## 1. Signatures and Cohomological induction

Setting: $G$ real red. Lie gp $\supset$ $K$ maximal cpt $\rightsquigarrow$ $\theta$ Cartan involution

$$\begin{array}{ccc} G & \supset & K \\ | & & | \\ \mathfrak{g}_0 & & \mathfrak{k}_0 \end{array}$$

$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C}$ etc.

**Problem:** Find all irred. Harish-Chandra modules admitting positive-definite invariant Hermitian forms.

**Q:** Classify admissible $(\mathfrak{g}, K)$ modules $\supset$ Harish-Chandra modules

Zuckerman, 1978: algebraic construction for admissible $(\mathfrak{g}, K)$ modules known as **cohomological induction:**

- $\mathfrak{g} \supset \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ $\theta$-stable parabolic subalgebra
- $L = N_G(\mathfrak{q})$ Levi subgroup

Cohomological induction is a two-step process:

$$\begin{array}{ccccc} \mathcal{C}(\mathfrak{l}, L\cap K) & & \xrightarrow{\text{j}^{\text{th}} \text{ cohomological induction functor } \mathcal{L}_j} & & \mathcal{C}(\mathfrak{g}, K) \\ & \xrightarrow[\text{Induction}]{①} & \mathcal{C}(\mathfrak{g}, L\cap K) & \xrightarrow[\text{j}^{\text{th}} \text{ Bernstein functor}]{②} & \\ \underbrace{V}_{\substack{\mathfrak{u} \circlearrowleft \text{ triv.} \\ \rightsquigarrow \mathcal{C}(\mathfrak{q}, L\cap K)}} & \longmapsto & \underbrace{U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} V}_{\substack{\textbf{generalized} \\ \textbf{Verma module}}} & \longmapsto & \Pi_j(U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} V) \end{array}$$

**Fact:** If $V$ has an invariant Hermitian form, then so does $\mathcal{L}_s V$ where $s = \dim \mathfrak{u} \cap \mathfrak{k}$.

**Want:** Relate signatures of forms on $V$, $\mathcal{L}_s V$.

**Theorem 1.1.** *(Vogan, Unitarizability of certain series of representations,* **Annals of Math.**, *1984: Theorem 1.3)*
*If $V \in \mathcal{C}(\mathfrak{l}, K)$ is unitary of infinitesimal character $\lambda - \rho(\mathfrak{u})$ and Re $\langle \lambda, \alpha^\vee \rangle \leqslant 1$ for every $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$, then $\mathcal{L}_s V$ is unitary also.*

Wallach, *On the unitarizability of derived functor modules*, **Inventiones Math.**, 1984:
Same result, less technical proof. Approach:

sig of $V$ $\rightsquigarrow$ sig of intermediate module (the GVM) $\rightsquigarrow$ sig for $\mathcal{L}_s V$

Extensions of Wallach's first computation:

Y, 2004: sig for irreducible Verma modules (any inf'l char a.e.)
Y, 2006: irreducible highest weight modules (any regular inf'l char)

Setup:

- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ $\theta$-stable Borel
- $\lambda \in \mathfrak{h}^* \rightsquigarrow M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$ (inf'l char $\lambda$)
- to have invariant Hermitian form on $M(\lambda)$ need: $\mathfrak{h}$ maximally compact, $\theta\Delta^+ = \Delta^+$, and $\lambda$ imaginary (recall $\mathfrak{b}$ $\theta$-stable)
- Invariant form on $M(\lambda)$ unique up to real scalar. Canonical form (i.e. $\langle v_{\lambda-\rho}, v_{\lambda-\rho}\rangle_\lambda = 1$) called the **Shapovalov** form.
- Invariance $\rightsquigarrow$ $\langle \cdot, \cdot \rangle_\lambda$ pairs $\lambda - \mu - \rho$, $\lambda + \bar{\mu} - \rho$ wt spaces
  finite-dimensional $\rightsquigarrow$ Can discuss *signatures* by restricting attention to $M(\lambda)_{\lambda-\mu-\rho}$ if $\mu$ imaginary, $M(\lambda)_{\lambda-\mu-\rho} \oplus M(\lambda)_{\lambda-+\bar{\mu}-\rho}$ if $\mu$ non-imaginary

**Signature:** encode in **signature character**:
On $M(\lambda)_{\lambda-\mu-\rho}$ where $\mu$ imaginary: let signature of matrix representing $\langle \cdot, \cdot \rangle_\lambda$ w.r.t. some basis be $(p(\mu), q(\mu))$.
Define **signature character** to be:
$$\sum_{\mu \in \Lambda_r^+ \text{ imaginary}} (p(\mu) - q(\mu)) e^{\lambda-\mu-\rho}$$

Why can we ignore non-imaginary $\mu$?

**Lemma 1.2.** *(Vogan, Unitarizability of certain series of representations,* **Annals of Math.**, *1984, Sublemma 3.18)*

*Let $E$ be a finite dimensional vector space carrying a non-degenerate invariant Hermitian form $\langle , \rangle$ of signature $(p, q)$. Let $S$ be a totally isotropic subspace of $E$ (that is, $\langle \cdot, \cdot \rangle |_S$ is zero), and set*

$$S^\perp = \{e \in E | \langle e, S \rangle = 0\}.$$

a) *The radical of $\langle \cdot, \cdot \rangle |_{S^\perp}$ is $S$; so $\langle \cdot, \cdot \rangle$ induces a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle_F$ on $F = S/S^\perp$.*
b) *Write $(p', q')$ for the signature of $\langle \cdot, \cdot \rangle_F$ and $m$ for the dimension of $S$. Then*

$$p = p' + m, \qquad q = q' + m.$$

If we apply this lemma to $M(\lambda)_{\lambda-\mu-\rho} \oplus M(\lambda)_{\lambda+\bar{\mu}-\rho}$ when $\mu$ is non-imaginary, observing that

$$\begin{aligned} \langle M(\lambda)_{\lambda-\mu-\rho}, M(\lambda)_{\lambda-\mu-\rho} \rangle_\lambda &= 0 \qquad \text{and} \\ \langle M(\lambda)_{\lambda+\bar{\mu}-\rho}, M(\lambda)_{\lambda+\bar{\mu}-\rho} \rangle_\lambda &= 0 \end{aligned}$$

we see that the number of positive and negative eigenvalues for a matrix representing $\langle \cdot, \cdot \rangle_\lambda$ on $M(\lambda)_{\lambda-\mu-\rho} \oplus M(\lambda)_{\lambda+\bar{\mu}-\rho}$ are equal, so "$p - q = 0$."

## 2. Signature Character Formulas That We Know

**Irreducible Verma Modules**:

**Theorem 2.1.** *(Y, The signature of the Shapovalov form on irreducible Verma modules,* **Representation Theory***, 2005: Theorems 4.6 and 6.12)*
*Let $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$ be the set of imaginary roots in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Subscripts or superscripts $i$ will refer to objects associated with $\Delta_i^+(\mathfrak{g}, \mathfrak{h})$. We will assume that everything (simple roots, reducibility hyperplanes, etc.) in this theorem is associated to the root system of imaginary roots. Choose the fundamental alcove $A_0^i$ of $W_a^i$ and the fundamental chamber $\mathfrak{C}_0^i$ of $W_i$ to contain $-\rho_i$. Let $\bar{\cdot} : W_a^i \to W_i$ be the homomorphism arising from the semidirect product structure $W_a^i = W_i \ltimes \Lambda_i$. Given $a \in W_a^i$, let $\tilde{a} \in W_i$ be such that $aA_0^i \in \tilde{a}\mathfrak{C}_0^i$. Let $aA_0^i = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \cdots \xrightarrow{r_\ell} C_\ell = \tilde{a}A_0^i$ be a path from $aA_0^i$ to $\tilde{a}A_0^i$. Then for imaginary $\lambda \in aA_0^i$:*

$$\begin{aligned} ch_s M(\lambda)|_{\mathfrak{a}_0} &= \lambda|_{\mathfrak{a}_0} \qquad \textit{and} \\ ch_s M(\lambda)|_{\mathfrak{t}_0} &= R^{aA_0}(\lambda|\mathfrak{t}_0) \\ &= \sum_{\substack{S=\{i_1<\cdots<i_k\} \\ \subset\{1,\ldots,\ell\}}} \varepsilon(S) 2^{|S|} \frac{e^{\overline{r_{i_1}}\,\overline{r_{i_2}}\cdots\overline{r_{i_k}} r_{i_k} r_{i_{k-1}} \cdots r_{i_1} \lambda|_{\mathfrak{t}_0} - \rho}}{\prod_{\alpha\in\Delta^+(\mathfrak{p},\mathfrak{t})}(1-e^{-\alpha}) \prod_{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})}(1+e^{-\alpha})} \end{aligned}$$

*where $\varepsilon(S) = \varepsilon(C_{i_1-1}, C_{i_1})\varepsilon(\overline{r_{i_1}} C_{i_2-1}, \overline{r_{i_1}} C_{i_2}) \cdots \varepsilon(\overline{r_{i_1}} \cdots \overline{r_{i_{k-1}}} C_{i_k-1}, \overline{r_{i_1}} \cdots \overline{r_{i_{k-1}}} C_{i_k})$, $\varepsilon(\varnothing) = 1$, and the formula for $\varepsilon(C, C')$ for alcoves $C$, $C'$ may be found in Theorem 6.12.*

Note: Wallach dealt with the case $\ell = 0$ for generalized Verma modules. See Lemma 2.3 of *On the unitarizability of derived functor modules*, **Inventiones Math.**, 1984.

**Irreducible Highest Weight Modules:**

**Theorem 2.2.** *(Y, Signatures of Invariant Hermitian Forms on Irreducible Highest Weight Modules,* **Duke Math. J.***, to appear, Theorem 3.2.3)*
*Let $\lambda$ be antidominant and regular. Let imaginary $\delta \in \mathfrak{h}^*$ be regular and let $w(\delta) \in W_\lambda$ be such that $\delta \in w(\delta)\mathfrak{C}_0$. Then for $x \in W_\lambda$ such that $x\lambda$ is imaginary:*

$$ch_s L(x\lambda) = \sum_{\substack{y_1<\cdots<y_j=x \\ y_k\lambda\text{'s imaginary}}} (-1)^{j-1} \left( \prod_{i=2}^{j} P^{\lambda, w(\delta)}_{w_\lambda y_i, w_\lambda y_{i-1}}(1) \right) \left( ch_s M(y_1\lambda + \delta t) e^{-\delta t} \right)$$

*for small $t > 0$. The $P^{\lambda,w}_{a,b}$'s are* **signed Kazhdan-Lusztig polynomials** *(defined later).*

## 3. Computing signatures for $(\mathfrak{g}, K)$-modules from signatures for the $(\mathfrak{g}, L\cap K)$-module to which the derived Bernstein functor is applied

Reference for this section:
Wallach, *On the unitarizability of derived functor modules*, **Inventiones Math.**, 1984.

**$(K)$ Signature character of $(\mathfrak{g}, K)$-module:**

| | | |
|---|---|---|
| For every $K$-type $\gamma$ : | $(p(\gamma),$ | $q(\gamma))$ |
| | # of copies of $\gamma$ for which form is pos def'n | # of copies of $\gamma$ for which form is neg def'n |

$$\boxed{\sum_{\gamma\in\hat{K}} (p(\gamma)-q(\gamma))e^{\gamma}}$$

Let $V \in \mathcal{C}(\mathfrak{g}, L\cap K)$ have an invariant Hermitian form $\rightsquigarrow$ invariant Hermitian form on $\Pi_*(V)$ naturally. In this pairing, $\Pi_{s-j}(V)$ is paired with $\Pi_{s+j}(V)$ where $s = \dim \mathfrak{u}\cap\mathfrak{k}$, from which we conclude

$$ch_s\Pi_*(V) = ch_s\Pi_s(V)$$

by Lemma 1.2. (Compare this with our previous application of this lemma to $M(\lambda)_{\lambda-\mu-\rho}\oplus M(\lambda)_{\lambda+\mu-\rho}$.)

Let $F_\gamma$ be a realization of $\gamma$ and let $\Gamma$ be the Zuckerman functor. Let $V\in\mathcal{C}(\mathfrak{g}, L\cap K)$ be irreducible. As $K$-rep:

$$\boxed{\Pi_s(V)} \simeq \Gamma^s(V) \simeq \bigoplus_{\gamma\in\hat{K}} \overbrace{\underbrace{\mathrm{Hom}_{\mathfrak{k},K}(F_\gamma,\Gamma^s(V))}_{\simeq\ \mathrm{Ext}^s_{\mathfrak{k},L\cap K}(F_\gamma,V)}}^{\text{trivial } K\text{-action}} \otimes F_\gamma$$

$$\boxed{\simeq \bigoplus_{\gamma\in\hat{K}} \underbrace{H^s(\mathfrak{k}, L\cap K; \mathrm{Hom}_{\mathbb{C}}(F_\gamma, V)_{L\cap K})}_{\substack{\text{sig of Herm form here}\\ \rightsquigarrow\ p(\gamma)-q(\gamma)}} \otimes F_\gamma}$$

Turns out that you can compute signature of form on $H^s(\dots)$ by looking at signature on $C^s(\dots)$ from chain complex:

$$\text{In } C^s: \qquad (Z^s)^\perp = B^s \qquad \text{and} \qquad (B^s)^\perp = Z^s$$

$$\Rightarrow \boxed{\mathrm{sig}\, C^s(\dots) = \mathrm{sig}\, Z^s/B^s = \mathrm{sig}\, H^s(\dots)} \qquad \text{by Lemma 1.2.}$$

**Computing signature of $C^s(\mathrm{Hom}_{\mathbb{C}}(F_\gamma, V)_{L\cap K})$:**

$$\begin{aligned} C^s\left(\mathrm{Hom}_{\mathbb{C}}(F_\gamma,V)_{L\cap K}\right) &= \mathrm{Hom}_{L\cap K}\left(\bigwedge^s(\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k}), \mathrm{Hom}_{\mathbb{C}}(F_\gamma,V)_{L\cap K}\right) \\ &= \mathrm{Hom}_{L\cap K}\left(\bigwedge^s(\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k}), F_\gamma^*\otimes V\right) \\ &= \left(\bigwedge^s(\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^*\otimes F_\gamma^*\otimes V\right)^{\mathfrak{l}\cap\mathfrak{k}} \end{aligned}$$

We wish to identify the trivial representations in $\bigwedge^s(\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^*\otimes F_\gamma^*\otimes V$. Recall the Weyl character formula: if $\xi\in(\mathfrak{l}\cap\mathfrak{k})\hat{}$ has highest weight $\mu$, then

$$D_{\mathfrak{l}\cap\mathfrak{k}}\, ch\,\xi = \sum_{s\in W_{\mathfrak{l}\cap\mathfrak{k}}} e^{s(\mu+\rho_{\mathfrak{l}\cap\mathfrak{k}})}$$

where $D_{\mathfrak{l}\cap\mathfrak{k}} = e^{\rho_{\mathfrak{l}\cap\mathfrak{k}}} \prod_{\alpha\in\Delta^+(\mathfrak{l}\cap\mathfrak{k},\mathfrak{t})} (1-e^{-\alpha})$ is the Weyl denominator. Observe that by multiplying the character of a representation by the Weyl denominator, we can identify the multiplicities of finite-dimensional representations of regular infinitesimal character by reading off the coefficient corresponding to the highest weight plus $\rho_{\mathfrak{l}\cap\mathfrak{k}}$ in the product. Therefore

$$\dim \operatorname{Hom}_{\mathfrak{l}\cap\mathfrak{k}}\left(triv, \bigwedge^s (\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^* \otimes F_\gamma^* \otimes V\right) = \text{coefficient of } e^{\rho_{\mathfrak{l}\cap\mathfrak{k}}} \text{ in } D_{\mathfrak{l}\cap\mathfrak{k}}\, ch\left(\bigwedge^s (\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^* \otimes F_\gamma^* \otimes V\right)$$

and similarly for signature characters. Therefore

$$ch_s \Pi_s V = \sum_{\gamma\in K^\wedge} (p(\gamma)-q(\gamma)) e^\gamma$$

where

$$\boxed{\begin{aligned} p(\gamma)-q(\gamma) &= \operatorname{sig} H^s(\ldots) = \operatorname{sig} C^s(\operatorname{Hom}_{\mathbb{C}}(F_\gamma, V)_{L\cap K}) \\ &= \operatorname{sig}\left(\bigwedge^s (\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^* \otimes F_\gamma^* \otimes V\right)^{\mathfrak{l}\cap\mathfrak{k}} \\ &= \text{coefficient of } e^{\rho_{\mathfrak{l}\cap\mathfrak{k}}} \text{ in } D_{\mathfrak{l}\cap\mathfrak{k}}\, ch_s\left(\bigwedge^s (\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^* \otimes F_\gamma^* \otimes V\right) \\ &= \text{coefficient of } e^{\rho_{\mathfrak{l}\cap\mathfrak{k}}} \text{ in } D_{\mathfrak{l}\cap\mathfrak{k}}\, ch_s \bigwedge^s (\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^*\, ch\, F_\gamma^*\, ch_s V \\ &= \text{coefficient of } e^0 \text{ in } \prod_{\alpha\in\Delta^+(\mathfrak{l}\cap\mathfrak{k},\mathfrak{t})} (1-e^{-\alpha}) \prod_{\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{k},\mathfrak{t})} (-1)^s(1-e^{\alpha})(1+e^{-\alpha})\, ch\, F_\gamma^*\, ch_s V. \end{aligned}}$$

(See Lemma 1.1 of Wallach's paper for the computation of $ch_s \bigwedge^s (\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k})^*$.)

## 4. Reducible Verma modules and signed Kazhdan-Lusztig polynomials

$M(\lambda) \supset J(\lambda)$ (=radical) $\rightsquigarrow M(\lambda)/J(\lambda) =: L(\lambda)$ **irreducible highest weight module**
deg form $\langle\cdot,\cdot\rangle_\lambda$ on $M(\lambda)$ $\underbrace{\rightsquigarrow \textbf{non-deg} \text{ form } \langle\cdot,\cdot\rangle_\lambda \text{ on } L(\lambda)}_{\text{sigs differ only by zero eigenvalues}}$ $\leftarrow$ Compute this sig

Structure of $M(\lambda)$ in terms of $L(\lambda)$'s:
For $\lambda \in \mathfrak{h}^*$ antidominant integral, $x, y \in W$:

$$\begin{aligned} \text{Composition factor multiplicity: } [M(x\lambda) : L(y\lambda)] &= P_{w_0x,w_0y}(1). \\ \text{Character formula: } ch\, M(x\lambda) &= \sum_{y\in W} P_{w_0x,w_0y}(1)\, ch\, L(y\lambda) \\ \text{Inversion formula: } ch\, L(x\lambda) &= \sum_{y\in W} (-1)^{\ell(x)-\ell(y)} P_{y,x}(1)\, ch\, M(y\lambda) \end{aligned}$$

If $\lambda$ is not integral, replace $W$ and its longest element $w_0$ with the integral Weyl group $W_\lambda$ and its longest element $w_\lambda$ in the formulas.

**Additional information encoded in Kazhdan-Lusztig polynomials:** structure of $j^{\text{th}}$ level of Jantzen filtration:

$$[M(x\lambda)_j : L(y\lambda)] = \text{coeff of } q^{\frac{\ell(x)-\ell(y)-j}{2}} \text{ in } P_{w_\lambda x, w_\lambda y}(q) \tag{4.1}$$

**The Jantzen filtration:**

- $\lambda_t := \lambda_0 + \delta t$ where $\lambda_0 \in H_{\alpha,n}$ and $\delta \in \mathfrak{h}^*$ regular, imaginary
- $\det\langle\cdot,\cdot\rangle_{\lambda_t} \neq 0$ for small $t \neq 0$ , $\det\langle\cdot,\cdot\rangle_{\lambda_0} = 0$

**Jantzen filtration:** $M = M(\lambda_0) = M^0 \supset M^1 \supset \cdots \supset M^N = \{0\}$

$v \in M^j \iff \exists f_v : (-\varepsilon,\varepsilon) \to M$ with
- $f_v(0) = v$ and
- $\langle f_v(t), v'\rangle_{\lambda_t}$ vanishes at least to order $j$ at $t = 0$

$\rightsquigarrow$ **non-degenerate** invariant Hermitian form $\lim_{t\to0^+} \frac{1}{t^j}\langle\cdot,\cdot\rangle_{\lambda_t}$ on $M_j := M^j/M^{j+1}$
$(p_j, q_j) :=$ signature of this form on $M_j$ then:

**Proposition 4.1.** *(Vogan, Unitarizability of certain series of representations,* **Annals of Math.***, 1984: Proposition 3.3)*

$$t > 0: \quad sig\ \langle\cdot,\cdot\rangle_{\lambda_t} = \left(\sum_{j=0}^{N} p_j, \sum_{j=0}^{N} q_j\right)$$

$$t < 0: \quad sig\ \langle\cdot,\cdot\rangle_{\lambda_t} = \left(\sum_{j\text{even}}^{N} p_j + \sum_{j\text{odd}}^{N} q_j, \sum_{j\text{even}}^{N} q_j + \sum_{j\text{odd}}^{N} p_j\right)$$

- $M(x\lambda)_j$ is semisimple: direct sum of $L(y\lambda)$'s with multiplicities given by (4.1)
- Proposition 4.1: $ch_s M(x\lambda + \delta t)$ = sum of sig chars of $L(y\lambda)$'s
- need to keep track of this sum

Introduce **signed Kazhdan-Lusztig polynomial** $P^{\lambda,\delta}_{w_\lambda x, w_\lambda y}$:

Each $L(y\lambda)$ in $M(x\lambda)_j \rightsquigarrow \begin{matrix} +1 \\ +1, -1, 0 \end{matrix}$ to coeff of $q^{\frac{\ell(x)-\ell(y)-j}{2}}$ in $\begin{matrix} P_{w_\lambda x, w_\lambda y} \\ P^{\lambda,\delta}_{w_\lambda x, w_\lambda y} \end{matrix}$

+1: sig is that of Shapovalov form on $L(y\lambda)$

$-1$: sig is "opposite" that of Shapovalov form on $L(y\lambda)$ (Recall inv Herm form on h.w.m. ! up to $\mathbb{R}$)

0: $L(y\lambda)$ paired with $L(-\overline{y\lambda})$ (which is possibly another copy of $L(y\lambda)$)

$$\text{Proposition 4.1} \rightsquigarrow ch_s \langle\cdot,\cdot\rangle_j = \sum_{y \in W_\lambda} \text{coeff of } q^{\frac{\ell(x)-\ell(y)-j}{2}} \text{ in } P^{\lambda,\delta}_{w_\lambda x, w_\lambda y} \times ch_s L(y\lambda)$$

$$\rightsquigarrow ch_s \langle\cdot,\cdot\rangle_{x\lambda+\delta t}\, e^{-\delta t} = \sum_{y \in W_\lambda} P^{\lambda,\delta}_{w_\lambda x, w_\lambda y}(1) ch_s L(y\lambda) \qquad \text{for } t > 0$$

$$\rightsquigarrow ch_s L(x\lambda) = \sum_{\substack{y_1 < \cdots < y_j = x \\ y_k\lambda\text{'s imaginary}}} (-1)^{j-1} \left(\prod_{i=2}^{j} P^{\lambda,\delta}_{w_\lambda y_i, w_\lambda y_{i-1}}(1)\right) \left(ch_s \langle\cdot,\cdot\rangle_{y_1\lambda+\delta t}\, e^{-\delta t}\right)$$

**Want:** Algorithm for computing $P^{\lambda,\delta}_{x,y}$.

The usual Kazhdan-Lusztig polynomials may be computed via $P_{x,x} = 1$, $P_{x,y} = 0$ when $x > y$, and by the recursive formulas:

a) $P_{w_\lambda x, w_\lambda y} = P_{w_\lambda xs, w_\lambda y}$ if $ys > y$ and $x, xs \geqslant y$, $s$ simple.

a') $P_{w_\lambda x, w_\lambda y} = P_{w_\lambda sx, w_\lambda y}$ if $sy > y$ and $x, sx \geqslant y$, $s$ simple.

b) If $y > ys$ then

$$q^c P_{w_\lambda xs, w_\lambda y} + q^{1-c} P_{w_\lambda x, w_\lambda y} = \sum_{z \in W_\lambda | zs > z} \mu(w_\lambda z, w_\lambda y) q^{\frac{\ell(z)-\ell(y)+1}{2}} P_{w_\lambda x, w_\lambda z} + P_{w_\lambda x, w_\lambda ys}$$

where $c = 1$ if $xs < x$, $c = 0$ if $xs > x$, and $\mu(w_\lambda z, w_\lambda y)$ is the multiplicity of $L(y\lambda)$ in $M(z\lambda)_1$.

**Theorem 4.2.** *(Y, "Signatures of invariant Hermitian forms on irreducible highest weight modules",* **Duke Math. J.***, to appear: Theorem 4.6.10) Letting $s = s_\alpha$ be a simple reflection, the signed Kazhdan-Lusztig polynomials are defined by the intial conditions $P^{\lambda,w}_{x,x} = 1$, $P^{\lambda,w}_{x,y} = 0$ for $x > y$ and the recursive formulas:*

a) $P^{\lambda,w}_{w_\lambda x, w_\lambda y} = sgn(-w\rho, x\alpha)\varepsilon(H_{x\alpha,-(\lambda,\alpha^\vee)}, xs) P^{\lambda,w}_{w_\lambda xs, w_\lambda y}$ *if $ys > y$ and $xs > x \geqslant y$*

a') $P^{\lambda,w}_{w_\lambda x, w_\lambda y} = sgn(-w\rho, \alpha)\varepsilon(H_{\alpha,(sx\lambda,\alpha^\vee)}, sx) P^{\lambda,w}_{w_\lambda sx, w_\lambda y}$ *if $sy > y$ and $sx > x \geqslant y$*

b) *If $x, y \in W_\lambda$ are such that $x < xs$ and $y > ys$ and $x > y$ then:*

$$\begin{aligned} &-(-1)^{\varepsilon((\lambda,\alpha^\vee)x\alpha)} P^{\lambda,w}_{w_\lambda xs, w_\lambda y}(q) + \operatorname{sgn}(\delta, x\alpha^\vee) q P^{\lambda,w}_{w_\lambda x, w_\lambda y}(q) \\ &= \sum_{z \in W_\lambda | z < zs} \operatorname{sgn}(\delta, z\alpha^\vee) a^{z\lambda,w}_{y,1} q^{\frac{\ell(z)-\ell(y)+1}{2}} P^{\lambda,w}_{w_\lambda x, w_\lambda z}(q) + \operatorname{sgn}(\delta, ys\alpha^\vee) P^{\lambda,w}_{w_\lambda x, w_\lambda ys}(q). \end{aligned}$$

The values of $\varepsilon(H_{\alpha,n}, w)$ are computed in "*The signature of the Shapovalov Form on Irreducible Verma Modules*", Representation Theory, 2004: Theorem 5.3.4 and Theorem 6.12.

## 5. Some Examples

Notation:

- $A(\lambda, w)$ where $\lambda \in \mathfrak{h}^*$ and $w \in W_\lambda$ is the alcove containing $\lambda + \delta t$ for $\delta \in w\mathcal{C}_0$ and small $t > 0$
- $\delta_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ is compact} \\ -1 & \text{if } \alpha \text{ is non-compact} \end{cases}$
- For an alcove $A$ and $\lambda \in \mathfrak{h}^*$, $R^A(\lambda) = ch_s M(\lambda)$ if $\lambda \in A$ is imaginary

**Example 1: $\mathfrak{g}_0 = \mathfrak{su}(2)$.** We have $\mathfrak{h} = \mathfrak{t}$. Let $\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1\}$ and let $\lambda_1$ be the corresponding fundamental weight.

Irreducible Verma modules: Choose $\lambda \in \mathfrak{h}^*$ so that $(\lambda, \alpha_1^\vee) \in (n, n+1)$ where $n \in \mathbb{Z}_{\geqslant 0}$. Then $\lambda \in A(n\lambda_1, w_0)$. The reducibility hyperplanes separating the alcove $aA_0$ containing $\lambda$ and $\tilde{a}A_0$ are $H_{\alpha_1,1}$, $H_{\alpha_1,2}$, ..., $H_{\alpha_1,n}$. In the setup of Theorem 2.1 we choose the path so that $r_1 = s_{\alpha_1,n}$, $r_2 = s_{\alpha_1,n-1}$, ..., $r_n = s_{\alpha_1,1}$. Suppose $S \subset \{1, 2, \ldots, n\}$ and $|S| \geqslant 2$. Then $\overline{r_{i_1}}C_{i_2-1}$ and $\overline{r_{i_1}}C_{i_2}$ lie in the Wallach region, and thus $\varepsilon(\overline{r_{i_1}}C_{i_2-1}, \overline{r_{i_1}}C_{i_2}) = 0$. Therefore $\varepsilon(S) = 0$ for $|S| \geqslant 2$. For our choice of path, note that $C_i \supset (n-i, n-i+1)$, whence $\varepsilon(\{i\}) = \varepsilon(C_{i-1}, C_i) = \varepsilon(H_{\alpha_1,n-i+1}, s_1) = \delta_{\alpha_1}^{n-i+1} = 1$ (see Lemma 5.2.17 or Theorem 6.12 of Y 2004). Substituting these values into Theorem 2.1:

$$\begin{aligned} R^{A(n\lambda_1,w_0)} = ch_s M(\lambda) &= \frac{\sum_{i=1}^n 2e^{\overline{r_i}r_i\lambda-\rho} + e^{\lambda-\rho}}{\prod_{\alpha\in\Delta^+(\mathfrak{p},\mathfrak{t})}(1-e^{-\alpha})\prod_{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})}(1-e^{-\alpha})} \\ &= \frac{\sum_{i=1}^n 2e^{\lambda-i\alpha_1-\rho} + e^{\lambda-\rho}}{1+e^{-\alpha_1}} \\ &= \frac{\sum_{i=1}^n e^{\lambda-(i-1)\alpha_1-\rho} + e^{\lambda-i\alpha_1-\rho}}{1+e^{-\alpha_1}} \\ &= e^{\lambda-\rho} + e^{\lambda-\rho-\alpha_1-\rho} + \cdots + e^{\lambda-(n-1)\alpha_1-\rho} + \frac{e^{\lambda-n\alpha_1-\rho}}{1+e^{-\alpha_1}}. \end{aligned}$$

Irreducible highest weight modules: Let $\lambda = -n\lambda_1$ for some $n \in \mathbb{Z}^+$. Since $\lambda$ is in the Wallach region, taking $n = 0$ in the above formula:

$$ch_s L(\lambda) = ch_s M(\lambda) = \frac{e^{\lambda-\rho}}{1+e^{-\alpha}}.$$

According to Theorem 4.2,

$$1 = P^{\lambda,w_0}_{w_0,w_0} = \text{sgn}(-w_0\rho, \alpha_1)\varepsilon(H_{\alpha_1,n}, s_1)P^{\lambda,w}_{w_0s_1,w_0} = \delta^n_{\alpha_1}P^{\lambda,w}_{w_0s_1,w_0} = P^{\lambda,w_0}_{w_0s_1,w_0}$$

by Lemma 5.2.17 or Theorem 6.12 of Y 2004. Substituting the values we have computed into Theorem 2.2:

$$\begin{aligned} ch_s L(s_1\lambda) &= R^{A(s_1\lambda,w_0)}(s_1\lambda) - P^{\lambda,w}_{w_0s_1,w_0}R^{A(\lambda,w_0)}(\lambda) \\ &= R^{A(n\lambda_1,w_0)}(s_1\lambda) - R^{A(-n\lambda_1,w_0)}(s_1\lambda - n\alpha_1) \\ &= R^{A(n\lambda_1,w_0)}(s_1\lambda) - R^{A(0\lambda_1,w_0)}(s_1\lambda - n\alpha_1) \\ &= \left(e^{s_1\lambda-\rho} + \cdots + e^{s_1\lambda-(n-1)\alpha_1-\rho} + \frac{e^{s_1\lambda-n\alpha_1-\rho}}{1+e^{-\alpha_1}}\right) - \left(\frac{e^{s_1\lambda-n\alpha_1-\rho}}{1+e^{-\alpha_1}}\right) \\ &= e^{s_1\lambda-\rho} + e^{s_1\lambda-\alpha_1-\rho} + \cdots + e^{s_1\lambda-(n-1)\alpha_1-\rho}. \end{aligned}$$

**Example 2: $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$.** We may proceed as in the previous example, but substitute $\delta_{\alpha_1} = -1$ instead of $\delta_{\alpha_1} = 1$.

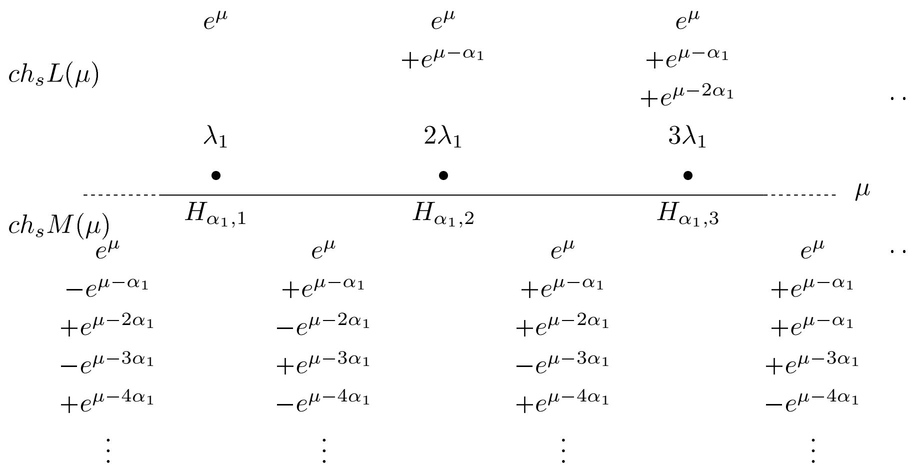


FIGURE 1. $\mathfrak{su}(2)$

Irreducible Verma modules: For $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \alpha_1^\vee) \in (n, n+1)$ where $n \in \mathbb{Z}_{\geqslant 0}$:

$$\begin{aligned}
ch_sM(\lambda) &= R^{A(n\lambda_1,w_0)}(\lambda) = \frac{\sum_{i=1}^{n}(-1)^{n-i+1}2e^{\overline{r_i}r_i\lambda-\rho} + e^{\lambda-\rho}}{\prod\limits_{\alpha\in\Delta^+(\mathfrak{p},\mathfrak{t})}(1-e^{-\alpha})\prod\limits_{\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})}(1+e^{-\alpha})} \\
&= \frac{\sum_{i=1}^{n}(-1)^i 2e^{\lambda-i\alpha_1-\rho} + e^{\lambda-\rho}}{1-e^{-\alpha_1}} \\
&= e^{\lambda-\rho} - e^{\lambda-\rho-\alpha_1-\rho} + \cdots + (-1)^{n-1}e^{\lambda-(n-1)\alpha_1-\rho} + (-1)^n\frac{e^{\lambda-n\alpha_1-\rho}}{1-e^{-\alpha_1}}.
\end{aligned}$$

Irreducible highest weight modules: For $\lambda = -n\lambda_1$ where $n \in \mathbb{Z}^+$:

$$ch_sL(\lambda) = ch_sM(\lambda) = \frac{e^{\lambda-\rho}}{1-e^{-\alpha_1}}.$$

Since $P^{\lambda,w_0}_{w_0s_1,s_0} = (-1)^n$, we have

$$\begin{aligned}
ch_sL(s_1\lambda) &= R^{A(s_1\lambda,w_0)}(s_1\lambda) - P^{\lambda,w_0}_{w_0s_1,w_0}R^{A(\lambda,w_0)}(\lambda) \\
&= \left(\sum_{i=0}^{n-1}(-1)^i e^{s_1\lambda-i\alpha_1-\rho} + (-1)^n\frac{e^{s_1\lambda-n\alpha_1-\rho}}{1-e^{-\alpha_1}}\right) - (-1)^n\left(\frac{e^{s_1\lambda-n\alpha_1-\rho}}{1-e^{-\alpha_1}}\right) \\
&= e^{s_1\lambda-\rho} - e^{s_1\lambda-\alpha_1-\rho} + \cdots + (-1)^{n-1}e^{s_1\lambda-(n-1)\alpha_1-\rho}.
\end{aligned}$$

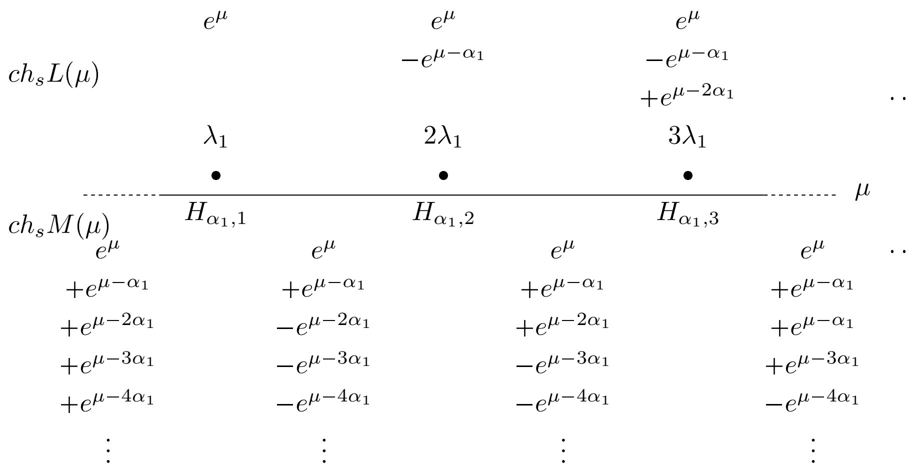


Figure 2. $\mathfrak{sl}(2)$