PARAMETRIZING REPRESENTATIONS OF K AFTER DAVID VOGAN

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Abstract. Let G be the real points of a complex connected reductive algebraic group $G_{\mathbb{C}}$. Let K be a maximal compact subgroup of G. We parametrize the set $\hat{K}$ of irreducible representations of K. The goal is to describe an algorithm for such a parametrization and to implement it as a package of the *Atlas of Lie groups and representations* software developed by Fokko du Cloux.

1. Introduction

Let $G_{\mathbb{C}}$ be a complex connected reductive algebraic group and G the set of real points of $G_{\mathbb{C}}$. Let θ be the Cartan involution of G which extends to an involution of $G_{\mathbb{C}}$. We denote by K a maximal compact subgroup of G. Then $G_{\mathbb{C}}^{\theta} = K_{\mathbb{C}}$ the complexification of K. We identify G with a root datum $(X^*, \Delta^+, X_*, (\Delta^+)^{\vee})$. So we would like to describe $\hat{K}$ in term of X^*.

Let H be a θ-stable Cartan subgroup of G and $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ the corresponding root system of $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G_{\mathbb{C}})$. Δ_{im} and Δ_{re} will denote the sets of imaginary and real roots in $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ respectively. Then $X^*(H_{\mathbb{C}})$ the character lattice of $H_{\mathbb{C}}$ is isomorphic to X^*. Finally, let $T = K \cap H$ a compact, possibly disconnected torus. We have the following lemma:

Lemma 1.1. *The set of characters of T is isomorphic to* $\frac{X^*(H_{\mathbb{C}})}{(1-\theta)X^*(H_{\mathbb{C}})}$.

Let ρ be half the sum of positive roots in $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and fix

$$\lambda \in \frac{X^*(H_{\mathbb{C}}) + \rho}{(1-\theta)X^*(H_{\mathbb{C}})}$$

We want $I(H, \Delta^+_{im}, \Delta^+_{re}, \lambda)$ to correspond to a virtual representation of G restricted to K. Consider a discrete series representation of G restricted to K for example. The main idea is to describe irreducible representations of K as lowest K-types.

Let $2\rho^{\vee}_{\mathbb{R}}$ be the sum of positive real coroots. Define

$$\Delta_T = \{\text{roots } \perp 2\rho^{\vee}_{\mathbb{R}}\}.$$

Key words and phrases.
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Then Δ_T is a θ-stable root system corresponding to a real Levi subgroup L of G with H fundamental in L. Fix Δ_T^+ containing Δ_{im}^+ and consider the set

$$\mathfrak{L} = \{\lambda \in \frac{X^*(H_{\mathbb{C}}) + \rho}{(1-\theta)X^*(H_{\mathbb{C}})}\}$$

such that

(1) λ is weakly dominant for Δ_T^+

(2) if α is a simple imaginary root and $\langle \lambda, \alpha^\vee \rangle = 0$ then α is non-compact.

(3) if β is a simple real root then $\langle \lambda, \beta^\vee \rangle$ is odd.

(1) ensures that $I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$ is a standard limit representation of G restricted to K.

(2) ensures that $I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$ is non-zero.

(3) ensures that $I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$ cannot be written using a more compact Cartan subgroup and Hecht-Schmid identities.

Given that λ is defined modulo $(1-\theta)X^*(H_{\mathbb{C}})$ to see that property (3) is well-defined one only needs to consider that for $\gamma \in X^*(H_{\mathbb{C}})$,

$$\langle \gamma - \theta\gamma, \beta^\vee \rangle = \langle \gamma, \beta^\vee - \theta\beta^\vee \rangle = \langle \gamma, 2\beta^\vee \rangle$$

which is even.

The main theorem is:

Theorem 1.2. *If $\lambda \in \mathfrak{L}$ then $I(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$ has a unique lowest K-type $\mu(H, \Delta_{im}^+, \Delta_{re}^+, \lambda)$. Hence*

$$\hat{K} = \coprod_{[\text{ H mod conjugation by K}]} \quad \coprod_{[\Delta_{im}^+ \text{ mod conjugation by } W(\mathrm{G},H)]} \mu(H, \Delta_{im}^+, \Delta_{re}^+, \lambda).$$

To see why conjugation under Δ_{re} does not interfere with this parametrization one has to observe that if $\lambda \simeq \lambda'$ for λ and $\lambda' \in \mathfrak{L}$ then for $\beta \in \Delta_{re}$

$$\lambda' = s_\beta(\lambda) - [\rho_{\mathbb{R}} - s_\beta(\rho_{\mathbb{R}})] = \lambda - (\langle \lambda, \beta^\vee \rangle + 1)\beta = \lambda - 2m\beta \text{ whith } m \in \mathbb{Z}$$

But $2m\beta = m\beta - \theta(m\beta) = m(1-\theta)\beta$.

2. Algorithm

To follow

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