Representations with Non-Integral Infinitesimal Character Atlas of Lie Groups Workshop 2006

Annegret Paul

13 July 2006

1 Setup

Let (G, τ) and $({}^\vee G, {}^\vee\tau)$ be given. Here G is an algebraic group with dual group ${}^\vee G$, τ an outer automorphism determining an inner class of real forms, $\delta \in Aut(G)$ a strong real form in the inner class determined by τ, chosen as in [3], and ${}^\vee\tau$, ${}^\vee\delta$ the corresponding objects for ${}^\vee G$. (Recall: δ is the unique involution in the chosen inner class fixing the chosen pinning; e. g., if τ is trivial then δ is the identity automorphism.) Form the extended groups $G^\Gamma = G \rtimes \Gamma = G \cup G\delta$ and ${}^\vee G^\Gamma = {}^\vee G \rtimes \Gamma = {}^\vee G \cup {}^\vee G\,{}^\vee\delta$. Recall that a *strong involution* of G is an element $x \in G\delta$ such that $x^2 \in Z(G)$, and $\theta_x = int(x)$ is the Cartan involution of the corresponding real group $G(\mathbb{R})$.

2 Integral L-Data

Recall from [1] (see also [2]) the sets of Integral L-Data parametrizing irreducible representations with integral infinitesimal character. These are septuples as follows:

$$(\mathcal{S}; \lambda) = (x, H, B, y, {}^d H, {}^d B; \lambda) \tag{1}$$

where x is a strong involution of G, $H \subset B$ a Cartan and a Borel subgroup of G such that H is θ_x-stable; and $y, {}^\vee H$, and ${}^\vee B$ are corresponding objects on the dual side. The data determine an isomorphism

$$\zeta : {}^d H \rightarrow {}^\vee H \tag{2}$$

which also identifies ${}^d\mathfrak{h}$ with ${}^\vee\mathfrak{h} = \mathfrak{h}^*$ taking the positive root system ${}^d\Psi^+$ corresponding to ${}^d B$ to the set ${}^\vee\Psi^+$ of coroots of the system of positive roots Ψ^+ determined by B.

The parameter λ is then an element of ${}^d\mathfrak{h} \simeq {}^\vee\mathfrak{h}$ such that $\exp(2\pi i\lambda) = y^2$, and dominant regular with respect to Ψ^+. The involutions θ_x and ${}^d\theta_y$ must be compatible in the sense that the corresponding involutions of $\mathfrak{h}$ and ${}^\vee\mathfrak{h}$ must satisfy ${}^d\theta_y = -\,{}^t\theta_x$. We make all these identifications and replace the superscripts d by $\vee$ everywhere. Conjugacy classes by $G \times {}^\vee G$ of sets of integral L-data correspond in a one-one fashion to irreducible admissible representations with integral regular infinitesimal character of strong real forms in the given inner class.

Since all pairs $H \subset B$ are conjugate by G (and similarly on the dual side), we may fix H, B, ${}^\vee H$, and ${}^\vee B$ and look at pairs (x, y), up to conjugation by $H \times {}^\vee H$ instead (see [3] for details). These pairs parametrize translation families of representions; giving λ amounts to choosing a particular representation in the family, with infinitesimal character λ.

Which representation is it? The element x specifies a real form $G(\mathbb{R})$, along with a conjugacy class of Cartan subgroups $H(\mathbb{R})$. To get a representation of $G(\mathbb{R})$, we need a character Λ of (a double cover of) $H(\mathbb{R})$; the representation is then obtained by parabolic induction. We have $\lambda = d\Lambda$. If $H(\mathbb{R})$ is connected, this determines the character uniquely. Otherwise, the element y determines it on the $\mathbb{Z}/2\mathbb{Z}$ factors. Details will be spelled out in the Dictionary (this is work in progress). The Atlas software produces the x's and y's using the 'kgb' command, and the compatible pairs (x, y) using the 'block' command.

Example 1 *$SL(2,\mathbb{R})$. We have $G = SL(2,\mathbb{C})$, ${}^\vee G = SO(3,\mathbb{C})$, which we think of as the isometry group of the form on $\mathbb{C}^3$ given by* $M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, *$\delta$ and ${}^\vee\delta$ trivial so that we can think of x and y as elements of G and ${}^\vee G$, rather than the extended groups. We choose the diagonal Cartan subgroups. Let* $t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2,\mathbb{C})$, *$s = diag(-1,-1,1) \in SO(3,\mathbb{C})$. Then up to conjugacy, a complete list of the pairs (x, y) for x giving the split real form $SL(2,\mathbb{R})$ are (t, M), $(-t, M)$, (w, I), and (w, s). The elements t and $-t$ give the compact Cartan which is connected, so only one y (giving the split Cartan on the dual side) is matched with these parameters. For $x = w$ we get the split Cartan $\simeq \mathbb{R}^\times$ of $SL(2,\mathbb{R})$, so there are two characters with the same differential, distinguished by the two matching choices I, s for y. If we fix λ as an integral multiple of ρ, the four representations of $SL(2,\mathbb{R})$ are the two discrete series, and the two principal series at infinitesimal character λ. If λ is an odd multiple of ρ, then (w, I) corresponds to the non-spherical principal series and (w, s) to the spherical one; if λ is even, these are switched.*

3 Non-Integral L-Data

According to [1], general (not necessarily integral) L-data are septuples

$$(\mathcal{S};\lambda) = (x, H, P, y, {}^\vee H, {}^\vee P; \lambda), \tag{3}$$

where $(x, H, {}^\vee H)$ is as in (1) (we assume we have chosen an isomorphism ζ as in (2) and made the appropriate identifications), $y \in {}^\vee G\, {}^\vee\delta$, $y^2 \in {}^\vee H$, ${}^\vee\theta_y$ normalizes ${}^\vee H$, θ_x and ${}^\vee\theta_y$ are compatible as above, ${}^\vee P$ is a positive set of roots for ${}^\vee G_{y^2} = Cent_{{}^\vee G}(y^2)$, P is the set of roots of G dual to ${}^\vee P$, $\lambda \in {}^\vee\mathfrak{h} = \mathfrak{h}^*$ is such that $\exp(2\pi i\lambda) = y^2$, and λ is dominant with respect to P. If λ is also regular and $\Psi = \Delta(\mathfrak{h}, \mathfrak{g})$ then $P = \{\alpha \in \Psi :< \lambda, {}^\vee \alpha > \in \mathbb{Z}_{>0}\}$. As in the integral case, conjugacy classes by $G \times {}^\vee G$ of these sets of data parametrize irreducible representations of all strong real forms in the given inner class with regular (not necessarily integral) infinitesimal character.

As before, fix H, $B \leftrightarrow \Psi^+$, ${}^\vee H$, and ${}^\vee B \leftrightarrow {}^\vee\Psi^+$. Then given (x, y), $P \subset \Psi^+$ and ${}^\vee P \subset {}^\vee\Psi^+$ are uniquely determined. Our parameters will be triples (x, y, λ) which specify representations, or pairs (x, y) which give translation families of representations.

3.1 First Approach: Use Integral Data for G

Assume that λ is real, i. e., $\lambda \in X^*(H) \otimes_{\mathbb{Z}} \mathbb{R}$, and regular. (The assumption that λ is real is not essential; we make it because this is the case we are most interested in, and so that we have a linear order on the elements. We could of course easily define such an order on $\mathbb{C}$.) If we require that λ is strictly dominant with respect to Ψ^+ (rather than just P), then representations will be in one-one correspondence with triples (x, y, λ) up to conjugation by $H \times {}^\vee H$.

Remark 2 *DAV: Although this does indeed give a one-one correspondence, it is a bad idea (mathematically) to require λ to be dominant with respect to the whole root system. Think about how else to account for equivalences by Weyl group elements taking P into Ψ^*...*

The parameters x are as for integral data, and hence the Atlas software computes them (using 'kgb'). Given a fixed x, what are the possible y and λ?

Let (x, y_I) be an integral pair, i. e., a pair giving an integral L-datum (listed in Atlas using the 'block' command). Any element y normalizing ${}^\vee H$ and compatible with x (i. e., mapping to the same twisted involution as y_I) is of the form $y = ty_I$ for some $t \in {}^\vee H$. Then

$$y^2 = ty_I t y_I = t\, {}^\vee\theta(t) y_I^2. \tag{4}$$

Now $t\,{}^\vee\theta(t) = \exp(X)$ for some $X \in (1 + {}^\vee\theta)\,{}^\vee\mathfrak{h} = {}^\vee\mathfrak{h}^{{}^\vee\theta}$. Consequently, the infinitesimal characters λ allowed (still assuming x fixed) are those of the form

$$\lambda = \lambda_0 + \lambda_I, \text{ where } \lambda_0 \in {}^\vee\mathfrak{h}^{{}^\vee\theta} \text{ and } \lambda_I \text{ integral.} \tag{5}$$

Proposition 3 *Fix $x \in \mathcal{X}$ (Fokko's one-sided parameter space [3]).*

1. *The possible infinitesimal characters of representations associated to x are those of the form $\lambda = \lambda_0 + \lambda_I$, where $\lambda_0 \in {}^\vee\mathfrak{h}^{{}^\vee\theta}$, λ_I is integral, i. e., $\exp(2\pi i\lambda_I) = y_I^2$ for some integral pair (x, y_I), and λ is regular dominant.*

2. *Suppose $\lambda = \lambda_0 + \lambda_I$ is as above (the decomposition is not unique; choose one). Write $t_{\lambda_0} = \exp(\pi i\lambda_0)$. If ${}^\vee G$ has trivial center then the representations with infinitesimal character λ associated to x are given by the pairs $(x, t_{\lambda_0}y)$ such that (x, y) is an integral pair (i. e., fixed x, vary y).*

Example 4 *$SL(2,\mathbb{R})$. If $x = \pm t$ then ${}^\vee\theta(X) = -X$ for $X \in {}^\vee\mathfrak{h}$, so ${}^\vee\mathfrak{h}^{{}^\vee\theta} = \{0\}$, and there are no non-integral infinitesimal characters (as expected since $H(\mathbb{R})$ is compact). If $x = w$ then $H(\mathbb{R}) \simeq \mathbb{R}^\times$, ${}^\vee\theta = 1$, ${}^\vee\mathfrak{h}^{{}^\vee\theta} = {}^\vee\mathfrak{h}$, and all (dominant) infinitesimal characters are allowed. Take $\lambda = \nu\rho$ for some $\nu > 0$, $\lambda = diag(\nu, -\nu, 0) \in {}^\vee\mathfrak{h}$. Then $t_{\lambda_0} = diag(e^{\pi i\nu}, e^{-\pi i\nu}, 1)$, so we get*

$$y_1 = t_{\lambda_0}I = diag(e^{\pi i\nu}, e^{-\pi i\nu}, 1) \text{ for the spherical principal series,} \tag{6}$$

$$y_2 = t_{\lambda_0}s = diag(-e^{\pi i\nu}, -e^{-\pi i\nu}, 1) = diag(e^{\pi i(\nu+1)}, e^{-\pi i(\nu+1)}, 1) \text{ for the nonspherical series.} \tag{7}$$

Notice that these reduce to the elements of Example 1 if ν is an integer.

Remark 5 *If the center of ${}^\vee G$ is not trivial, there are fewer representations than there are integral pairs. Working guess for the general case (this works for $SO(2,1)$, $SO(3,2)$ and a split torus, e. g.): Fix λ as in part 1 of Proposition 3 and a particular decomposition $\lambda = \lambda_0 + \lambda_I$, and write $z = \exp(2\pi i\lambda_I) \in Z({}^\vee G)$. Then the representations with infinitesimal character λ which are associated to x are given by the pairs $(x, t_{\lambda_0}y)$ such that (x, y) is an integral pair with $y^2 = z$.*

Example 6 *$SO(2,1)$. This is the dual picture to $SL(2,\mathbb{R})$. The principal series are given by pairs (x, y) with $x = M$, and there are four choices for $y : \pm I, \pm t$. The first two satisfy $y^2 = I$, the second two $y^2 = -I$. For a given infinitesimal character $\nu\rho$, there are two non-isomorphic principal series of $SO(2,1)$ parametrized by*

$$(x, y_1) = \left(M, diag(e^{\pi i\frac{\nu}{2}}, e^{-\pi i\frac{\nu}{2}})\right) \tag{8}$$

and

$$(x, y_2) = \left(M, diag(-e^{\pi i \frac{\nu}{2}}, -e^{-\pi i \frac{\nu}{2}})\right), \tag{9}$$

which we can get either by multiplying $\pm I$ *by* $t_{\lambda_0} = diag\left(e^{\pi i \frac{\nu}{2}}, e^{-\pi i \frac{\nu}{2}}\right)$*, or by multiplying* $\pm t$ *by* $t_{\lambda_0} = diag\left(e^{\pi i \frac{\nu+1}{2}}, e^{-\pi i \frac{\nu+1}{2}}\right)$.

3.2 Second Approach: Reduce to a Smaller Group E

Idea: An infinitesimal character λ determines (as above) the sets P, ${}^\vee P$. Then $(X, P, {}^\vee X, {}^\vee P)$ is a based root datum. The corresponding group E (an endoscopic group for G) is not necessarily a subgroup of G; however, the dual group ${}^\vee E$ is the subgroup of ${}^\vee G$ corresponding to the subrootsystem ${}^\vee P$ of ${}^\vee \Psi^+$. The x's for G may be identified with certain x_E for E, and integral pairs (x_E, y) for E should then parametrize representations for G with infinitesimal character λ. Details need to be worked out; in particular this identification $x \mapsto x_E$, eliminating duplication, dealing with centers, and keeping track of the correct inner class and real forms in E. Stay tuned...

References

[1] Jeffrey Adams, *Parameters for Representations of Real Groups*, Notes for a series of talks given during the second Atlas workshop at AIM in Palo Alto, CA, July 2004, updated for workshop July 2005 (available at http://atlas.math.umd.edu).

[2] Jeffrey Adams and David Vogan, *Lifting of Characters and Harish-Chandra's Method of Descent*, preprint.

[3] Fokko du Cloux, *Combinatorics for the representation theory of real reductive groups*, Notes for a series of talks during the third meeting of the *Atlas of Lie Groups* workshop at AIM, July 2005 (available at http://atlas.math.umd.edu).