-^{\{\mu,\mu\}}$ have the same restriction to $D_k \times D_{2n-k}$, so

$$\chi_+^{\{\mu,\mu\}} \downarrow_{D_k \times D_{2n-k}}^{D_{2n}} = \chi_-^{\{\mu,\mu\}} \downarrow_{D_k \times D_{2n-k}}^{D_{2n}} = \frac{1}{2} \sum_{\alpha,\beta,\psi,\theta} c_{\alpha\psi}^\mu c_{\beta\theta}^\mu \, \chi^{\{\alpha,\beta\}} \times \chi^{\{\psi,\theta\}}.$$

The ordered pairs (α, ψ) and (β, θ) are never equal (given that k is odd), so the above expansion could be rewritten without a factor of 1/2 as a sum over unordered pairs of ordered pairs.

If k is even, the above expansion is a first approximation to the restriction of $\chi_\pm^{\{\mu,\mu\}}$, but now there are correction terms of the form

$$\pm\frac{1}{2} \sum_{\alpha,\psi} c_{\alpha\psi}^\mu \; \chi_\pm^{\{\alpha,\alpha\}} \times \; \chi_\pm^{\{\psi,\psi\}}$$

for each of the four ways to choose an even number of '$-$' signs (in the case of $\chi_+^{\{\mu,\mu\}}$), or an odd number (in the case of $\chi_-^{\{\mu,\mu\}}$). It follows that the net multiplicity of a split product in a split restriction is

$$\langle \chi_\pm^{\{\mu,\mu\}} \downarrow_{D_k \times D_{2n-k}}^{D_{2n}}, \, \chi_\pm^{\{\alpha,\alpha\}} \times \; \chi_\pm^{\{\psi,\psi\}} \rangle = \begin{cases} \binom{c+1}{2} & \text{if the number of '}-\text{' is even}, \\ \binom{c}{2} & \text{if the number of '}-\text{' is odd}, \end{cases} \tag{5}$$

where $c = c_{\alpha\psi}^\mu$. This can be deduced from Theorem 7.5 in [**S**].

E. Permutations representations induced by reflection subgroups.

Let W be a reflection subgroup of D_n, say

$$S^{\pm}_{\alpha_1} \times S^{\pm}_{\alpha_2} \times \cdots \times D_{\beta_1} \times D_{\beta_2} \times \cdots ,$$

allowing the '$\pm$' so that we are sure to reach every conjugacy class. (Recall that this matters only when all of the β_i's are 0 and all of the α_i's are even). All such permutation representations are type D induction products involving terms of the form $1 \uparrow^{D_m}_{S^{\pm}_m}$ and $1 \uparrow^{D_m}_{D_m} = \chi^{\{m,0\}}$. We know that the former is a sum of terms of the form $\chi^{\{k,m-k\}}$, plus a correction term of the form $\chi^{\{m/2,m/2\}}_{\pm}$ if m is even (see §3C).

Thus permutation character decompositions can be deduced from rules for decomposing products of the form $\chi^{\{k,m-k\}} \cdot \theta$ and $\chi^{\{m/2,m/2\}}_{\pm} \cdot \theta$, for $\theta \in \widehat{D}_n$. We claim that most of these calculations can be carried out in the induction ring R^B, based on the following:

$$\begin{aligned} \big(\chi^{\{\mu,\nu\}} \uparrow^{B_n}_{D_n}\big) \downarrow^{B_n}_{D_n} &= \big(\chi^{(\mu,\nu)} + \chi^{(\nu,\mu)}\big) \downarrow^{B_n}_{D_n} = 2\chi^{\{\mu,\nu\}}, \\ \big(\chi^{\{\mu,\mu\}}_{\pm} \uparrow^{B_n}_{D_n}\big) \downarrow^{B_n}_{D_n} &= \chi^{(\mu,\mu)} \downarrow^{B_n}_{D_n} = \chi^{\{\mu,\mu\}}_{+} + \chi^{\{\mu,\mu\}}_{-}. \end{aligned}$$

This shows that if we seek the multiplicity of $\chi^{\{\mu,\nu\}}$ in (say) $\chi \cdot \theta$ (where $\mu \neq \nu$), then we can do the calculation up in R^B and divide by 2.

On the other hand, if we want the multiplicity of $\chi^{\{\mu,\mu\}}_{\pm}$ in $\chi \cdot \theta$, notice that the rules for split branching in §3D show that these multiplicities are the same unless both factors χ and θ are of the split type as well. In the not-both-split case, we can again induce the calculation up to R^B and this time take the multiplicity of $\chi^{(\mu,\mu)}$ (without dividing by 2).

In the both-factors-split case, with $\chi = \chi^{\{m/2,m/2\}}_{\pm}$, rule (5) specializes to

$$\langle \chi^{\{\mu,\mu\}}_{\pm},\ \chi^{\{m/2,m/2\}}_{\pm} \cdot \chi^{\{\alpha,\alpha\}}_{\pm} \rangle = \begin{cases} 1 & \text{if the number of '$-$' is even and } \mu \in H_{m/2}(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

4. The Exceptional Groups

For the exceptional groups, I prefer to use my Maple package `coxeter`.

Download it here: ⟨`www.math.lsa.umich.edu/~jrs/maple.html`⟩.

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