Signatures of Invariant Hermitian Forms on Highest Weight Modules

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**Motivation**

**Unitary Dual Problem:** Classify the unitary irreps of a group

- abelian group: Pontrjagin
- compact, connected Lie group: Weyl, 1920s
- locally compact group—eg. reductive Lie group: open except for some special cases
- study a broader family of representations: those which admit an invariant Hermitian form
- real reductive Lie group: equivalent to classifying the irreducible Harish-Chandra modules (admissible, finitely-generated $(\mathfrak{g}, K)$-modules) which admit a positive-definite invariant Hermitian form
- Zuckerman 1978: construct all admissible $(\mathfrak{g}, K)$-modules by cohomological induction
• $G$ - real reductive Lie group
• $K$ - a maximal compact subgroup of $G$
• $g_0, k_0$ corresponding Lie algebras and $g, k$ their complexifications
• $q = l \oplus u$ parabolic subalgebra
• $\theta$ - Cartan involution corresponding to $K$

begin with an $(l, L \cap K)$-module $V$ where $L$ is a Levi subgroup of $G$ and $l$ its complexified Lie algebra

Step 1: extend to a rep of $q = l \oplus u$ by allowing $u$ to act trivially, then apply induction functor

$$\text{ind}_{(q, L \cap K)}^{(g, L \cap K)}(V) = U(g) \otimes_{U(q)} V$$

Step 2: apply a Zuckerman functor $\Gamma^j = j$th derived functor of the left exact covariant functor $\Gamma$ which takes the $K$-finite part of a representation
• $\text{ind}_{(\mathfrak{g},L\cap K)}^{\mathfrak{g},L\cap K}(V)$ is a **generalized Verma module**, hence our interest in highest weight modules

• Strategy: relate the signature of invariant Hermitian form on $V$ to signature of cohomologically induced module $\Gamma^j \text{ind}_{(\mathfrak{g},L\cap K)}^{\mathfrak{g},L\cap K}(V)$

• **1984, Vogan:** Suppose $\mathfrak{q}$ is $\theta$-stable. For an irreducible, unitarizable $(\mathfrak{l}, L \cap K)$-module $V$ with infinitesimal character $\lambda \in \mathfrak{h}^*$, if

$$\text{Re}(\alpha, \lambda - \rho(u)) \geq 0 \quad \forall \alpha \in \Delta(u, \mathfrak{h})$$

then $\Gamma_m(\text{Hom}_q(U(\mathfrak{g}), V \otimes \Lambda^{top}u))$ is also unitarizable, where $m = \dim u \cap \mathfrak{k}$.

(Fact: $\text{pro}^\mathfrak{g}_\mathfrak{h}(V^h) := \text{Hom}_q(U(\mathfrak{g}), V^h) \simeq (\text{ind}^\mathfrak{g}_\mathfrak{h}V^h)$.)

• **1984, Wallach:** more elementary proof of same result by computing the signature of the Shapovalov form on generalized Verma modules (invariant Hermitian form on the module obtained in Step 1 of cohomological induction)
• Potentially useful for unitary dual problem: signature of Shapovalov form on a generalized Verma module, when it exists, with no restrictions on value of infinitesimal character

• Today: irreducible Verma modules, irreducible highest weight modules of regular infinitesimal character

**Invariant Hermitian Forms**

**Definition:** Invariant Hermitian form \( \langle \cdot, \cdot \rangle \) on \( V \):

For all \( v, w \in V \)

- rep of \( G \): \( \langle gv, w \rangle = \langle v, g^{-1}w \rangle \) for all \( g \in G \)
- \( \mathfrak{g} \)-module: \( \langle Xv, w \rangle + \langle v, \bar{X}w \rangle = 0 \) for every \( X \in \mathfrak{g} \), where \( \bar{X} \) denotes the complex conjugate of \( X \) with respect to the real form \( \mathfrak{g}_0 \)

- sesquilinear

When does a Verma module admit an invariant Hermitian form?
**Theorem:** An irreducible representation $(\pi, V)$ admits a non-degenerate invariant Hermitian form if and only if it is isomorphic to a subrepresentation of its Hermitian dual $(\pi^h, V^h)$.

Let $b = h + n$ be a Borel subalgebra of $g$ and $\Delta^+(g, h)$ the corresponding system of positive roots.

$$M(\lambda) = \text{ind}_b^g(\mathbb{C}_\lambda) \text{ so } M(\lambda)^h = \text{pro}_{\bar{h}}^g(\mathbb{C}_{\bar{\lambda}}) = \text{Hom}_b(U(g), \mathbb{C}_{-\bar{\lambda}})$$

We see that $M(\lambda)$ embeds into $M(\lambda)^h$ if $\bar{\lambda} = -\lambda$ and $\Delta^+(g, h) = -\Delta^+(g, h)$. When does this happen?

For $\mu \in h^*$, define: $(\theta \mu)(H) = \mu(\theta^{-1} H) \quad (\bar{\mu})(H) = \overline{\mu(H)}$

Then: $\theta_{\bar{g}_\alpha} = g_{\theta \alpha} \quad \bar{g}_\alpha = g_\alpha$

**Theorem:** If $h$ is $\theta$-stable and maximally compact, $\lambda$ is imaginary, and $\theta \Delta^+(g, h) = \Delta^+(g, h)$, then $M(\lambda)$ admits a non-degenerate invariant Hermitian form.
by $\mathfrak{h}$-invariance, the $\lambda - \mu$ weight space is orthogonal to the $\lambda - \nu$ weight space if $\nu \neq -\bar{\mu}$

- each weight space is finite dimensional, so it makes sense to talk about signatures and the determinants

Constructing the form:

For $X \in \mathfrak{g}$, let $X^* = -\bar{X}$ and extend $X \mapsto X^*$ to an involutive anti-automorphism of $U(\mathfrak{g})$ by $1^* = 1$ and $(xy)^* = y^*x^*$.

We have the decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})n + \mathfrak{n}^{op}U(\mathfrak{g}))$.

Let $p$ be the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ under this direct sum.

- For $x, y \in U(\mathfrak{g})$, by invariance, $\langle xv_{\lambda}, yv_{\lambda} \rangle \lambda = \langle y^*xv_{\lambda}, v_{\lambda} \rangle \lambda$.
- $\langle (U(\mathfrak{g})n + \mathfrak{n}^{op}U(\mathfrak{g}))v_{\lambda}, v_{\lambda} \rangle = \{0\}$.
- $\langle xv_{\lambda}, yv_{\lambda} \rangle \lambda = \langle p(y^*x)v_{\lambda}, v_{\lambda} \rangle \lambda = \lambda(p(y^*x)) \langle v_{\lambda}, v_{\lambda} \rangle \lambda$
- See that an invariant Hermitian form on a Verma module is unique up to a real scalar. When $\langle v_{\lambda}, v_{\lambda} \rangle \lambda = 1$ : Shapovalov form
**Theorem:** (Shapovalov determinant formula) The determinant of the Shapovalov form on the $\lambda - \mu$ weight space is

$$\prod_{\alpha \in \Delta^+(g, \mathfrak{h})} \prod_{n=1}^{\infty} ((\lambda + \rho, \alpha^\vee) - n)^{P(\mu - n\alpha)}$$

up to multiplication by a scalar, where $P$ denotes Kostant’s partition function. (Assumption: $\mathfrak{h}$ is compact.)

- radical of Shapovalov form = unique maximal submodule of $M(\lambda)$
- form non-degenerate precisely for the irreducible Verma modules
- according to Shapovalov determinant formula, $M(\lambda)$ is reducible on the affine hyperplanes $H_{\alpha, n} := \{\lambda + \rho \mid (\lambda + \rho, \alpha^\vee) = n\}$ where $\alpha$ is a positive root and $n$ is a positive integer
- in any connected set of purely imaginary $\lambda$ avoiding these reducibility hyperplanes, as the Shapovalov form never becomes degenerate, the signature corresponding to fixed $\mu$ remains constant
Definition: The largest of such regions, which we name the Wallach region, is the intersection of the negative open half spaces
\[ \left( \bigcap_{\alpha \in \Pi} H_{\alpha,1}^- \right) \bigcap H_{\tilde{\alpha},1}^- \]
with \(ih_0^*\), where \(\tilde{\alpha}^\vee\) is the highest coroot, \(\Pi\) = simple roots corresponding to \(\Delta^+\), and \(H_{\tilde{\beta},n} = \{ \lambda + \rho | (\lambda + \rho, \beta^\vee) < n \}\).

Definition: If the signature of the Shapovalov form on \(M(\lambda)_{\lambda-\mu}\) is \((p(\mu), q(\mu))\), the signature character of \(\langle \cdot, \cdot \rangle_\lambda\) is
\[ ch_{\lambda}M(\lambda) = \sum_{\mu \in \Lambda^+} (p(\mu) - q(\mu)) e^{\lambda - \mu} \]
Pick \(\lambda, \xi\) so that \(\lambda + t\xi\) stays in the Wallach region for \(t \geq 0\). An asymptotic argument (degree of \(t\) on the diagonal > degree off the diagonal) leads to:
Theorem: (Wallach) The signature character of $M(\lambda)$ for $\lambda + \rho$ in the Wallach region is

$$ch_s M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+(p,t)} (1 - e^{-\alpha}) \prod_{\alpha \in \Delta^+(t,t)} (1 + e^{-\alpha})}.$$ 

Goal: be able to find the signature everywhere.

Idea: determine how the signature changes as you cross a reducibility hyperplane. Combine this with induction.

• take $\lambda$ s.t. $\lambda + \rho$ lies in exactly one reducibility hyperplane $H_{\alpha,n}$
• for reg $\xi$ and non-zero $t$ in a nbd of 0, $\langle \cdot, \cdot \rangle_{\lambda+t\xi}$ is non-degenerate
• $\langle \cdot, \cdot \rangle_{\lambda}$ has radical isom to the irreducible Verma module $M(\lambda - n\alpha)$
• therefore signature must change by plus or minus the signature of $\langle \cdot, \cdot \rangle_{\lambda-n\alpha}$ across $H_{\alpha,n}$

This can be made rigorous by using the Jantzen filtration.
• the $H_{\alpha,n}$’s where $\alpha$ is a root, $n$ an integer, partition $\mathfrak{h}^*$ into alcoves

**Definition:** For an alcove $A$, $\exists$ constants $c^A_{\mu}$ for $\mu \in \Lambda_r^+$ such that

$$R^A(\lambda) := \sum_{\mu \in \Lambda_r^+} c^A_{\mu} e^{\lambda - \mu}$$

is the signature character of $\langle \cdot, \cdot \rangle_\lambda$ when $\lambda + \rho$ lies in the alcove $A$.

Our description of how signatures change as you cross a reducibility hyperplane may be expressed:

**Lemma 1:** If $A, A'$ are adjacent alcoves separated by $H_{\alpha,n}$,

$$R^A(\lambda) = R^{A'}(\lambda) + 2\varepsilon(A, A') R^{A-n}(\lambda - n\alpha)$$

where $\varepsilon(A, A')$ is zero if $H_{\alpha,n}$ is not a reducibility hyperplane and plus or minus one otherwise.

• use $R(\lambda)$ to denote common signature character for alcoves in Wallach region
We use the affine Weyl group, whose action on $\mathfrak{h}^\ast$ partitions $\mathfrak{h}^\ast$ into precisely the alcoves with walls $H_{\alpha,n}$ as described above.

**Definition**  The **fundamental alcove** is

$$A_0 = \{ \lambda + \rho \mid (\lambda + \rho, \alpha^\vee) < 0 \quad \forall \alpha \in \Pi, \quad (\lambda + \rho, \check{\alpha}^\vee) > -1 \}.$$

- reflections through walls of $A_0$ generate the affine Weyl group, $W_a$: reflections $s_{\alpha,0}$ for each simple root $\alpha$ and $s_{\check{\alpha},-1}$ generate $W_a$
- omit $s_{\check{\alpha},-1} \rightarrow$, generate the Weyl group $W$ as a subgroup of $W_a$
- these generators compatible with reflection through walls of the fundamental Weyl chamber $C_0$, which we choose to contain $A_0$:

$$C_0 = \bigcap_{\alpha \in \Pi} H_{\alpha,0}.$$
Definition  We will define two maps $\tau$ and $\tilde{\tau}$ from the affine Weyl group to the Weyl group as follows:

- $\tau$ comes from structure of $W_a$ as semidirect product of translation by the root lattice and the Weyl group: $w = s$ if $w = ts$ with $t =$ translation by an element of $\Lambda_r$, $s \in W$
- We let $\tilde{w}$ be such that $wA_0$ lies in the Weyl chamber $\tilde{w}C_0$.

- $\tau$ is a group homomorphism
- $\tilde{\tau}$ is not a group homomorphism
- $s_{a,n} = s_a$, and $s_{a,0}s_{a,n}\mu = \mu - n\alpha$
Observe that we can rewrite Lemma 1 as
\[ R^{wA_0}_w(\lambda) = R^{w' A_0}_w(\lambda) + 2 \varepsilon(wA_0, w'A_0) R^{s_{\alpha,n}s_{\alpha,n} A_0}_{s_{\alpha,n}s_{\alpha,n} A_0}(s_{\alpha,n} A_0) (s_{\alpha,n} A_0) \]

For \( w \) in the affine Weyl group, let \( wA_0 = C_0 \stackrel{r_1}{\rightarrow} C_1 \stackrel{r_2}{\rightarrow} \cdots \stackrel{r_\ell}{\rightarrow} C_\ell = \tilde{w}A_0 \) be a (not necessarily reduced) path from \( wA_0 \) to \( \tilde{w}A_0 \). Applying (1), \( \ell \) times, we obtain
\[ R^{wA_0}_w(\lambda) = R^{\tilde{w}A_0}_w(\lambda) + \sum_{j=1}^{\ell} \varepsilon(C_{j-1}, C_j) 2 R^{r_j C_j}_{r_j C_j} (r_j C_j) \]

Observe that a path from \( r_j C_j \) to \( r_j C_\ell \) is
\[ r_j C_j r_j C_j r_j C_j + \cdots r_j C_\ell. \]

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Applying induction on path length, we arrive at the following:

**Theorem 2:** For \( w \) in the affine Weyl group, let 

\[ wA_0 = C_0 \overset{r_1}{\rightarrow} C_1 \overset{r_2}{\rightarrow} \cdots \overset{r_\ell}{\rightarrow} C_\ell = \tilde{w}A_0 \]

be a (not necessarily reduced) path from \( wA_0 \) to \( \tilde{w}A_0 \).

\[ R^{wA_0}(\lambda) \text{ equals } \sum_{S=\{i_1<\cdots<i_k\}\subset\{1,\ldots,\ell\}} \varepsilon(S)2^{|S|} R^{r_{i_1} \cdots r_{i_k} \tilde{w}A_0} \left( r_{i_1}r_{i_2} \cdots r_{i_k}r_{i_{k-1}} \cdots r_{i_1} \lambda \right) \]

where \( \varepsilon(\emptyset) = 1 \) and 

\[ \varepsilon(S) = \varepsilon(C_{i_1-1}, C_{i_1}) \varepsilon(r_{i_1}C_{i_2-1}, r_{i_1}C_{i_2}) \cdots \varepsilon(r_{i_1} \cdots r_{i_{k-1}}C_{i_{k-1}-1}, r_{i_1} \cdots r_{i_{k-1}}C_{i_k}) \].

Calculating \( \varepsilon \): difficult.
Calculating $\varepsilon$

The strategy for computing $\varepsilon$ is as follows:

- We show that for a fixed hyperplane $H_{\alpha,n}$, the value of $\varepsilon$ for crossing from $H_{\alpha,n}^+$ to $H_{\alpha,n}^-$ depends only on the Weyl chamber to which the point of crossing belongs.

- We consider rank 2 root systems of types $A_2$ and $B_2$, generated by simple roots $\alpha_1$ and $\alpha_2$, and calculate the values for $\varepsilon$ by calculating changes that occur at the Weyl chamber walls. Our proofs do not depend on simplicity of the $\alpha_i$.

- For an arbitrary positive root $\gamma$ in a generic irreducible root system which is not type $G_2$, we develop a formula for $\varepsilon$ inductively by replacing the $\alpha_i$ from the previous step with appropriate roots. Key in the induction is the independence of our rank 2 arguments from the simplicity of the $\alpha_i$. 
Let's begin with something simple: calculate \( \varepsilon \) for \( \alpha \) simple.

**Lemma 2:** Let \( \delta_\alpha \) be \(-1\) if \( \alpha \) is noncompact, and \( 1 \) if it is compact.

If \( \alpha \) is simple and \( n \) is positive and if \( H_{\alpha,n} \) separates \( wA_0 \) and \( w'A_0 \) with \( wA_0 \subset H^+_{\alpha,n} \) and \( w'A_0 \subset H^-_{\alpha,n} \), then \( \varepsilon(wA_0, w'A_0) = \delta_\alpha^n. \)

**Proof:** Choose \( X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha} \), and \( H_\alpha = [X_\alpha, Y_\alpha] \), a standard triple so that \( \mu(H_\alpha) = (\mu, \mu)(\lambda) \forall \mu \in \mathfrak{h}^* \). We may arrange so that

\[
-\overline{Y_\alpha} = \delta_\alpha X_\alpha.
\]

The \( \lambda - n\alpha \) weight space of \( M(\lambda) \) is one-dimensional and spanned by the vector \( Y^n_\alpha v_\lambda \). We know that

\[
\langle Y^n_\alpha v_\lambda, Y^n_\alpha v_\lambda \rangle_{\lambda} = \delta_\alpha^n \langle v_\lambda, X^n_\alpha Y^n_\alpha v_\lambda \rangle_{\lambda}
\]

\[
= \delta_\alpha^n n! \langle v_\lambda, H_\alpha (H_\alpha - 1) \cdots (H_\alpha - (n-1))v_\lambda \rangle_{\lambda}
\]

from \( \mathfrak{sl}_2 \) theory. We conclude that

\[
\varepsilon(wA_0, w'A_0) = \delta_\alpha^n.
\]
Dependence on Weyl Chambers

**Proposition 1:** Suppose $\alpha$ is a positive root and $n \in \mathbb{Z}^+$ and suppose $H_{\alpha,n}$ separates adjacent alcoves $wA_0$ and $w'A_0$, with $wA_0 \subset H_{\alpha,n}^+$ and $w'A_0 \subset H_{\alpha,n}^-$. The value of $\varepsilon(w, w')$ depends only on $H_{\alpha,n}$ and $\tilde{w}(= \tilde{w}')$.

We begin by refining Theorem 2: if we take an arbitrary $C_\ell$, the formula becomes

$$R^{wA_0}(\lambda) = \sum_{I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, \ell\}} \varepsilon(I) 2^{|I|} R^{r_{i_1} \cdots r_{i_k} C_\ell} (r_{i_1} \cdots r_{i_k} r_{i_k} \cdots r_{i_1} \lambda).$$

If we choose in particular $C_\ell = C_0$, we have

$$R^{C_0}(\lambda) = \sum_{I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, \ell\}} \varepsilon(I) 2^{|I|} R^{r_{i_1} \cdots r_{i_k} C_0} (r_{i_1} \cdots r_{i_k} r_{i_k} \cdots r_{i_1} \lambda). \quad (2)$$
We begin by proving the proposition in the special case where
\( wA_0 = C_i \) and \( w'A_0 = C_{i+1} \) as described in the following figure:

\[ \mathcal{C} = \{ C_0, \ldots, C_5 \} \]

![Figure 1: Type A_2](image-url)
Lemma 3: Let \( C = \{C_i\}_{i=0,\ldots,k-1} \) be a set of alcoves that lie in the interior of some Weyl chamber and suppose the reflections \( \{r_j\}_{j=1,\ldots,k} \) preserve \( C \). If \( w, v \in W_\alpha \) are generated by the \( r_j \) then

\[
(w^{-1}w = v^{-1}v) \iff w = v.
\]

Proof: \( \Rightarrow \): By simple transitivity of the action of \( W_\alpha \) on the alcoves, \( w^{-1}w = v^{-1}v \iff w^{-1}wC = v^{-1}vC \) for any alcove \( C \). Choose in particular \( C = C_i \). The alcoves \( w^{-1}wC_i \) and \( v^{-1}vC_i \) belong to the same Weyl chamber as they are the same alcove. As the \( r_j \)'s preserve \( C \) which lies in the interior of some Weyl chamber, \( wC_i \) and \( vC_i \) belong to the same Weyl chamber. Thus \( w^{-1} = v^{-1} \), whence \( w = v \). The other direction is trivial.

Note: \( C \) in the figure satisfies the conditions of Lemma 3.
To prove the proposition for the figure, we need to show that
\( \varepsilon(C_i, C_{i+1}) + \varepsilon(C_{i+3}, C_{i+4}) = 0, \) \((C_6 = C_0)\).

For \( I = \{i_1 < \cdots < i_k\} \), we define \( w_I = r_{i_k} r_{i_{k-1}} \cdots r_1 \). We rewrite (2) as
\[
\sum_{\emptyset \neq I \subset \{1, \ldots, \ell\}} 2^{|I|} \varepsilon(I) R^{\overline{w_I}^{-1} C_0} \overline{w_I}^{-1} w_I \lambda = 0 \tag{3}
\]

Using Lemma 3 and the partial ordering on \( \Lambda \), we obtain
\[
\sum_{\emptyset \neq I \subset \{1, \ldots, \ell\}} 2^{|I|} \varepsilon(I) = 0 \tag{4}
\]
for every \( \mu \in \Lambda \).

Suppose \( \mu = m\alpha_1 \). The subsets \( I \) of length less than 3 for which \( \overline{w_I}^{-1} w_I = \mu \) are \( I = \{1\}, \{4\} \). By considering equation (4) modulo 8, we obtain \( \varepsilon(C_0, C_1) + \varepsilon(C_3, C_4) = 0 \) which gives the desired result for \( H_{\alpha_1, m} \). The same proof can be used for the other hyperplanes and also for type \( B_2 \).
Generalization: $\mathcal{C} = \{\text{alcoves containing } \mu_0 \text{ in their closures}\}$.
Conditions of Lemma 3 satisfied, argue as before.
$C = \{ C_0, \ldots, C_5 \}$

$\alpha_1, 0$

$H_{\alpha_1,0}$

$H_{\alpha_2,0}$

$\alpha_1$

$\alpha_2$

$s_{\alpha_1}$

$s_{\alpha_2}$

$s_{\alpha_1+s_{\alpha_2}}$

$s_{\alpha_1+s_{\alpha_2}}$

$s_{\alpha_1+s_{\alpha_2}+\alpha_2}$

$s_{\alpha_1+s_{\alpha_2}+\alpha_2}$

$s_{\alpha_1+s_{\alpha_2}+\alpha_2}+\alpha_2$

$s_{\alpha_1+s_{\alpha_2}+\alpha_2}+\alpha_2$

$s_{\alpha_1+s_{\alpha_2}+\alpha_2}+\alpha_2$

$s_{\alpha_1+s_{\alpha_2}+\alpha_2}+\alpha_2$
Calculating $\varepsilon$ for Type $A_2$

- know how to calculate $\varepsilon$ for hyperplanes corresponding to simple roots, so we know how to calculate $\varepsilon$ in the Weyl chambers adjacent to the fundamental Weyl chamber
- again, changes along a closed path should sum to zero
- so previous diagram, where $C$ overlaps with two Weyl chambers, allows you to relate values of $\varepsilon$ in one chamber to values in an adjacent chamber

<table>
<thead>
<tr>
<th>Weyl chamber walls in $C$</th>
<th>Equations</th>
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| $H_{\alpha_1,0}$         | $\varepsilon(C_2, C_3) + \varepsilon(C_5, C_6) = 0$  
                          | $\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_2, C_3)\varepsilon(\pi_7 C_4, \pi_7 C_5) = 0$ |
| $H_{\alpha_2,0}$         | $\varepsilon(C_6, C_1) + \varepsilon(C_3, C_4) = 0$  
                          | $\varepsilon(C_1, C_2) + \varepsilon(C_4, C_5) + 2\varepsilon(C_0, C_1)\varepsilon(\pi_7 C_1, \pi_7 C_2) = 0$ |
| $H_{\alpha_1 + \alpha_2,0}$ | $\varepsilon(C_6, C_1) + \varepsilon(C_3, C_4) = 0$  
                          | $\varepsilon(C_2, C_3) + \varepsilon(C_5, C_6) = 0$ |
Final Formula for $\varepsilon$

Notation: $\varepsilon(H, N, s) = \varepsilon(A, A')$ where $A \subset H^+_N$, $A' \subset H^-_N$, $A$ and $A'$ are adjacent, and $A \subset sC_0$.

Using induction on height:

**Theorem 3:** Let $\gamma$ be a positive root, and let $\gamma = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$ be such that $\text{ht}(s_{i_j} \cdots s_{i_{k-1}} \alpha_{i_k})$ decreases as $j$ increases. Let $w_\gamma = s_{i_1} \cdots s_{i_k}$. If $\gamma$ hyperplanes are positive on $sC_0$, then

$$
\varepsilon(H, N, s) = (-1)^N \# \{ \text{noncompact } \alpha_{i_j} : |\alpha_{i_j}| \geq |\gamma| \} 
\times (-1)^{\# \{ \beta \in \Delta(w^{-1}_\gamma) : |\beta| = |\gamma|, \beta \neq \gamma, \text{ and } \beta, s_\beta \gamma \in \Delta(s^{-1}) \}} 
\times (-1)^{\# \{ \beta \in \Delta(w^{-1}_\gamma) : |\beta| \neq |\gamma| \text{ and } \beta, -s_\beta \gamma, \beta \in \Delta(s^{-1}) \}}.
$$

Extending results so that we know how to compute signature characters for non-compact Cartan subalgebras: use formulas for singular vectors.
Irreducible Highest Weight Modules

- the Shapovalov form on $M(\lambda)$ descends to an invariant
  Hermitian form on the irreducible highest weight module $L(\lambda)$

Let $\lambda$ be antidominant, regular, and $x \in W_\lambda$. The Jantzen filtration of $M(x \cdot \lambda)$ ($x \cdot \lambda = x(\lambda + \rho) - \rho$) is

$$M(x \cdot \lambda) = M(x \cdot \lambda)^0 \supset M(x \cdot \lambda)^1 \supset \cdots \supset M(x \cdot \lambda)^N = \{0\}$$

where, for fixed $\delta$ regular,

$$M(x \cdot \lambda)^j = \left\{ \text{vectors } av_{x,\lambda} \in M(x \cdot \lambda) \mid \langle av_{x,\lambda+\delta t}, bv_{x,\lambda+\delta t} \rangle_{x \cdot \lambda+\delta t} \text{ vanishes at least to order } j \text{ at } t = 0 \forall b \in U(n^{op}) \right\}.$$
\( M(x \cdot \lambda)_j = M(x \cdot \lambda)_j^j / M(x \cdot \lambda)^{j+1} \) is semisimple.

- Kazhdan-Lusztig polynomials tell you:
  \[ [M(x \cdot \lambda)_j : L(y \cdot \lambda)] = \text{coefficient of } q^{(\ell(x) - \ell(y) - j)/2} \]
  in \( P_{w \lambda x, w \lambda y}(q) \)

- Jantzen filtration does not depend on choice of \( \delta \).

- Get a non-degenerate invariant Hermitian form \( \langle \cdot, \cdot \rangle_j \) on \( M(x \cdot \lambda)_j \).

- Define analogous polynomials keeping track of signatures:
  form on each copy of \( L(y \cdot \lambda) \) in \( j \)th level of filtration has signature \( \pm \) signature of the Shapovalov form on \( L(y \cdot \lambda) \).

- Form on \( j \)th level, however, does:
  \( \text{ch}_s M(x \cdot \lambda + \delta t) \) equals:
  \[ \sum_j \text{ch}_s \langle \cdot, \cdot \rangle_j \] for small \( t > 0 \)
  \[ \sum_{j \text{ even}} \text{ch}_s \langle \cdot, \cdot \rangle_j - \sum_{j \text{ odd}} \text{ch}_s \langle \cdot, \cdot \rangle_j \] for small \( t < 0 \)
More precisely, the signature of the form depends on the (integral) Weyl chamber containing $\delta$: if $\delta \in wC_0$, there are integers $a_{y,j}^{x,\lambda,w}$ such that
\[
ch_s \langle \cdot, \cdot \rangle_j = \sum_{y \leq w} a_{y,j}^{x,\lambda,w} ch_s L(y \cdot j)
\]
\[
R^{wA_0 + x\lambda}(x\lambda) = \sum_j a_{y,j}^{x,\lambda,w} ch_s L(y \cdot j)
\]

Proposition: Letting $a_y^{x,\lambda,w} = \sum_j a_{y,j}^{x,\lambda,w}$, then
\[
ch_s L(x\lambda) = \sum_{y_1 < \cdots < y_j = x} (-1)^{j-1} \left( \prod_{i=2}^{j} a_{y_{i-1}}^{w,\lambda,w} \right) R^{y_1\lambda + wA_0}(y_1\lambda).
\]
The usual Kazhdan-Lusztig polynomials may be computed via the inductive formulas:

a) \( P_{w_\lambda, x, w_\lambda} = P_{w_\lambda x s, w_\lambda y} \) if \( y s > y \) and \( x, x s \geq y \), \( s \) simple

a') \( P_{w_\lambda x, w_\lambda y} = P_{w_\lambda s x, w_\lambda y} \) if \( s y > y \) and \( x, s x \geq y \), \( s \) simple

b) If \( y > y s \) then

\[
q^c P_{w_\lambda x s, w_\lambda y} + q^{1-c} P_{w_\lambda x, w_\lambda y} = \sum_{z \in W_\lambda | z s > z} \mu(w_\lambda z, w_\lambda y)q^{(\ell(z) - \ell(y) + 1)}
\]

Signed versions: inductive formulas similar. Have to include some signs which depend on \( x, \lambda, w, s = s_\alpha \).
We would like to extend this work to generalized Verma modules for the purpose of studying invariant Hermitian forms on Harish-Chandra modules. Open problems which need to be solved for this purpose:

- reducibility of generalized Verma modules
  - computing the determinant of the Shapovalov form in some special cases: Khomenko-Mazorchuk
  - sufficient conditions for certain principal series representations: Speh-Vogan

- determining the composition series of a generalized Verma module (i.e. what are the irreducible factors, and what are their multiplicities)
  - composition series for generalized principal series representations ⇒ determine reducibility of representation induced from a parabolic subgroup