Unitarity of non-spherical principal series

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Minimal Principal Series

- G: a real split semisimple Lie group
- θ: Cartan involution; $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: Cartan decomposition of $\mathfrak{g}$
- $\mathfrak{a}$: maximal abelian subspace of $\mathfrak{p}$, $A = \exp_G(\mathfrak{a})$, $M = Z_K(\mathfrak{a})$
- $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$: the set of restricted roots
- (δ, V^δ): an irreducible tempered unitary representation of M
- $P = MAN$: a *minimal parabolic* in G with Levi factor MA
- ν: a *real* character of A, *strictly dominant* for N

$\Rightarrow$ $X_P(\delta, \nu) = \mathrm{Ind}_P^G(\delta \otimes \nu)$ **a minimal principal series for** G

Langlands quotient representation

If $\bar{P}$ is the opposite parabolic, there is an intertwining operator

$$A = A(\bar{P} \colon P \colon \delta \colon \nu) \colon X_P(\delta, \nu) \longrightarrow X_{\bar{P}}(\delta, \nu).$$

Define **the Langlands quotient representation** to be the closure of the image of this operator:

$$\bar{X}(\delta, \nu) = \overline{\mathrm{Im}(A(\bar{P} \colon P \colon \delta \colon \nu))}.$$

$\bar{X}(\delta, \nu)$ is the unique irreducible quotient of $X_P(\delta, \nu)$.

Unitarity of $\bar{X}_P(\delta, \nu)$

- $\bar{X}_P(\delta \otimes \nu)$ has **a non-degenerate invariant Hermitian form** if and only if there exists $\omega \in K$ satisfying

$$\omega P \omega^{-1} = \bar{P} \qquad \omega \cdot \delta \simeq \delta \qquad \omega \cdot \nu = -\nu.$$

- Every non-degenerate invariant Hermitian form on $\bar{X}_P(\delta \otimes \nu)$ is a real multiple of the form induced by the Hermitian operator

$$B = \delta(\omega) \circ R(\omega) \circ A(\bar{P} : P : \delta : \nu)$$

from $X_P(\delta \otimes \nu)$ to $X_P(\delta \otimes -\nu)$.

$\bar{X}_P(\delta \otimes \nu)$ is unitary if and only if B is semidefinite

Computing the signature of B

This is a very hard problem. Two reductions are possible:

1^{st} reduction: a K-type by K-type calculation

For all μ in the principal series, we get an operator

$$R_\mu(\omega, \nu)\colon\ \mathrm{Hom}_K(E_\mu, X_P(\delta \otimes \nu)) \to \mathrm{Hom}_K(E_\mu, X_P(\delta \otimes -\nu))$$

which, by Frobenius reciprocity and the minimality of P, becomes

$$R_\mu(\omega, \nu)\colon\ \mathrm{Hom}_M(E_\mu\,|_M, V^\delta) \to \mathrm{Hom}_M(E_\mu\,|_M, V^\delta).$$

Computing the signature of B

2^{nd} reduction: a rank-one reduction

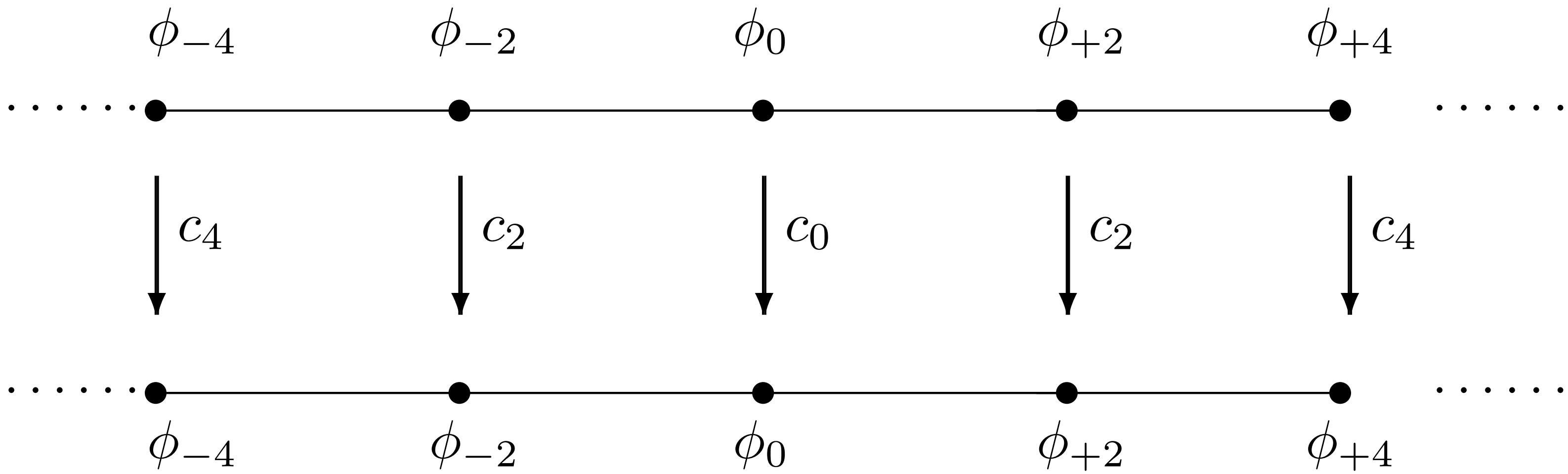
- Decompose ω into a product of simple reflections: $w = \prod_{i=1}^{l} s_{\alpha_i}$
- By Gindikin-Karpelevic, $R_\mu(\omega, \nu)$ decomposes accordingly:

$$\boxed{R_\mu(\omega, \nu) = \prod_{i=1}^{l} R_\mu(s_{\alpha_i}, \gamma_i)}$$

- A factor $R_\mu(s_{\alpha_i}, \gamma_i)$ is induced from the corresponding intertwining operator for the $SL(2, \mathbb{R})$ associated to α_i
- The intertwining operator for $SL(2, \mathbb{R})$ is known, so we get explicit formulas for the factors $R_\mu(s_{\alpha_i}, \gamma_i)$

$SL(2,\mathbb{R})$: the spherical case

The intertwining operator $X_P(triv \otimes \gamma) \to X_P(triv \otimes -\gamma)$ acts by:

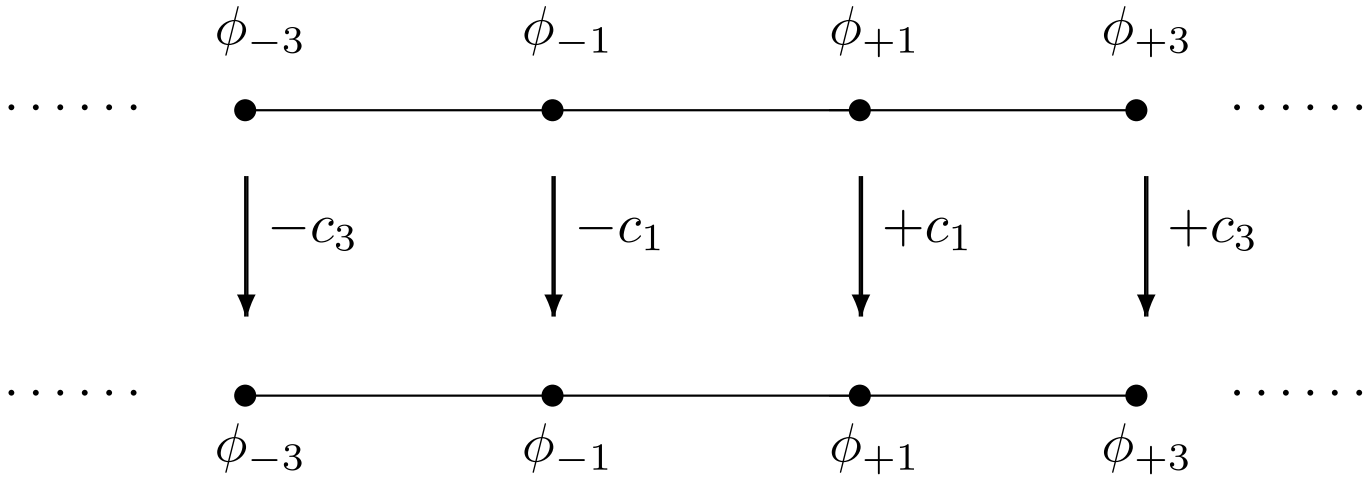


We can normalize the operator so that $c_0 = 1$ and

$$c_{2n} = \frac{\Pi_{j=1}^{n}((2j-1) - \langle \gamma, {}^\vee\alpha \rangle)}{\Pi_{j=1}^{n}((2j-1) + \langle \gamma, {}^\vee\alpha \rangle)} \qquad \forall\, n \geq 1$$

$SL(2,\mathbb{R})$: the non spherical case

The intertwining operator $X_P(sign \otimes \gamma) \rightarrow X_P(sign \otimes -\gamma)$ acts by:



We can normalize the operator so that $c_1 = 1$ and

$$c_{2n+1} = \frac{(2-\gamma)(4-\gamma)\cdots(2n-\gamma)}{(2+\gamma)(4+\gamma)\cdots(2n+\gamma)} \qquad \forall\, n \geq 1$$

G split: the spherical case

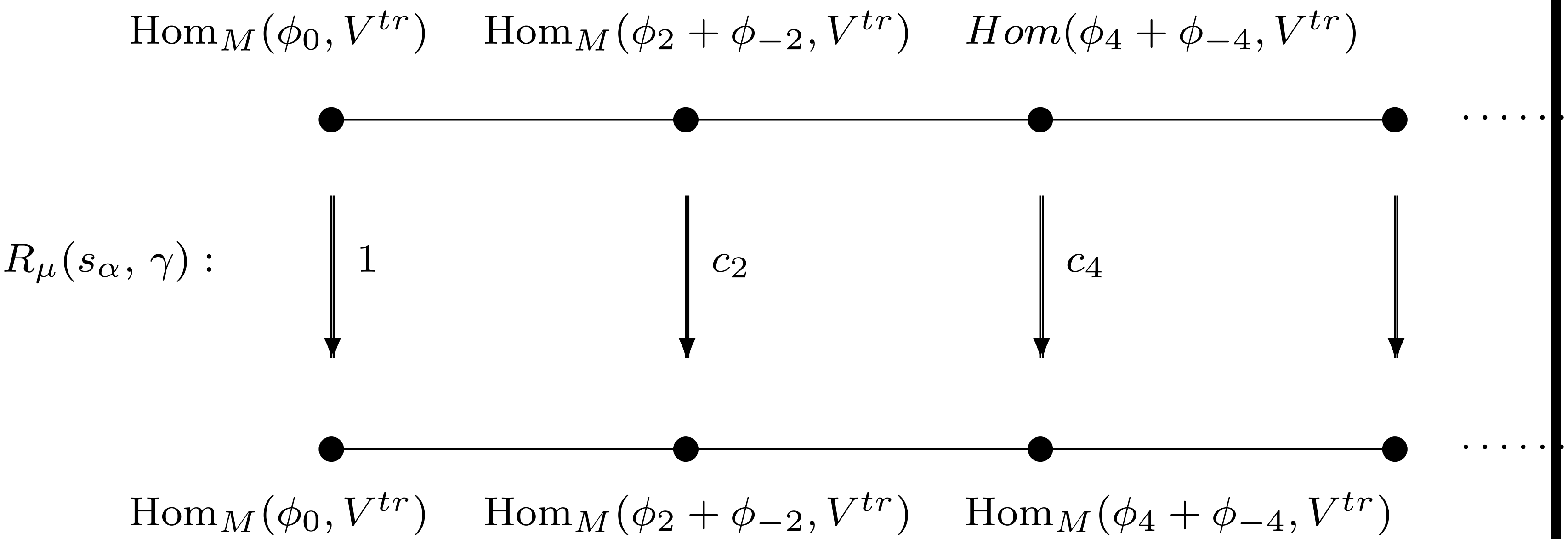
- The operator $R_\mu(\omega,\nu) :\ = (E_\mu^*)^M \to (E_\mu^*)^M$ decomposes in operators corresponding to simple reflections
- Every factor $R_\mu(s_\alpha,\gamma)$ is again an operator on $(E_\mu^*)^M$
- If $\mu\mid_{K^\alpha} = \bigoplus_{j\in\mathbb{Z}} \phi_j$ is the decomposition of μ w.r.t. the $SO(2)$-subgroup K^α attached to α, then

$$\boxed{(E_\mu^*)^M = \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}_M(\phi_{2n}+\phi_{-2n}, V^{triv})}$$

  is the decomposition of $(E_\mu^*)^M$ in MK^α-invariant subspaces

- $R_\mu(s_\alpha,\gamma)$ preserves this decomposition, and acts on $\mathrm{Hom}_M(\phi_{2n}+\phi_{-2n}, V^{triv})$ by the scalar c_{2n}

The operator $R_\mu(s_\alpha, \gamma)$, for δ trivial



$R_\mu(s_\alpha, \gamma)$ **depends on the decomposition of** μ **w.r.t** K^α

$R_\mu(s_\alpha,\gamma)$, for δ trivial and μ petite

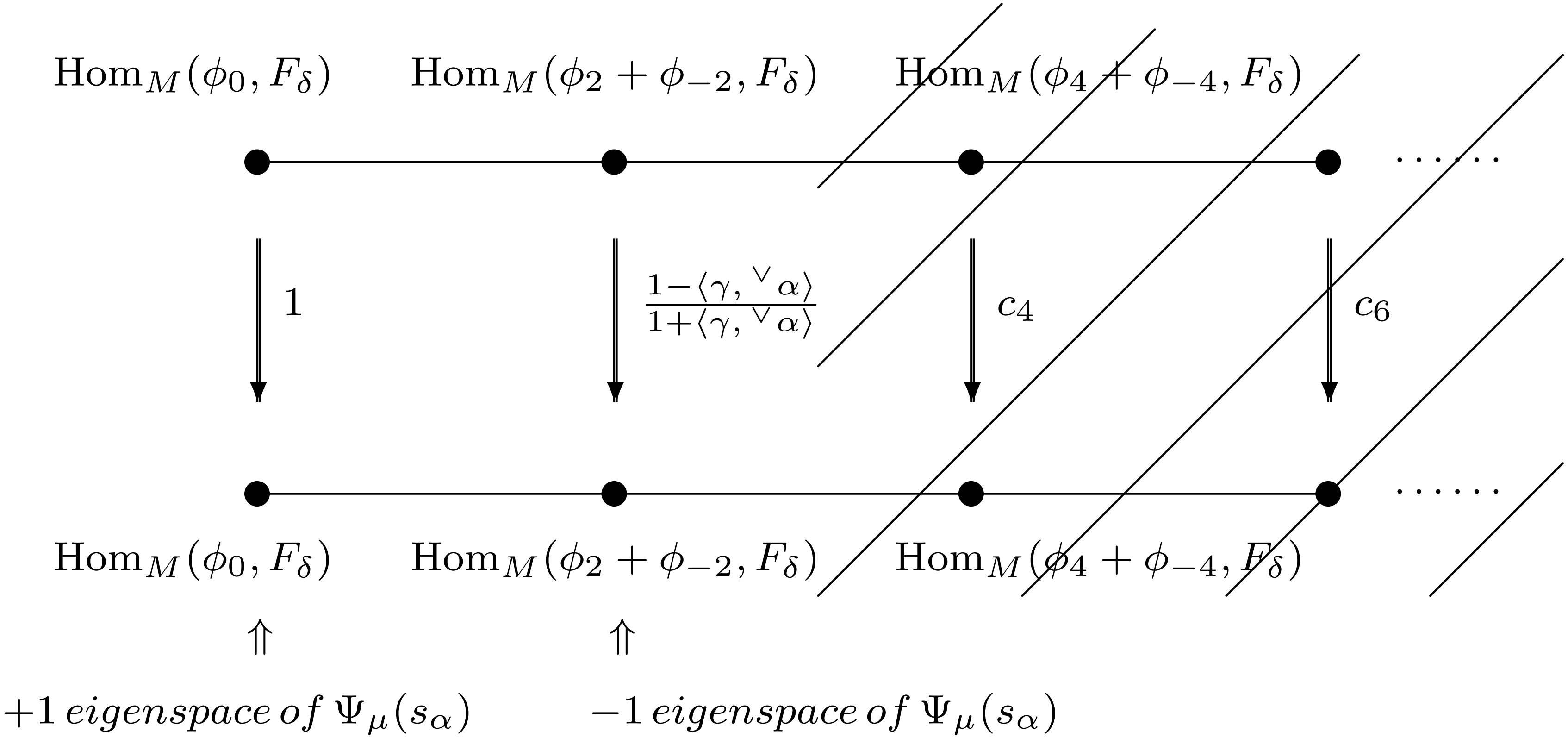
- If μ is petite,
$(E_\mu^*)^M = \mathrm{Hom}_M(\phi_0,\, V^{tr}) \oplus \mathrm{Hom}_M(\phi_{-2}+\phi_{+2},\, V^{tr})$
- $(E_\mu^*)^M = \mathrm{Hom}_M(E_\mu, V^{tr})$ carries a representation Ψ_μ of W,

$$\mathrm{Hom}_M(\phi_0,\, V^\delta) \equiv \text{ (+1)-eigenspace of } \Psi_\mu(s_\alpha)$$
$$\mathrm{Hom}_M(\phi_{-2}+\phi_{+2},\, V^\delta) \equiv \text{ (−1)-eigenspace of } \Psi_\mu(s_\alpha)$$

- $R_\mu(s_\alpha,\gamma)$ acts by c_0 on the $(+1)$-eigenspace of $\Psi_\mu(s_\alpha)$, and by c_2 on the (-1)-eigenspace of $\Psi_\mu(s_\alpha)$. This gives:

$$\boxed{R_\mu(s_\alpha,\gamma) = \left(\frac{c_0+c_2}{2}\right) Id + \left(\frac{c_0-c_2}{2}\right)\Psi_\mu(s_\alpha)}$$

$R_\mu(s_\alpha, \gamma)$, for δ trivial and μ petite



$$R_\mu(s_\alpha,\ \gamma) = \begin{cases} +1 & \textbf{on the } (+1)\textbf{-eigenspace of } \Psi_\mu(s_\alpha) \\ \frac{1-\langle\gamma,\check{\alpha}\rangle}{1+\langle\gamma,\check{\alpha}\rangle} & \textbf{on the } (-1)\textbf{-eigenspace of } \Psi_\mu(s_\alpha) \end{cases}$$

Relevant W-types

- In the p-adic case, the spherical representation $\bar{X}(\nu)$ is unitary if and only if the operator $R_\tau(\omega,\nu)$ is positive semidefinite, for every representation τ of W
- Dan and Dan have determined a subset of W-representations (the **relevant set**) that detects unitarity:

  $\bar{X}(\nu)$ is unitary $\Leftrightarrow$ $R_\tau(\omega,\nu)$ is pos. semidef. for all τ relevant
- If μ is petite and W acts on $(E_\mu^*)^M$ by τ, then

$$\boxed{\text{the real operator } R_\mu(\omega,\nu) = \text{ the p-adic operator } R_\tau(\omega,\nu)}$$

Relevant W-types

A_{n-1}	(n-k, k), j ≤ n/2
B_n, C_n	(n-j, j)×(0), j ≤ n/2, (n-k)×(k), k=0...n
D_n	(n-j, j)×(0), j ≤ n/2, (n-k)×(k), k < n/2
	(n/2)×(n/2)$^{\pm}$ if n is even
F_4	1_1, 2_3, 8_1, 4_2, 9_1,
E_6	1_p, 6_p, 20_p, 30_p, 15_q,
E_7	1_a, $7'_a$, 27_a, $56'_a$, $21'_b$, 35_b, 105_b,
E_8	1_x, 8_z, 35_x, 50_x, 84_x, 112_z, 400_z, 300_x, 210_x.

Relevant K-types

Barbasch has proved that for every G real and split, there is a set of petite K-types such that the $(E_\mu^*)^M$'s realize all the relevant W-representations (**relevant K-types**).

These relevant petite K-types provide **non-unitarity certificates** for spherical Langlands quotients:

> $\bar{X}(triv. \otimes \nu)$ **is unitary only if** $R_\mu(\omega, \nu)$ **is positive semidefinite for every relevant K-type** μ

For relevant K-types, the operator $R_\mu(\omega, \nu)$ can "easily" be constructed by means of Weil groups computations.

Classical groups

The spherical unitary dual for a **classical** split reductive group is independent of whether the field is real or p-adic. Therefore

$\bar{X}(triv \otimes \nu)$ is unitary in the real case

$\Updownarrow$

$\bar{X}(\nu)$ is unitary in the p-adic case

$\Updownarrow$

$R_\tau(\omega, \nu)$ is positive semidefinite, for all relevant W-types τ

$\Updownarrow$

$R_\mu(\omega, \nu)$ is positive semidefinite, for all relevant K-types μ

Spherical - Not Spherical

S: $R_\mu(\omega,\nu)$ is an endomorphism of $\mathrm{Hom}_M(E_\mu, V^{triv})$, and this space has a W-representation

N-S: $R_\mu(\omega,\nu)$ is an endomorphism of $\mathrm{Hom}_M(E_\mu, V^{\delta})$, and this space does not have a W-representation

S: Every factor $R_\mu(s_\alpha,\gamma)$ acts on $\mathrm{Hom}_M(E_\mu, V^{triv})$

N-S: $R_\mu(s_\alpha,\gamma)$ carries $\mathrm{Hom}_M(E_\mu, V^{\delta})$ into $\mathrm{Hom}_M(E_\mu, V^{s_\alpha\cdot\delta})$

S: The action of $R_\mu(s_\alpha,\gamma)$ is "easy"

N-S: the action of $R_\mu(s_\alpha,\gamma)$ is more complicated

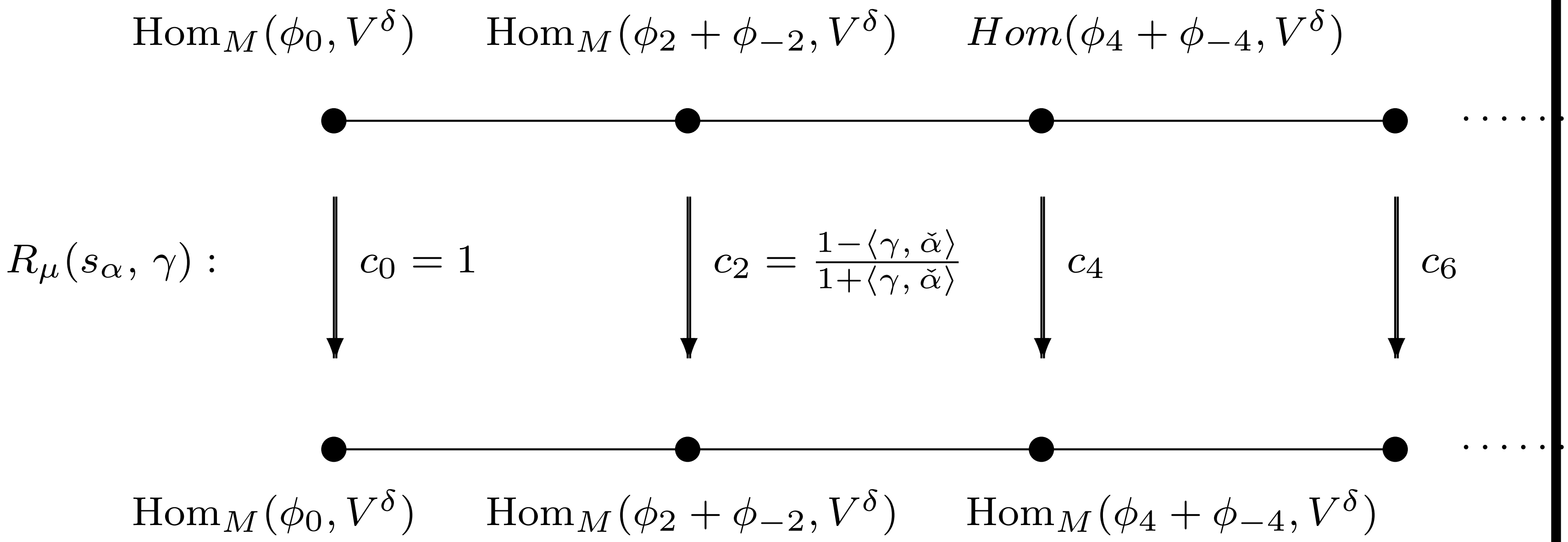
The action of $R_\mu(s_\alpha, \gamma)$, for α good

If α is a good root, $R_\mu(s_\alpha, \gamma)$ behaves just like in the spherical case:

- $R_\mu(s_\alpha, \gamma)$ is an endomorphism of $\mathrm{Hom}_M(E_\mu\,|_M, V^\delta)$
- $\mathrm{Hom}_M(E_\mu\,|_M, V^\delta) = \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}_M(\phi_{2n} + \phi_{-2n}, V^\delta)$
- $R_\mu(s_\alpha, \gamma)$ acts on $\mathrm{Hom}_M(\phi_0, V^\delta)$ by 1. For all $n \geq 1$, it acts on $\mathrm{Hom}_M(\phi_{2n} + \phi_{-2n}, V^\delta)$ by the scalar:

$$c_{2n} = \frac{\Pi_{j=1}^{n}((2j-1) - \langle \gamma, {}^\vee\alpha \rangle)}{\Pi_{j=1}^{n}((2j-1) + \langle \gamma, {}^\vee\alpha \rangle)}$$

The action of $R_\mu(s_\alpha, \gamma)$, for α good



When μ is petite (of level ≤ 3):

$$R_\mu(s_\alpha, \gamma) = \begin{cases} +1 & \text{on the } (+1)\text{-eigenspace of } \Psi_\mu(s_\alpha) \\ \frac{1-\langle\gamma, \breve{\alpha}\rangle}{1+\langle\gamma, \breve{\alpha}\rangle} & \text{on the } (-1)\text{-eigenspace of } \Psi_\mu(s_\alpha) \end{cases}$$

Ψ_μ is the representation of W_δ^0 on $\mathrm{Hom}_M(E_\mu, V^\delta)$

The action of $R_\mu(s_\alpha, \gamma)$, for α bad

- The reflection s_α may fail to stabilize δ, so

$$R_\mu(s_\alpha,\, \gamma)\colon\ \mathrm{Hom}_M(E_\mu,\, V^\delta) \to \mathrm{Hom}_M(E_\mu,\, V^{s_\alpha\cdot\delta})$$

may fail to be an endomorphism

- The operator $R_\mu(s_\alpha,\, \gamma)$ carries

$$\mathrm{Hom}_M(\phi_{2n+1} + \phi_{-2n-1},\, V^\delta) \to \mathrm{Hom}_M(\phi_{2n+1} + \phi_{-2n-1},\, V^{s_\alpha\cdot\delta})$$

- For T in $\mathrm{Hom}_M(\phi_{2n+1} + \phi_{-2n-1},\, V^\delta)$, $R_\mu(s_\alpha,\, \gamma)T$ is the map

$$\phi_{2n+1} + \phi_{-2n-1} \to V^{s_\alpha\cdot\delta},\ (v_+ + v_-) \mapsto c_{2n+1}\, T(v_+ - v_-)$$

The operator τ_μ^α

For every K-type μ and every root α, we have an operator

$$\tau_\mu^\alpha \colon \operatorname{Hom}_M\left(E_\mu, V^\delta\right) \to \operatorname{Hom}_M\left(E_\mu, V^{s_\alpha\cdot\delta}\right), \; S \mapsto S \circ \mu(\sigma_\alpha^{-1})$$

For any integer $k \geq 0$, set $U_k = \operatorname{Hom}_M(\phi_k + \phi_{-k}, V^\delta)$. Then

- If α is good, $\operatorname{Hom}_M(E_\mu, V^\delta) = \operatorname{Hom}_M(E_\mu, V^{s_\alpha\cdot\delta}) = \bigoplus_{n\in\mathbb{N}} U_{2n}$. The operator τ_μ^α acts on U_{2n} by $(-1)^n$

- If α is bad, $\operatorname{Hom}_M(E_\mu, V^\delta) = \bigoplus_{n\in\mathbb{N}} U_{2n+1}$. The image of an element T in U_{2n+1}, is the map

$$\phi_{2n+1} + \phi_{-2n-1} \to V^{s_\alpha\cdot\delta}, \; (v_+ + v_-) \mapsto (-1)^{n+1} i\, T(v_+ - v_-)$$

The action of $R_\mu(s_\alpha, \gamma)$, for α bad

$\mathrm{Hom}_M(\phi_1 + \phi_{-1}, V^\delta)$ $\mathrm{Hom}_M(\phi_3 + \phi_{-3}, V^\delta)$ $Hom(\phi_5 + \phi_{-5}, V^\delta)$

$\bullet$ —— $\bullet$ —— $\bullet \cdots\cdots$

$R_\mu(s_\alpha, \gamma)$: $\downarrow +ic_1\,\tau_\mu^\alpha$ $\downarrow -ic_3\tau_\mu^\alpha$ $\downarrow -ic_5\,\tau_\mu^\alpha$

$\bullet$ —— $\bullet$ —— $\bullet \cdots\cdots$

$\mathrm{Hom}_M(\phi_1 + \phi_{-1}, V^{s_\alpha \cdot \delta})$ $\mathrm{Hom}_M(\phi_3 + \phi_{-3}, V^{s_\alpha \cdot \delta})$ $\mathrm{Hom}_M(\phi_5 + \phi_{-5}, V^{s_\alpha \cdot \delta})$

If μ is petite of level ≤ 2, then $R_\mu(s_\alpha, \gamma) = ic_1\tau_\mu^\alpha$

A very special case

Assume that there is a minimal decomposition of ω in simple roots that involves **only good roots.**

If μ is petite, every factor $R_\mu(s_\alpha, \gamma)$ behaves like in the spherical case, and depends only on the representation ψ_μ of the Weyl group of the good co-roots on $\mathrm{Hom}_M(\delta, \nu)$.
The full intertwining operator $R_\mu(\omega, \nu)$ can be constructed in terms of ψ_μ, and coincides with the p-adic operator R_{ψ_μ}.

This gives a way to compare the non-spherical principal series for our real split group with a spherical principal series for the p-adic split group associated to the root system of the good co-roots.

An example

Let G be the double cover of the real split E_8, and let δ be the genuine representation. Because $W_\delta^0 = W$, we are in the "very special case". A direct computation shows that not every relevant $W(E_8)$-type can be realized as the representation of W on $\mathrm{Hom}_M(E_\mu, V^\delta)$. Therefore

*it **might** happen that the non-spherical genuine principal series $X(\delta \otimes \nu)$ for the real E_8 is unitary, even if the spherical principal series $X(\nu)$ for the p-adic E_8 is not unitary.*

The unitarity of $X(\nu)$ could be ruled out exactly by the W-type that we are unable to match.

Another kind of matching...

Most often, a minimal decomposition of ω involves both good and bad roots. We would still like to construct the intertwining operator in terms of the representation ψ_μ of W_δ^0 on $\mathrm{Hom}_M(E_\mu, V^\delta)$. When is this possible?

Claim: *If w belongs to W_δ^0 and μ is a petite K-type of level at most two ($\diamondsuit$), then the intertwining operator $R_\mu(\omega, \nu)$ for G coincides with the p-adic operator R_{ψ_μ} for the split group associated to the root system of the good co-roots.*

The hypothesis $\diamondsuit$ can be weakened: the characters ± 3 of K^α are allowed for the simple good roots α that appear in a minimal decomposition of ω as an element of W_δ^0.

Conclusions

Suppose that

- The split group corresponding to W_δ^0 is a classical group
- Every relevant W_0^δ type appears in some $\mathrm{Hom}_M(E_\mu, V^\delta)$
- The matching of the intertwining operators is possible (in particular, ω is in W_δ^0).

Then you obtain non-unitarity certificates for $\bar{X}_P(\delta, \nu)$.

In the cases of (E_8, δ_{135}), (E_6, δ_{27}), (E_6, δ_{36}), (F_4, δ_3), (F_4, δ_{12}), there is a complete matching of W_δ^0-types, w_0 always belongs to W_δ^0 and the good roots determine a classical split group, so we expect to be able to get non-unitarity certificates.