Parameters for Representations of Real Groups
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The basic references are [7] and [6]. The parameters given in these notes only exist in the unpublished preprint [4]. The case of regular integral infinitesimal character is discussed in [1]. Everything appears, sometimes in somewhat different form, in [2].

1 Algebraic Groups and Root Data

A root datum is a quadruple

$$D = (X, \Delta, X^\vee, \Delta^\vee)$$

where $X, X^\vee$ are free abelian groups of finite rank, and $\Delta, \Delta^\vee$ are finite subsets of $X, X^\vee$, respectively. In addition there is a perfect pairing $\langle , \rangle : X \times X^\vee \to \mathbb{Z}$ so $X^\vee \simeq \mathrm{Hom}(X, \mathbb{Z})$. There must exist a bijection $\alpha \to \alpha^\vee : \Delta \to \Delta^\vee$ such that for all $\alpha \in \Delta$,

$$\langle \alpha, \alpha^\vee \rangle = 2,\ s_\alpha(\Delta) = \Delta,\ s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee.$$

Heer $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ and $s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$ $(x \in X, y \in X^\vee)$.

By [3, Lemma VI.1.1] (applied to $\mathbb{Z}\langle\Delta\rangle$ and $\mathbb{Z}\langle\Delta^\vee\rangle$) the conditions determine the bijection uniquely once Δ and $\Delta^\vee$ are given. In particular $(X, \Delta, X^\vee, \Delta^\vee)$ is determined by (X, Δ) if $Z\langle\Delta\rangle = X$. This condition holds if and only if the corresponding group is semisimple.

Suppose Δ^+ is a set of positive roots of Δ. Then $\Delta^{\vee+} = \{\alpha^\vee \mid \alpha \in \Delta^+\}$ is a set of positive roots of $\Delta^\vee$, and

$$D_b = (X, \Delta^+, X^\vee, \Delta^{\vee+})$$

is a *based root datum*. Alternatively we may replace Δ^+ with a set Π of simple roots.

Two root systems are isomorphic if there exists an isomorphism $\phi : X \to X'$ such that $\phi(\Delta) = \Delta'$ and $\phi^t(\Delta'^\vee) = \Delta^\vee$. Here $\phi^t : X'^\vee \to X$ is given by

$$\langle \phi(x), y' \rangle = \langle x, \phi^t(y') \rangle \quad (x \in X, y' \in X'^\vee). \tag{1.1}$$

Let $\mathbb{G}$ be a connected reductive algebraic group and choose a Cartan subgroup $\mathbb{T}$. The corresponding root data is

$$D = (X^*(\mathbb{T}), \Delta, X_*(\mathbb{T}), \Delta^\vee)$$

where $X^*(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$, $X_*(\mathbb{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$, $\Delta = \Delta(\mathbb{G}, \mathbb{T})$ is the set of roots of $\mathbb{T}$ in $\mathbb{G}$, and $\Delta^\vee = \Delta^\vee(\mathbb{G}, \mathbb{T})$ is the set of co-roots.

If $\mathbb{T}'$ is another Cartan subgroup the associated root data is isomorphic to the given one. This isomorphism is canonical up to the Weyl group $W = W(\mathbb{G}, \mathbb{T})$.

Given a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$ we get a set of positive roots Δ^+, and corresopnding positive coroots $\Delta^{\vee+}$. Associated to this is a based root datum

$$D = (X^*(\mathbb{T}), \Delta^+, X_*(\mathbb{T}), \Delta^{\vee+}).$$

Given another choice of $\mathbb{T}' \subset \mathbb{B}'$ there is a *canonical* isomorphism of associated based root data.

There is an exact sequence

$$1 \to \mathrm{Int}(\mathbb{G}) \to \mathrm{Aut}(\mathbb{G}) \to \mathrm{Out}(\mathbb{G}) \to 1 \tag{1.2}$$

where $\mathrm{Int}(\mathbb{G})$ is the group of inner automorphisms of $\mathbb{G}$, $\mathrm{Aut}(\mathbb{G})$ is the automorphism group of $\mathbb{G}$, and and $\mathrm{Out}(\mathbb{G}) \simeq \mathrm{Aut}(\mathbb{G})/\mathrm{Int}(\mathbb{G})$ is the group of outer automorphisms. If we let $Z(\mathbb{G})$ be the center of $\mathbb{G}$ then $\mathrm{Int}(\mathbb{G}) \simeq \mathbb{G}/Z(\mathbb{G})$, also known as $\mathbb{G}_{ad}$ (which is a semisimple group).

A *splitting datum* for $\mathbb{G}$ is a set

$$(\mathbb{B}, \mathbb{T}, \{X_\alpha\}) \tag{1.3}$$

where $\mathbb{B}$ is a Borel subgroup, $\mathbb{T}$ is a Cartan subgroup contained in $\mathbb{B}$, Π is the set of simple roots associated to $\mathbb{B}$, and $\{X_\alpha \mid \alpha \in \Pi\}$ is a set of simple root vectors. This is also referred to as an *epinglage* or a *pinning.*

The group $\mathrm{Int}(\mathbb{G})$ acts simply transitively on the set of splitting data. It follows that if $S = (\mathbb{B}, \mathbb{T}, \{X_\alpha\})$ is a splitting datum S then

$$Stab_{\mathrm{Aut}(\mathbb{G})}(S) \simeq \mathrm{Out}(\mathbb{G})$$

and this isomorphism gives a splitting of the exact sequence (1.2). Furthermore since any automorphism may be modified by an inner automorphism to fix $\mathbb{B}$ and $\mathbb{T}$ (as sets) and act as a permutation on $\{X_\alpha\}$. It follows that

$$Out(\mathbb{G}) \simeq Aut(D_b)$$

In particular If $\mathbb{G}$ is semisimple then $\mathrm{Out}(\mathbb{G})$ is isomorphic to the automorphisms of the Dynkin diagram of $\mathbb{G}$.

$$\mathrm{Out}(\mathbb{G}) \simeq \mathrm{Aut}(D_b). \tag{1.4)(a}$$

We also have

$$\mathrm{Out}(\mathbb{G}) \simeq \mathrm{Aut}(D)/W. \tag{1.4)(b}$$

Fix $\gamma \in \mathrm{Out}(\mathbb{G})$. Define

$$Z(\mathbb{G})^\gamma = \{z \in Z(\mathbb{G}) \mid s(\gamma) z s(\gamma)^{-1} = z\} \tag{1.5}$$

This is independent of the choice of a splitting s of (1.2).

If $\mathbb{G} = \mathbb{T}$ is a torus an automorphism θ is determined by an automorphism of $X_*(\mathbb{T})$, i.e. an element of $GL(n, \mathbb{Z})$. If θ has order 2 then there is a basis

$$x_1, \dots, x_r, y_1, \dots y_s, z_1, z_1', \dots, z_t, z_t'$$

of $X_*(\mathbb{T})$ so that $\theta(x_i) = x_i$, $\theta(y_i) = -y_i$, and $\theta(z_i) = z_i', \theta(z_i') = z_i$.

In general $\mathbb{G} = \mathbb{T}\mathbb{G}_d$ where $\mathbb{T} = Z(\mathbb{G})^0$ is a central torus, and an automorphism is given by automorphisms of $\mathbb{T}$ and $\mathbb{G}_d$, which agree on $\mathbb{T} \cap \mathbb{G}_d$.

2 The Dual Group and the Dual Automorphism

Suppose we are given $\mathbb{G}$ with corresponding root data $D = (X, \Delta, X^\vee, \Delta^\vee)$. The *dual root data* is $D^\vee = (X^\vee, \Delta^\vee, X, \Delta)$, and the *dual group* is the group $\mathbb{G}^\vee$ defined by $D^\vee$. Alternatively we may describe $\mathbb{G}$ and $\mathbb{G}^\vee$ in terms of their based root data D_b and $D_b^\vee$.

If $\tau \in \mathrm{Aut}(D)$ then $-\tau^t \in \mathrm{Aut}(D^\vee)$ cf. Section 1).

Note that if $\tau \in \mathrm{Aut}(D_b)$ then $-\tau^t$ is probably contained in $\mathrm{Aut}(D_b^\vee)$. However $-w_0\tau^t \in \mathrm{Aut}(D_b^\vee)$ where w_0 is the long element of the Weyl group. We define $\tau^\vee = -w_0\tau^t$, this defines an isomorphism

$$\mathrm{Aut}(D_b) \ni \tau \to \tau^\vee \in \mathrm{Aut}(D_b^\vee)$$

By (1.4)(a) we obtain a bijection (not a group homomorphism) also denoted $\tau \to \tau^\vee$

$$\mathrm{Out}(\mathbb{G}) \simeq \mathrm{Out}(\mathbb{G}^\vee).$$

Definition 2.1 *For $\gamma \in Out(\mathbb{G})$ define $\gamma^\vee \in Out(\mathbb{G}^\vee)$ by* (**??**).

3 Real Forms of $\mathbb{G}$

To say that $\mathbb{G}$ is defined over $\mathbb{R}$ means that there is an anti–holomorphic involution σ of $\mathbb{G}(\mathbb{C})$. Then $G(\mathbb{R}) = \mathbb{G}(\mathbb{C})^\sigma$, and we will write $G = \mathbb{G}(\mathbb{R})$. We say σ is equivalent to σ' if $\sigma' = int(g) \circ \sigma \circ \mathrm{int}(g^{-1})$ for some $g \in \mathbb{G}$, i.e.

$$\sigma(x) = g\sigma(g^{-1}xg)g^{-1} \quad (x \in \mathbb{G}(\mathbb{C})).$$

An involution of $\mathbb{G}$, i.e. an algebraic automorphism of $\mathbb{G}$ of order 2, may be considered a holomorphic involution of $\mathbb{G}(\mathbb{C})$. We say involutions θ, θ' are equivalent if $\theta = int(g) \circ \theta' \circ \mathrm{int}(g^{-1})$ for some $g \in \mathbb{G}$.

Suppose $\mathbb{G}$ is defined over $\mathbb{R}$, with corresponding anti–holomorphic involution σ. We may choose an involution θ of $\mathbb{G}$, a "Cartan involution", such that $K = G^\theta$ is a maximal compact subgroup of G. Then $\mathbb{K} = \mathbb{G}^\theta$ is the algebraic group corresponding to K, and $\mathbb{K}(\mathbb{C}) = \mathbb{G}(\mathbb{C})^\theta$.

Lemma 3.1 *The map taking an anti–holomorphic involution σ to a corresponding Cartan involution θ is a bijection between equivalence classes of real forms and equivalence classes of involutions.*

We work entirely with Cartan involutions.

Definition 3.2 *We say two involutions θ, θ' are inner if they have the same image in Out($\mathbb{G}$), i.e. there exists $g \in G$ such that $\theta' = int(g) \circ \theta$, or*

$$\theta'(x) = g\theta(x)g^{-1}. \quad (x \in \mathbb{G}).$$

This is an equivalence relation, and an equivalence class is called an inner class. Such a class is determined by an involution $\gamma \in$ Out($\mathbb{G}$), and we refer to γ as an inner class.

Note that if θ is equivalent ot θ' then θ is inner to θ'.

Definition 3.3 *We say two real forms of $\mathbb{G}$ are inner if their Cartan involutions θ, θ' are inner.*

In fact two real forms are inner to each other if and only if the have the "same" fundamental (i.e. most compact) Cartan subgroup.

4 Basic Data

Fix $\mathbb{G}$. By Definition 3.2 an inner class of real forms is given by an involution $\gamma \in \mathrm{Out}(\mathbb{G})$.

Thus our basic data will be a pair $(\mathbb{G}, \gamma)$ where γ is an involution in $\mathrm{Out}(\mathbb{G})$. By Section 2 we obtain $(\mathbb{G}^\vee, \gamma^\vee)$.

5 Principal and Distinguished Involutions

Definition 5.1 *An involution θ of $\mathbb{G}$ is principal if the corresponding real group G is quasisplit, i.e. contains a Borel subgroup.*

Lemma 5.2 *The following conditions are equivalent*

1. *θ is a principal involution*
2. *There is a θ–stable Cartan subgroup $\mathbb{T}$ with no imaginary roots,*
3. *There are a θ–stable Cartan subgroup $\mathbb{T}$ and a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$, such that every simple root of $\mathbb{T}$ is complex or non–compact imaginary.*

K Every real form is inner to a quasiplit group:

Lemma 5.3 *Any inner class of involutions contains a principal involution, which is unique up to conjugation by* $\mathbb{G}$*.*

That is given $\theta_0 \in \mathrm{Out}(\mathbb{G})$ there exists a principal involution $\theta \in \mathrm{Aut}(\mathbb{G})$ with image θ_0, and if θ, θ' are two such, then $\theta' = \mathrm{int}(g)\theta\mathrm{int}(g)^{-1}$ for some $g \in \mathbb{G}$.

Definition 5.4 *An involution is said to be distinguished if there are* θ*–stable Cartan and Borel subgroups* $\mathbb{T} \subset \mathbb{B}$ *so that every simple imaginary root is compact (equivalently: every simple root is compact imaginary or complex). A real form is said to be distinguished if its Cartan involution is distinguished.*

Every real group has a distinguished inner form:

Lemma 5.5 *Any inner class of involutions contains a distinguished involution, and any two such are conjugate by* $\mathbb{G}$*.*

6 Encoding real forms

Fix $(\mathbb{G}, \gamma)$ as in Section 4. Let $\Gamma = \{1, \sigma\} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Choose a involution θ_0 in the inner class of γ. Consider the group $\mathbb{G} \rtimes \Gamma$ where the action of σ on $\mathbb{G}$ is by θ_0. That is $\mathrm{int}(\sigma) = \theta_0$.

Suppose θ is a Cartan involution of a real form in the same inner class. Then $\theta = \mathrm{int}(g) \circ \theta_0$. Let $\delta = g\sigma \in \mathbb{G} \rtimes \Gamma - \mathbb{G}$. Then

$$\theta = \mathrm{int}(\delta).$$

That is every Cartan involution in this inner class is given by conjugation by an element of $\mathbb{G} \rtimes \Gamma$.

It is natural to take θ_0 to be either a principal involution or a distinguished involution in the inner class (cf. Section 5).

Note that

$$\delta^2 = g\sigma(g) \in Z(\mathbb{G})^\gamma$$

(cf. 1.5).

7 L-Groups: Version 1

Fix $(\mathbb{G}, \gamma)$ as in Section 4.

Roughly speaking the L–group of $\mathbb{G}$ is the semidirect product $\mathbb{G}^\vee \rtimes \Gamma$ where σ acts on $\mathbb{G}$ by a distinguished involution in the inner class of $\gamma^\vee \in \mathrm{Aut}(\mathbb{G}^\vee)$ (Definition 2.1).

More precisely we need to incorporate a conjugacy class of such splittings into the data:

Definition 7.1 *An L–group for $\mathbb{G}$ is a pair $(\mathbb{G}^{\vee\Gamma}, \mathcal{S})$, where $\mathbb{G}^{\vee\Gamma}$ fits in an exact sequence*

$$1 \to \mathbb{G}^\vee \to \mathbb{G}^{\vee\Gamma} \to \Gamma \to 1$$

and $\mathcal{S}$ is a $\mathbb{G}^\vee$–conjugacy class of splittings of this exact sequence, such that for $s \in \mathcal{S}$, $int(s(\sigma))$ is a distinguished involution in the inner class of $\gamma^\vee$.

Remark 7.2 There is a unique quasisplit group G in the given inner class (in fact a unique strong inner form, cf. Section 9). This has a distinguished representation π_0: the spherical principal series with infinitesimal character 0.

The Weil group (cf. Section 16) maps to Γ, and therefore a homomorphsim $\phi : \Gamma \to \mathbb{G}^{\vee\Gamma}$ defines an irreducible representation of G (in fact an L–packet, which is a singleton in this case). There is not necessarily a distinguished homomorphism $\phi : \Gamma \to \mathbb{G}^{\vee\Gamma}$. The choice of L–group structure is such a homomorphism ϕ, and the choice of L–group struture amounts to declaring that ϕ corresponds to π_0.

8 Basic Data Revisited

Fix $(\mathbb{G}, \gamma)$ as in Section 4. We obtain $\mathbb{G}^\vee$ and $\gamma^\vee \in \mathrm{Out}(\mathbb{G})$ as in Section 2. We may therefore think of this as a quadruple

$$(\mathbb{G}, \gamma, \mathbb{G}^\vee, \gamma^\vee).$$

We may define $(\mathbb{G}^{\vee\Gamma}, \mathcal{S}^\vee)$, as in Section 7. The same definition applied to $(\mathbb{G}^\vee, \gamma^\vee)$ gives us a group $(\mathbb{G}^\Gamma, \mathcal{S})$.

9 Strong Real Forms

Fix $(\mathbb{G},\gamma)$ as in Section 4, and $\mathbb{G}^\Gamma,\mathbb{G}^{\vee\Gamma}$ as in Section 8. We apply the discussion of Section 6 to $\mathbb{G}^\Gamma$.

Definition 9.1 *A strong real form of* $\mathbb{G}$ *is an element* $x\in\mathbb{G}^\Gamma-\mathbb{G}$ *satisfying* $x^2\in Z(\mathbb{G})$. *We say two strong real forms* x,x' *are equivalent if* x *is* $\mathbb{G}$*–conjugate to* x'.

Lemma 9.2 *If* x *is a strong real form of* $\mathbb{G}$ *let* $\theta_x=int(x)$. *This is the Cartan involution of a real form in the inner class* γ. *This map is surjective onto the real forms in this inner class. If* $\mathbb{G}$ *is adjoint it is a bijection.*

10 Representations

Fix $(\mathbb{G},\gamma)$ as in Section 4, Fix $\mathbb{G}$, an inner class $\gamma\in\mathrm{Out}(\mathbb{G})$, and $(\mathbb{G}^\Gamma,\mathbb{G}^{\vee\Gamma})$ as in Section 8.

Definition 10.1 *A representation of a strong real form of* $\mathbb{G}$ *is a pair* (x,π) *where* x *is a strong real form of* $\mathbb{G}$ *and* π *is a* $(\mathfrak{g},\mathbb{K}_x)$*–module.*

We say (x,π) *is equivalent to* (x',π') *if there exists* $g\in\mathbb{G}$ *such that* $gxg^{-1}=x'$ *and* $g\cdot\pi\simeq\pi'$. *Here* $g\cdot\pi(h)=\pi(g^{-1}hg)$ *for* $h\in\mathbb{K}_{x'}$, *and* $g\cdot\pi(X)=\pi(Ad(g^{-1})X)$ *for* $X\in\mathfrak{g}$.

Suppose ζ is a distinguished isomorphism. Then ζ induces bijections:

$$\Delta^\vee(\mathbb{G},\mathbb{T})\simeq\Delta(\mathbb{G}^\vee,{}^d\mathbb{T}) \tag{10.2}$$

$$\Delta(\mathbb{G},\mathbb{T})\simeq\Delta^\vee(\mathbb{G}^\vee,{}^d\mathbb{T}) \tag{10.3}$$

11 Distinguished Isomorphisms

Fix $(\mathbb{G},\gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma,\mathcal{S}),(\mathbb{G}^{\vee\Gamma},\mathcal{S}^\vee))$ as in Section 8.

Suppose $\mathbb{T}$ is a Cartan subgroup of $\mathbb{G}$, and ${}^d\mathbb{T}$ is a Cartan subgroup of $\mathbb{G}^\vee$. By the construction of $\mathbb{G}^\vee$ there are isomorphisms

$$X_*(\mathbb{T}^\vee)\simeq X_*({}^d\mathbb{T})$$

and

$$\mathbb{T}^\vee \simeq {}^d\mathbb{T}, \quad \mathfrak{t}^\vee \simeq {}^d\mathfrak{t}.$$

Given Borel subgroups $\mathbb{B}, {}^d\mathbb{B}$ containing $\mathbb{T}, {}^d\mathbb{T}$ respectively, we obtain isomorphisms

$$\zeta(\mathbb{B}, {}^d\mathbb{B}) : \mathbb{T}^\vee \simeq {}^d\mathbb{T}, \quad \mathfrak{t}^\vee \simeq {}^d\mathfrak{t}.$$

Also recall $X^*(\mathbb{T}) = X_*(\mathbb{T}^\vee)$ and $\mathfrak{t}^* = \mathfrak{t}^\vee$ (canonically). So ζ may be interpreted as an isomorphism

$$\zeta : \mathfrak{t}^* \simeq {}^d\mathfrak{t} \tag{11.1}$$

Definition 11.2 *We say an isomorphism $\zeta : \mathbb{T}^\vee \simeq {}^d\mathbb{T}$ is distinguished if it is equal to $\zeta(\mathbb{B}, {}^d\mathbb{B})$ for some $\mathbb{B}, {}^d\mathbb{B}$.*

Now suppose θ is an involution of $\mathbb{T}$, and ${}^d\theta$ is an involution of ${}^d\mathbb{T}$. Then (cf. Section 2) $\theta^\vee$ is an involution of $\mathbb{T}^\vee$. Suppose $\zeta : \mathbb{T}^\vee \simeq {}^d\mathbb{T}$ is a distinguished isomorphism. We define an involution $\zeta^*(\theta)$ by carrying the involution $\theta^\vee$ of $\mathbb{T}^\vee$ to ${}^d\mathbb{T}$ via ζ, i.e.

$$\zeta^*(\theta)(t) = \zeta(\theta^\vee(\zeta^{-1}(t))) \quad (t \in {}^d\mathbb{T}).$$

12 Integral L–data

Fix $(\mathbb{G}, \gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma, \mathcal{S}), (\mathbb{G}^{\vee\Gamma}, \mathcal{S}^\vee))$ as in Section 8.

Here is the data which will parametrize representations with integral infinitesimal character.

Definition 12.1 *Fix $(\mathbb{G}, \gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma, \mathcal{S}), (\mathbb{G}^{\vee\Gamma}, \mathcal{S}^\vee))$ as in Section 8.*

A set of weak integral L–data is a 6-tuple $(x, \mathbb{T}, \mathbb{B}, y, \mathbb{T}^\vee, \mathbb{B}^\vee)$ where

1. *$\mathbb{T} \subset \mathbb{B} \subset \mathbb{G}$ are a Cartan and Borel subgroup, respectively,*
2. *$x^2 \in Z(\mathbb{G})$,*
3. *$\mathbb{T}$ is θ_x–stable where $\theta_x = int(x)$,*

4. $\mathbb{T}^\vee \subset \mathbb{B}^\vee \subset \mathbb{G}^\vee$ *are a Cartan and Borel subgroup, respectively,*

5. $y^2 \in Z(\mathbb{G}^\vee)$,

6. $\mathbb{T}^\vee$ *is* $\theta_y^\vee$*–stable where* $\theta_y^\vee = int(y)$,

7. *The isomorphism* $\zeta = \zeta(\mathbb{B}, \mathbb{B}^\vee)$ *satisfies* $\zeta^*(\theta_x) = \theta_y^\vee$,

A set of (integral) L–data is a pair (S, λ) *where* $S = (x, \mathbb{T}, \mathbb{B}, y, \mathbb{T}^\vee, \mathbb{B}^\vee)$ *is a set of weak L–data,* $\lambda \in \mathfrak{t}^\vee$*, and* $exp(2\pi i\lambda) = y^2$.

If (S, λ) is a set of strong integral L–data let $\zeta = \zeta(\mathbb{B}, {}^d\mathbb{B})$, and identify λ with an element of $\mathfrak{t}^*$ via (11.1).

13 L–data

Fix $(\mathbb{G}, \gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma, \mathcal{S}), (\mathbb{G}^{\vee\Gamma}, \mathcal{S}^\vee))$ as in Section 8.

We generalize the construction of the previous section to include representations with non–integral infinitesimal character.

Definition 13.1 *Fix* $(\mathbb{G}, \gamma)$ *as in Section 4, and* $((\mathbb{G}^\Gamma, \mathcal{S}), (\mathbb{G}^{\vee\Gamma}, \mathcal{S}^\vee))$ *as in Section 8.*

A set of weak L–data is a 6*-tuple* $(x, \mathbb{T}, P, y, \mathbb{T}^\vee, P^\vee)$ *where*

1. $\mathbb{T} \subset \mathbb{G}$ *is a Cartan subgroup,*

2. $x^2 \in Z(\mathbb{G})$,

3. $\mathbb{T}$ *is* θ_x*–stable where* $\theta_x = int(x)$,

4. P *is contained in a set of positive roots of* $\Delta(\mathbb{T}, \mathbb{G})$,

5. $\mathbb{T}^\vee \subset \mathbb{G}^\vee$ *is a Cartan subgroup,*

6. $y^2 \in \mathbb{T}^\vee$

7. $\mathbb{T}^\vee$ *is* $\theta_y^\vee$*–stable where* $\theta_y^\vee = int(y)$*, an involution of* $\mathbb{G}^\vee_{y^2} = Cent_{\mathbb{G}^\vee}(y^2)$,

8. $\mathbb{B}^\vee$ *is a Borel subgroup of* $\mathbb{G}^\vee_{y^2}$ *containing* $\mathbb{T}$,

9. *There is a distinguished isomorphism* ζ *satisfying:* $\zeta^*(\theta_x) = \theta_y^\vee$ *and* $\Delta(\mathbb{B}^\vee, \mathbb{T}^\vee) = \{\zeta(\alpha^\vee) \mid \alpha \in P\}$.

A set of L–data is a pair (S,λ) where $S=(x,\mathbb{T},\mathbb{B},y,\mathbb{T}^\vee,\mathbb{B}^\vee)$ is a set of weak L–data, $\lambda\in\mathfrak{t}^\vee$, and $exp(2\pi i\lambda)=y^2$.

If (S,λ) is a set of L–data let ζ be any distinguished isomorphism as in (9). Then we identify λ with an element of $\mathfrak{t}^*$ via (11.1).

14 Final Limit L–Data

Suppose $X=(S,\lambda)$ is a set of L–data, integral or not. Associated to X is a standard representation $I(X)$ of a real form of $\mathbb{G}$. Let $J(X)$ be the *socle* of X, i.e. the set of irreducible subrepresentations of $I(X)$. If λ is regular then $J(X)$ is a single irreducible representation. Otherwise this may fail.

For example $I(X)$ might be the reducible principal series representation of $SL(2,\mathbb{R})$ with infinitesimal character 0 and odd K–types; this is the direct sum of two limits of discrete series representations $\pi^\pm$. This realization as limits of discrete series shows how to obtain each $\pi^\pm$ as some $J(Y^\pm)$.

We need to do this in general: put a restriction on the parameters which are allowed, so that $J(X)$ is always irreducible, and we obtain every irreducible precisely once. There is the *final limit* construction of [9, Definition 2.4]. We describe the resulting formulation in terms of our parameters.

So suppose (S,λ) is a set of L–data as in Definition 13.1. As at the end of SEction 13 choose a distinguished isomorphism $\zeta:\mathfrak{t}^\vee\rightarrow\mathfrak{t}^*$ and use it to identify λ with an element of $\mathfrak{t}^*$. If $\alpha\in\Delta(\mathbb{G},\mathbb{T})$ is an imaginary root (with respect to $\theta=\theta_x$ then $\langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}$. It follows that $\alpha\in P$; then P defines positive and simple roots of $\Delta_{im}(\mathbb{G},\mathbb{T})$.

On the other hand $B^\vee$ defines a set of positive roots for the set of imaginary roots (with respect to $\theta_y^\vee$) of $\Delta(G_{y^2}^\vee,T^\vee)$. We therefore have a notion of simple roots of $\Delta_{im}(\mathbb{G}_{y^2}^\vee,T^\vee)$.

Definition 14.1 *Suppose (S,λ) is a set of L–data (Definition 13.1). We say (S,λ) is a set of final limit L–data if the following conditions hold.*

- *Suppose α is a simple root of $\Delta_{im}(\mathbb{G},\mathbb{T})$ and $\langle\zeta(\lambda),\alpha^\vee\rangle=0$. Then α is non–compact.*
- *Supose β is a simple roots of $\Delta_{im}(\mathbb{G}_{y^2}^\vee,T^\vee)$ and $\langle\lambda,\beta\rangle=0$. Then β is non–compact.*
- *(No condition on complex roots?)*

15 Parametrization of Representations

Fix $(\mathbb{G}, \gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma, \mathcal{S}), (\mathbb{G}^{\vee\Gamma}, \mathcal{S}^\vee))$ as in Section 8.

Suppose (S, λ) is a set of strong L–data. Associated to (S, λ) is a standard $(\mathfrak{g}, \mathbb{K}_x)$–module $I(S, \lambda)$ with infinitesimal character (the $\mathbb{G}$–orbit of) λ. As at the end of SEction 13 choose a distinguished isomorphism $\zeta : \mathfrak{t}^\vee \to \mathfrak{t}^*$ and use it to identify λ with an element of $\mathfrak{t}^*$.

Theorem 15.1 *Suppose (x, π) is an irreducible representation of a strong real form of $\mathbb{G}$ (Section 10) with regular infinitesimal character. Then $\pi \simeq J(S, \lambda)$ for some $S = (x, \dots)$ and λ. Two non–zero representations $(x, J(S, \lambda))$ and $(x, J(S', \lambda'))$ are isomorphic if and only if (S, λ) is $\mathbb{G} \times \mathbb{G}^\vee$ conjugate to (S', λ').*

15.1 General Infinitesimal Character

Let $J(S, \lambda)$ be the socle of $I(S, \lambda)$, i.e. the direct sum of the irreducible subrepresentations of $I(s, \lambda)$. [Q: we need to define this using the translation principle?]

We say (S, λ) is M–regular if $\langle \lambda, \alpha^\vee \rangle \neq 0$ for all imaginary roots (with respect to θ_x) of $\Delta(\mathbb{G}, \mathbb{T})$ [there may be a ρ–shift missing here].

Theorem 15.2 *Suppose (x, π) is an irreducible representation of a strong real form of $\mathbb{G}$ (Section 10). Then there exists strong, M–regular, L–data (S, λ) so that π is a subrepresentation of $J(S, \lambda)$. If (S', λ') also satisfies these conditions then (S', λ') is $\mathbb{G} \times \mathbb{G}^\vee$–conjugate to (S, λ).*

This gives a finite to one map from equivalence classes of strong, M–regular, L–data (S, λ) to equivalence classes (x, π) of representations of strong real forms of $\mathbb{G}$. This map is a bijection in the case of regular infinitesimal character. In Section 17 we will describe how to compute the fiber of this map.

16 Sketch of the Construction of $I(S, \lambda)$

Fix $(\mathbb{G}, \gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma, \mathcal{S}), (\mathbb{G}^{\vee\Gamma}, \mathcal{S}^\vee))$ as in Section 8. Let (S, λ) be a set of L–data, where $S = (x, \mathbb{T}, P, y, \mathbb{T}^\vee, \mathbb{B}^\vee)$.

Recall the Weil group is $W_\mathbb{R} = \langle \mathbb{C}^*, j \rangle$ where $j^2 = -1$ and $jzj^{-1} = \overline{z}$.

The data $(y, \mathbb{T}^\vee, \mathbb{B}^\vee, \lambda)$ defines an L–homomorphism $\phi : W_\mathbb{R} \to \mathbb{G}^{\vee\Gamma}$ as follows:

$$\phi(z) = z^\lambda \overline{z}^{Ad(y)\lambda}$$
$$\phi(j) = exp(-\pi i \lambda) \tag{16.3}$$

where $z^\lambda = exp(\lambda \log(z))$ $(z \in \mathbb{C}^* \subset W_\mathbb{R})$ (it requires a short argument that $\phi(z)$ is well defined).

Then $\phi : W_\mathbb{R} \to \langle \mathbb{T}^\vee, y \rangle$. This is not necessarily isomorphic to the L–group of T. It is isomorphic to an E–group $\mathbb{T}^\vee\Gamma$ of $\mathbb{T}$, and maps into $\mathbb{T}^\vee\Gamma$ parametrize characters of the ρ–cover $T(\mathbb{R})_\rho$ of $T(\mathbb{R})$.

The extra data in S gives us an isomorphism of $\langle \mathbb{T}, y \rangle \simeq \mathbb{T}^\vee\Gamma$, and hence a character Λ of $T(\mathbb{R})_\rho$.

For example in the case of a discrete series representation Λ is a character with differential the Harish–Chandra parameter λ; recall that $\lambda - \rho$ (and not necessarily λ) exponentiates to the compact Cartan.

If λ is regular Λ is all that is needed to define a standard module $I(\Psi, \Lambda)$ as in [5, Definition 8.27]. If Λ is singular an extra choice of positive real roots is necessary. This is included in the data of S.

The module $I(\Psi, \Lambda)$ may be written as cohomological induction from a principal series representation of a quasisplit group L. The reducibility of $J(S, \lambda)$ (for singular λ) comes from the reducibility of the corresponding standard module for L.

Therefore the fiber of the map described in Theorem 15.2 is obtained from the discussion in the next section applied to L.

17 R–Groups

We need some definitions and results from [7, Chapter 4].

Suppose G is quasiplit and $H = TA$ is the maximally split Cartan subgroup. Let $M = Cent_G(A)$; this is an abelian group. We say a character δ of M is fine if its restriction to $(G^d \cap M)^0$ is trivial [7, Definition 4.3.8]. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$, the (non–zero) real roots, and $\overline{\Delta} \subset \Delta$ the reduced root system of Δ. We say a root α of Δ is real if it is the restriction of a real root of $\Delta(\mathfrak{g}, \mathfrak{h})$, and complex otherwise.

$$W = Norm_K(A)/M \tag{17.1}$$
$$= W(\overline{\Delta}). \tag{17.2}$$

Let $\overline{\Delta}_\delta$ be the good roots ([7, Definition 4.3.11]). That is

$$\overline{\Delta}_\delta = \{\alpha \in \overline{\Delta} \,|\, \alpha \text{ is complex or } \alpha \text{ is real and } \delta(m_\alpha) = 1\}$$

Fix $\nu \in \hat{A}$. As in [7, Definitions 4.3.13 and 4.4.9] define

$$\begin{aligned}
W_\delta &= Stab_W(\delta) \\
W_\delta^0 &= W(\overline{\Delta}_\delta) \\
R_\delta &= W_\delta/W_\delta^0 \\
W(\nu) &= Stab_W(\nu) \\
W_\delta(\nu) &= Stab_W(\delta \otimes \nu) \\
W_\delta^0(\nu) &= Stab_{W_\delta^0}(\delta \otimes \nu) \\
R_\delta(\nu) &= W_\delta(\nu)/W_\delta^0(\nu) \subset R_\delta \\
R_\delta^\perp(\nu) &= \text{annihlator of } R_\delta(\nu) \text{ in } \widehat{R_\delta}
\end{aligned}$$

Note that $\widehat{R_\delta}/R_\delta^\perp(\nu) \simeq \widehat{R_\delta(\nu)}$.

Definition 17.3 *Suppose* $(S.\lambda)$ *is a set of strong L–data. The R–group for* S*, denoted* $R(S,\lambda)$ *is* $\widehat{R_\delta(\nu)}$ *computed on* L *. . . [Assignment part 1: make this precise! It comes down to the real roots - a computation involving the principal series of the quasisplit group* L*].*

If λ is regular then $R(S,\lambda) = 1$.

Lemma 17.4 *The fiber of the map of Theorem 15.2 is naturally parametrized by* $R(S,\lambda)$ *[Assignment part 2: so that this lemma holds].*

18 L–packets and Blocks

Fix $(\mathbb{G},\gamma)$ as in Section 4, and $((\mathbb{G}^\Gamma,\mathcal{S}),(\mathbb{G}^{\vee\Gamma},\mathcal{S}^\vee))$ as in Section 8.

Fix $y,\mathbb{T}^\vee,\mathbb{B}^\vee$ as in the definition of L–data, and λ satisfying $exp(2\pi i\lambda) = y^2$. Recall (Section 16) this data defines an L–homomorphism $\phi : W_\mathbb{R} \to \mathbb{G}^{\vee\Gamma}$.

We assume λ is regular.

Definition 18.1 *An L–packet is the set of representation* $J(S,\lambda)$ *where* $(S = (x,\mathbb{T},P,y,\mathbb{T}^\vee,\mathbb{B}^\vee),\lambda)$ *is a set of L–data.*

This is sometimes called a “super” L–packet; it includes representations on various strong real forms. Its restriction to a single strong real form is a conventional L–packet.

[Question: singular infinitesimal character?]

Definition 18.2 *Fix* x, y *satisfying* $x^2 \in Z(\mathbb{G})$*. The* $\mathbb{Z}$*–spane of the representation* $J(S,\lambda)$ *where* $(S = (x,\mathbb{T},P,y,\mathbb{T}^\vee,\mathbb{B}^\vee),\lambda)$ *is a set of L–data is a block.*

Again this is sometimes called a super–block. The restriction to a strong real form is a block. This is a minimal subspace of the Grothendieck group which is spanned by both irreducible and standard modules. Thus the Kazhdan–Lusztig polynomials are defined on blocks.

19 Example: $SL(2)$

Let $\mathbb{G} = SL(2)$. Then $\mathrm{Out}(\mathbb{G}) = 1$ so $\gamma = 1$. We have $(\mathbb{G},\gamma,\mathbb{G}^\vee,\gamma^\vee) = (SL(2),1,PSL(2),1)$.

We fix some notation. Let $\mathbb{B}^\pm$ be the upper and lower triangular matrices in $SL(2)$ respectively. Let $\mathbb{T}$ be the diagonal Cartan subgoup. Write $\mathbb{B}^\pm$ and $\mathbb{T}$ for $PSL(2)$ as well. (We abuse notation slightly and write $PSL(2)$ as 2×2 matrices.)

Let $t(z) = diag(z,1/z), m_\rho = t(i)$. Note that in $PGL(2)$ $t(z) = t(-z)$. Let $\lambda(z) = diag(z,-z) \in \mathfrak{t}^\vee$.

The group $\mathbb{G}^{\vee\Gamma}$ is generated by $\mathbb{G}^\vee$ and an element $\delta^\vee$ satisfying $(\delta^\vee)^2$ and $\delta^\vee g \delta^{\vee -1} = m_\rho g m_\rho^{-1}$.

The group $\mathbb{G}^\Gamma$ is generated by $\mathbb{G}$ and δ, where $\delta^2 = -I$ and $\delta g\delta^{-1} = m_\rho g m_\rho^{-1}$.

There is a unique L–group structure $(\mathbb{G}^{\vee\Gamma},\{\delta^\vee,\mathbb{B}^+\})$. Here $\{\delta^\vee,\mathbb{B}^+\}$ denotes the $\mathbb{G}^\vee$ conjugacy class of $(\delta^\vee,\mathbb{B}^+)$.

There are two L–group structures $(\mathbb{G}^\Gamma,\{\pm\delta,\mathbb{B}^+)\}$. Note that $(\delta,\mathbb{B}^+)$ is conjugate to $(-\delta,\mathbb{B}^-)$. This corresponds to the fact that $PGL(2,\mathbb{R})$ has two one–dimensional representations, and this choice amounts to choosing one of these. Dually this corresponds to choosing a discrete series representation of $SL(2,\mathbb{R})$ with infinitesimal character ρ.

There are three inequivalent strong real forms of $\mathbb{G}$, given by $x = \delta, \pm m_\rho \delta$. The corresponding real groups are $SL(2,\mathbb{R})$ and $SU(2)$, respectively. These may be though of as $SU(2,0), SU(1,1)$ and $SU(0,2)$.

There are two inequivalent strong real forms of $\mathbb{G}^\vee$, since it is adjoint, corresponding to $PGL(2,\mathbb{R})$ and $SO(3)$, respectively.

20 Other Parametrizations

There are several other ways to parametrize the standard and irreducible representations of real groups. The problem is how to conveniently write down characters of Cartan subgroups; disconnectedness is the main issue.
Assignment: Carefully write down how to go back and forth between these classifications.
1 θ–stable data $(\mathfrak{q}, H, \delta, \nu)$ ([7, Definition 6.5.1] This realizes the standard modules as derived functor modules from a minimal principal series of a quasipslit group L.
2 Character data (H, γ) with $\gamma = (\Gamma, \overline{\gamma})$, [7, Definition 6.6.1]. Here Γ is a character of H, not of a two–fold cover as in (43). The infinitesimal character is $\overline{\gamma}$, which is $d\Gamma+$ a ρ–shift.
3 Cuspidal data (M, δ, ν) [7, Definition 6.6.11]. Here M is a real Levi factor and δ is a (relative) discrete series representation of M. This is the original Langlands version of the classification.
4 $I(\Psi, \Lambda)$ ([5, Definition 8.27 and Theorem 8.29]) Here Λ is a character of the ρ–cover of H, and the infinitesimal character is $d\Lambda$.

21 Vogan Duality

The irreducible representations of strong real forms of $\mathbb{G}$ are parametrized by integral L–data $(x, \mathbb{T}, \mathbb{B}, y, \mathbb{T}^\vee, \mathbb{B}^\vee)$ with $x^2 \in Z(\mathbb{G}), y^2 \in Z(\mathbb{G}^\vee)$. This data is symmetric: $(y, \mathbb{T}^\vee, \mathbb{B}^\vee, x, \mathbb{T}, \mathbb{B})$ is L–data with the roles of $\mathbb{G}, \mathbb{G}^\vee$ reversed, and this defines a representation of a strong real form of $\mathbb{G}$ with integral infinitesimal character. This realizes Vogan duality [8], analogous to duality for Verma modules given by multiplication by the long element of the Weyl group.

Now suppose λ is regular but not integral. Then L–data satisfies x^2 is central in $\mathbb{G}$, but y^2 is not necessarily central in $\mathbb{G}^\vee$. To recover Vogan duality we have to allow x^2 not central in $\mathbb{G}$. This can be done, but requires some extra work. See [4].

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