Computing K-type multiplicities in standard representations (after Vogan)

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The first part of these notes is an updating (and correction) of [Khat] and is devoted to a paremetrization of the irreducible representations of the (generally disconnected) maximal compact subgroup of a real group in Harish-Chandra's class. The second part describes how to use that paremetrization of the first part to compute K-type multiplicities in standard modules. (By Frobenius reciprocity, this is equivalent to Blattner's formula and branching from K to $K \cap M$ which is complicated significantly by the disconnectedness of the groups in question.) An interlude between the first and second parts describes which K-types are relevant for determining unitarity for irreducible Hermitian $(\mathfrak{g}, K)$ modules.

All of this material is intended as a report on ideas of David Vogan.

Part I. A parametrization of irreducible representations of K

Let G be a real reductive group in Harish-Chandra's class. It may be instructive and useful to weaken that hypothesis, but we content ourselves with it here. It certainly contains the class of groups obtained as the real points of a connected reductive algebraic group defined over $\mathbb{R}$. (Henceforth we shall call these groups "algebraic".)

Let K be the maximal compact subgroup of G. The point of these notes is to recall a parametrization of $\widehat{K}$ (i.e. equivalence classes of irreducible representations of K) due to David Vogan. Note that even if G is algebraic, the description of $\widehat{K}$ is not covered by [duCloux]: the group K need not belong to the class considered there.

For orientation one should consult [branch]. Those notes provide provide a completely different perspective, essentially that of Cartan-Weyl, and parametrize $\widehat{K}$ in terms of irreducible representations of a large Cartan subgroup. By contrast, these notes intricately use the fact that our K is the maximal compact subgroup of G.

Theorem 1 *Let $\widehat{G}^{\mathrm{temp},\circ}$ denote the set of irreducible tempered representations with real infinitesimal character. Then the map*

$$\widehat{G}^{\mathrm{temp},\circ} \longrightarrow \widehat{K}$$

obtained by taking lowest K-types is a well-defined bijection. More precisely, if $\overline{X} \in \widehat{G}^{\mathrm{temp},\circ}$, then

1. *$\overline{X}$ has a unique lowest K-type;*

2. *Two irreducible tempered representations with real infinitesimal character whose lowest K types coincide are necessarily isomorphic; and*

3. *Each K type $\mu \in \widehat{K}$ arises as the lowest K-type of an element of $\widehat{G}^{\text{temp},\circ}$*

Thus $\widehat{K}$ is parametrized by $\widehat{G}^{\text{temp},\circ}$. A parametrization of this latter set in terms of (more or less) combinatorial data is given in Proposition 10. Putting them together we get the parametrization we seek (Corollary 11).

Sketch. Fix $\mu \in \widehat{K}$ and let T denote a maximal torus in K. Let $\mathfrak{t}_\circ$ denote the Lie algebra of T. We seek to find a irreducible unique tempered representation with real infinitesimal character and lowest K-type μ. This is a consequence of the Vogan-Zuckerman classification. The classification attaches an element $\lambda(\mu) \in i\mathfrak{t}_\circ^*$ to μ. An algorithm to compute $\lambda(\mu)$ is given in [Vgr, Proposition 5.3.3]; see also Definition 6.6.4 and Lemma 6.6.5 of [Vgr]. Later improvements of the algorithm due to Carmona are summarized in [SV]. Let $\mathfrak{q}$ denote the θ-stable parabolic subalgebra of $\mathfrak{g}$ (the complexified Lie algebra of G) defined by $\lambda(\mu)$ and write L for the analytic subgroup of G corresponding to $\mathfrak{q} \cap \bar{\mathfrak{q}}$. The group L is quasisplit. Let μ_L denote the $L \cap K$ type generated by a T highest weight space of μ. (It is not immediately obvious that the highest weight space generates an irreducible $L \cap K$ representation. But it does.) Then μ_L is a fine $L \cap K$ type for L. The classification provides (via cohomological induction using $\mathfrak{q}$) a bijection between irreducible representations of G with lowest K-type μ and irreducible representations of L with lowest $(L \cap K)$-type μ_L. The bijection restricts to a bijection of tempered representations. This is sketched in [Vgr, Section 6.7], although it should not be difficult to prove it directly without reference to the Langlands classification.

In any event, the bijection of the classification reduces matters to G quasisplit and μ fine (i.e. to the case that $\lambda(\mu)$ is central). More precisely, it is enough to show that for each fine K-type μ there exists a unique tempered representation with real infinitesimal character which has μ as its unique lowest K-type. This case is treated in [Vgr, Chapter 4]. Here is a sketch. Let MA denote a maximally split θ-stable Cartan subgroup in G. Write

$$\mu|_M = \delta_1 \oplus \cdots \oplus \delta_k. \tag{2}$$

This decomposition is multiplicity free since μ is fine. Choose an M-type δ_i appearing in the restriction. Consider the principal series

$$I(\delta_i) = \text{ind}_{MAN}^G(\delta_i \otimes \mathbb{1} \otimes \mathbb{1});$$

we will have no occassion to specify N. Obviously $I(\delta_i)$ is tempered (since it is induced from a discrete series). By Frobenius reciprocity (and the fact that the restriction in (2) is multiplicity-free), it follows that μ appears with multiplicity one in $I(\delta_i)$. Let $X(\mu)$ denote the constituent of $I(\delta_i)$ containing μ. The main results of [Vgr, Chapter 4] imply that $X(\mu)$ is well-defined independent of the choice of δ_i and that the (unique) lowest K-type of $X(\mu)$ is μ. This completes the case of μ fine, and hence the sketch of the theorem. □

The perspective offered by Theorem 1 has a number of wonderful advantages. It appears to be the right kind of "data structure" from the point of view of du Cloux's existing software. In Remark 14 we explain how to compute lowest K-types of irreducible representations using this parametrization of $\widehat{K}$. We also remark that in the case that G is algebraic, the computation of the character lattice of a large Cartan subgroup of K seems to be tractable using du Cloux's software. This suggests the interesting (and important) auxilliary problem of implementing a translation between the parametrization of Theorem 1 and the Cartan-Weyl parametrization.

Example 3 For orientation, we include the example of $G = \mathrm{SL}(2,\mathbb{R})$. Of course $K = \mathrm{SO}(2)$ and $\widehat{K}$ is parametrized by $\mathbb{Z}$. In the obvious notation, write $\mu_n \in \widehat{K}$. The relevant observation is that for $|n| \geq 2$, μ_n is the lowest K type of a (tempered) discrete series. Meanwhile the two representations $\mu_{\pm 1}$ arise as the lowest K types of the two (tempered) irreducible limits of discrete series. (We may also realize the pair $\mu_{\pm 1}$ as the the lowest K types of the reducible nonspherical (tempered) principal series with infinitesimal character zero; this is where the R-group shenanigans first appear.) Finally, the trivial representation μ_0 is the lowest K type of the (tempered) irreducible spherical principal series with infinitesimal character zero. In particular, we see that by passage to lowest K-types, we obtain a bijection from the set of irreducible tempered representation of G with real infinitesimal character to $\widehat{K}$. □

Example 4 It's also a good idea to keep $\mathrm{GL}(2,\mathbb{R})$ in mind; here K is the disconnected orthogonal group $\mathrm{O}(2)$. This time $\widehat{K}$ is parametrized by strictly positive integers, together with trivial and sgn representations which we denote $\mu_0^{\pm}$. For $n > 0$, the K type μ_n arises as the lowest K type of relative discrete series (or in the case of μ_1, relative limits of discrete series). To account for $\mu_0^{\pm}$, note that there are four tempered principal series with infinitesimal character zero corresponding to the four characters of M. Two of these principal series are isomorphic and isomorphic to a relative limit of discrete series; so we have already accounted for them. The other two are distinct; one has lowest K type μ_0^+, the other μ_0^-. (See Example 12 below for a sharper treatment of $\mathrm{GL}(2)$.) □

Example 5 Next we consider $\mathrm{U}(1,1)$ to illustrate how the parametrization behaves in the rank one case even when we restrict to connected groups. In this case $K = \mathrm{U}(1) \times \mathrm{U}(1)$ and a K-type is thus a pair of integers $(a,b) \in \mathbb{Z}^2$. Suppose $\lambda = (\lambda_1, \lambda_2)$ is the Harish-Chandra parameter of a discrete series or a limit of discrete series. So $\lambda_i \in \frac{1}{2} + \mathbb{Z}$, and there are two cases:

$\lambda_1 \geq \lambda_2$. The lowest K type of the corresponding discrete series (or limit) is

$$\lambda + \rho = \left(\lambda_1 + \frac{1}{2}, \lambda_2 - \frac{1}{2}\right).$$

Hence we obtain all K types of the form $(a,b) \in \mathbb{Z}^2$ with $a > b$.

$\lambda_1 \leq \lambda_2$. The lowest K type of the corresponding discrete series (or limit) is

$$\lambda + \rho = \left(\lambda_1 - \frac{1}{2}, \lambda_2 + \frac{1}{2}\right).$$

Hence we obtain all K types of the form $(a, b) \in \mathbb{Z}^2$ with $a < b$.

Thus the discrete series and limits parameterize all K types of the form (a, b) with $a \neq b$. We are missing those with $a = b$ and according to the parametrization we need to look for them in representations induced from the Borel (with real infinitesimal character). The split Cartan H is naturally isomorphic to $\mathbb{C}^\times$, and the characters of $\mathbb{C}^\times$ that give real infinitesimal character are of the form $\chi_n(z) = (z/|z|)^n$ for $n \in \mathbb{Z}$. The lowest K types of the corresponding standard representations, say $X(\chi_n)$, are

$$(n/2, n/2) \text{ if } n \text{ is even,}$$

and

$$((n \pm 1)/2, (n \mp 1)/2) \text{ if } n \text{ is odd.}$$

In the latter case, the induced representation contains two limits of discrete series, and we have already accounted for them. (Roughly speaking, in the terminology introduced below, $X(\chi_n)$ fails condition (F2).) For n odd, the lowest K types of the various $X(\chi_n)$ account for the missing K-types of the form (a, a). $\square$

Now we turn to parametrizing $\widehat{G}^{\text{temp},\circ}$. To begin, we need to discuss how to parametrize *all* irreducible admissible representations of G. We are going to trot out pseudocharacters; these are different from the parameters in [duCloux], but translating between the two is tractable. (This problem will be taken up in the future by Paul, du Cloux, and others.) The main point is that all the conditions we impose on our parameters also translate nicely into the framework of [duCloux].

Write θ for the Cartan involution of G. Let $\mathfrak{h}_\circ = \mathfrak{t}_\circ \oplus \mathfrak{a}_\circ$ denote a θ-stable Cartan in $\mathfrak{g}_\circ$, the Lie algebra of G. As usual drop $\circ$ subscripts to denote complexifications. Let H denote the centralizer of $\mathfrak{h}_\circ$ in G. The decomposition $\mathfrak{h}_\circ = \mathfrak{t}_\circ \oplus \mathfrak{a}_\circ$ implies that $H = TA$ where $T = H \cap K$ and $A = \exp(\mathfrak{a}_\circ)$ is a vector group. A *regular pseudocharacter* of H is a pair

$$\gamma = (\Gamma, \bar{\gamma})$$

subject to the following conditions:

(R1) Γ is an irreducible representation of H and $\bar{\gamma} \in \mathfrak{h}^*$;

(R2) Suppose α is an imaginary root of $\mathfrak{h}$ in $\mathfrak{g}$. Then $\langle \bar{\gamma}, \alpha \rangle$ is real and nonzero, and hence $\bar{\gamma}$ defines a system of positive roots Ψ (making $\bar{\gamma}$ dominant);

(R3) If we write $\rho(\Psi)$ for the half-sum of the elements of Ψ and $\rho_c(\Psi)$ for the half-sum of the compact ones, then

$$d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi). \tag{6}$$

Write $\mathcal{P}_{\text{reg}}(H)$ for the set of regular pseudocharacters attached to H.

To each $\gamma = (\Gamma, \bar{\gamma}) \in \mathcal{P}_{\text{reg}}(H)$, we may build a standard module $X(\gamma)$ as follows. Let $L = MA$ denote the centralizer in G of A. Conditions (R2) and (R3) imply that $\bar{\gamma}|_{\mathfrak{t}}$ is the Harish-Chandra parameter of a discrete series for M. Since M may be disconnected, the Harish-Chandra parameter need not determine a single discrete series. We may, however, specify such a discrete series, say X^M, by requiring that its lowest $M \cap K$ type have highest weight $\Gamma|_T$. Next choose a real parabolic subgroup MN so

$$\text{the real part of } \bar{\gamma} \text{ restricted to } \mathfrak{a} \text{ is negative on the roots of } \mathfrak{a} \text{ in } \mathfrak{n}. \tag{7}$$

Meanwhile write ν for the character of (the simply connected group) A corresponding to $\bar{\gamma}|_{\mathfrak{a}}$. Define

$$X(\gamma) = \operatorname{ind}_{MAN}^{G}(X^M \otimes \nu \otimes \mathbb{1}).$$

Then $X(\gamma)$ has infinitesimal character $\bar{\gamma}$ and the condition in (7) guarantees that the (possibly reducible) Langlands subrepresentations occurs as a submodule.

The standard modules $X(\gamma)$ for $\gamma \in \mathcal{P}_{\text{reg}}$ are enough for some purposes, but not enough for a classification. For instance, for $\mathrm{SL}(2, \mathbb{R})$, the only way to get the two limits of discrete series is as the two constituents of the (reducible) nonspherical principal series with infinitesimal character zero. Thus if we are interested in a map from our standard modules to irreducibles, it must be multivalued. To remedy this, we must enlarge the class of standard modules (by considering "limit" pseudocharacters); in the SL(2) case, this amounts to including the two limits of discrete series as standard modules. To make a bijection between standard and irreducibles, we must then throw out some standard modules our (by restricting to "final" limit pseudocharacters); in the SL(2) case, this amounts to throwing out the nonspherical principal series with infinitesimal character zero since their constituents are already accounted for by the addition of the limits of discrete series as standard modules. (In general, we will throw out the standard modules corresponding to the "less compact" terms in a Hecht-Schmid character identity.)

We begin with the enlarged set of standard modules. A *limit pseudocharacter* of H is a triple

$$\gamma = (\Psi, \Gamma, \bar{\gamma})$$

with the following properties:

(L1) Ψ is a positive system for the imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$, Γ is an irreducible representation of H; and $\bar{\gamma} \in \mathfrak{h}^*$;

(L2) If $\alpha \in \Psi$, then $\langle \bar{\gamma}, \alpha \rangle \geq 0$;

(L3) $d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi)$.

Let $\mathcal{P}_{\text{lim}}(H)$ denote the limit pseudocharacters for H. Clearly $\mathcal{P}_{\text{reg}}(H) \subset \mathcal{P}_{\text{lim}}(H)$: the regular pseudocharacter $(\Gamma, \bar{\gamma})$ gets mapped to $(\Psi, \Gamma, \bar{\gamma})$ where Ψ is specified by (R2) above.

But the notion of limit pseudocharacter allows for more: the infinitesimal character $\bar{\gamma}$ can now be singular according to (L2). In this case, Ψ is not uniquely specified of course; indeed (in the connected case) we should think of Ψ specifying a chamber of discrete series from which we translate to infinitesimal character $\bar{\gamma}$ of a limit of discrete series for M. To each $\gamma \in \mathcal{P}_{\lim}(H)$, we may define a standard module $X(\gamma)$ as above; this time X^M is a limit of discrete series and the choice of N (as in (7)) is a little messier. Two issues present themselves: the limit of discrete series X^M may be zero; and (as in the case of $\mathrm{SL}(2,\mathbb{R})$), we need to rule out certain reducibilities among the standard modules $X(\gamma)$ accounted for by Hecht-Schmid character identities. The conditions (F1) and (F2) below are designed with these respective issues in mind, and determine the standard modules we want to throw out.

A limit pseudocharacter is *final* if

(F1) If α is a simple root in Ψ such that $\langle \alpha, \bar{\gamma} \rangle = 0$, then α is noncompact;

(F2) If α is a real root of $\mathfrak{h}$ in $\mathfrak{g}$ and $\langle \alpha, \bar{\gamma} \rangle = 0$, then α does not satisfy the Speh-Vogan parity condition.

We write $\mathcal{P}_{\mathrm{fin}}(H)$ for the set of final limit pseudocharacters for H and $\mathcal{P}_{\mathrm{fin}}$ for the union of all $\mathcal{P}_{\mathrm{fin}}(H)$. Note that K acts on $\mathcal{P}_{\mathrm{fin}}$ by componentwise conjugation.

To each $\gamma \in \mathcal{P}_{\mathrm{fin}}(H)$, we attach a standard module $X(\gamma)$. This time $\Gamma|_T$ and Ψ define a limit of discrete series X^M of M (which is nonzero by (F1)). As above, write ν for the character of A determined by $\gamma|_{\mathfrak{a}_\circ}$. The associated standard module is given by parabolic induction,

$$X(\gamma) = \mathrm{ind}_{MAN}^{G}(X^M \otimes \nu \otimes \mathbb{1}).$$

Note that we have been somewhat sloppy and have not specified N precisely. But we have no occasion to do so. (What matters for us is that the global character and K-type spectrum of $X(\gamma)$ are independent of the choice of N.) The condition (F2) insures that $X(\gamma)$ has a unique irreducible constituent that contains all of the lowest K-types of $X(\gamma)$. We write $\overline{X}(\gamma)$ for that constituent. (Note that the definition of $\overline{X}(\gamma)$ does not depend on the choice of N.)

Here is the sharpened version of the Langlands-Knapp-Zuckerman classification as interpreted by Vogan (e.g. [Vunit, Section 2]).

Theorem 8 *Retain the notation above.*

(a) Suppose $\gamma_1, \gamma_2 \in \mathcal{P}_{\mathrm{fin}}$. Then

$$\overline{X}(\gamma_1) \simeq \overline{X}(\gamma_2)$$

if and only if γ_1 and γ_2 are conjugate by K.

(b) Suppose $\overline{X}$ is an irreducible Harish-Chandra module for G. Then there exists $\gamma \in \mathcal{P}_{\mathrm{fin}}$ such that $\overline{X} \simeq \overline{X}(\gamma)$.

Consequently (equivalence classes of) irreducible Harish-Chandra module for G are parametrized by K orbits on $\mathcal{P}_{\mathrm{fin}}$.

Here is where the tempered modules fit in the above classification.

Definition 9 Define $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$ to be the subset of $\gamma \in \mathcal{P}_{\mathrm{fin}}(H)$ such that the restriction of $\bar{\gamma}$ to $\mathfrak{a}$ is identically zero. Let $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}$ denote the union of all $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$. (Note again that K acts on $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}$.) We say that $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}$ is the set of *tempered final limit pseudocharacters with real infinitesimal character*. The terminology is explained by the following result.

Proposition 10 *Fix $\gamma \in \mathcal{P}_{\mathrm{fin}}$. Then $\overline{X}(\gamma)$ is tempered if and only if $\gamma \in \mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}$. In this case,*

$$\overline{X}(\gamma) = X(\gamma);$$

that is, the standard module corresponding to $\gamma \in \mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}$ is already irreducible.

Corollary 11 *The set of equivalence classed of irreducible admissible representations of K is parametrized by K orbits on $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$ (Definition 9). The parametrization takes an orbit $K \cdot (\gamma, H)$ in $\mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}(H)$ to the lowest K-type of $X(\gamma)$.*

The corollary follows immediately from Proposition 10 and Theorem 1.

Example 12 Let $G = \mathrm{GL}(n, \mathbb{R})$. We begin with some structure theoretic facts. Begin by noting that the K-conjugacy classes of Cartan subalgebras in G are parametrized by partitions $n = a + 2b$ so that if H denotes a corresponding representative of such a class,

$$H \simeq (\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b.$$

Under the Cartan decomposition, $H = TA$ with

$$T \simeq (\mathbb{Z}/2)^a \times (S^1)^b,$$

or perhaps more suggestively

$$T \simeq \mathrm{O}(1)^a \times \mathrm{SO}(2)^b,$$

and $A \simeq (\mathbb{R}_{>0})^{a+b}$. Thus a character Γ of H is given by the data of an a-tuple of signs,

$$\epsilon = (\epsilon_1, \ldots, \epsilon_a) \in (\mathbb{Z}/2)^a,$$

and a b-tuple of integers

$$(m_1, \ldots, m_b) \in \mathbb{Z}^b,$$

and an $(a + b)$-tuple of complex numbers $(\nu_1, \ldots, \nu_a)$.

The imaginary roots of $\mathfrak{h} \in \mathfrak{g}$ (in the standard coordinates for $\mathrm{SO}(2)^b$) are simply $\{\pm 2e_1, \cdots \pm 2e_b\}$; all of them are noncompact. (This assertion really amounts to one about GL(2) where it is easy to check.)

Now let $(\Psi, \Gamma, \bar{\gamma})$ be a final limit pseudocharacter with real infinitesimal character (Definition 9) with Γ a character of H. We claim that such a pseudocharacter amounts to the following data

(a) a pair of nonnegative integers a and b so that $n = a + 2b$;

(b) a b-tuple of nonnegative integers $(m_1, \ldots, m_b)$; and

(c) a single sign ϵ (if $a \neq 0$).

Here is a sketch. The choice of H (up to K-conjugacy) is the data of the partition in (a). Let Γ be given by the tuples (m_i), (ϵ_j), and (ν_k) as described above. The real infinitesimal character requirement (that $\bar{\gamma}$ restricted to $\mathfrak{a}$ is zero) together with (L3) means that each $\nu_k = 0$. The data of Ψ in (L1) together with the requirements of (L2) means that we may take each $m_i \geq 0$; this is the data of (b). Since there are no compact imaginary roots, (F1) is empty. Meanwhile (F2) means that all of the signs ϵ_j must all be $+$ or all be $-$. (If a pair of signs differ, they specify a real root and we would be able to apply a Hecht-Schmid identity to arrive at a more compact Cartan.) So indeed the data of the tuple ϵ reduces to a single sign ϵ; this is the sign in (c).

Thus to the data of (a)-(c) we may attach an irreducible tempered representation of G with real infinitesimal character. What is the corresponding lowest K-type? Here is the answer. Consider

$$L = \mathrm{SO}(2)^b \times \mathrm{O}(a) \subset K = \mathrm{O}(n).$$

So a representation of L is given by a b-tuple of integers tensored with a representation of $\mathrm{O}(a)$. Then the corresponding lowest K-type μ is the unique K-type so that

(i) The restriction of μ to L contains the L representation specified by

$$(m_1 + 1, \ldots, m_b + 1) \otimes \mathbb{1}$$

if ϵ is $+$;

$$(m_1 + 1, \ldots, m_b + 1) \otimes \det$$

if ϵ is $-$; and

(ii) the restriction of μ to $\mathrm{SO}(2)^b$ contains no weight higher than $(m_1 + 1, \ldots, m_b + 1)$.

The case of GL(2) (Example 4) is especially instructive. □

Remark 13 Suppose that γ' is a limit pseudocharacter that satisfies (F1) but not (F2). Then we may still form the standard module $X(\gamma)$ as described above. Loosely speaking, the failure of (F2) can be resolved after crossing a sequence of walls, and each wall corresponds to a Hecht-Schmid character identity. More precisely, one may apply a sequence of (multivalued) inverse Cayley transforms to arrive at a set $\{\gamma_i'\}$ with each $\gamma_i' \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}$. (The number of possible inverse Cayley transforms is bounded by the rank of $\mathfrak{g}$ and each can produce at most two new pseudocharacters, so the cardinality of $\{\gamma_i'\}$ is at most 2 to the rank of $\mathfrak{g}$, and typically much less.) The Hecht-Schmid identities imply that

$$[X(\gamma')] = \sum_i [X(\gamma_i')];$$

here the right-hand side is effectively computable. Thus if (F2) fails for γ', the standard module $X(\gamma')$ may be written as a sum of standard modules corresponding to elements in $\mathcal{P}_{\text{fin}}^{\text{temp},\circ}$.

Remark 14 As promised, we conclude by describing how to compute lowest K types in terms of this parametrization. Suppose X is any irreducible admissible representation of G. Using the classification above, we may write $X = \overline{X}(\gamma)$ for some $\gamma \in \mathcal{P}_{\text{fin}}$. Then X is the lowest K type constituent of $X(\gamma)$. Now modify γ by making the e^{ν} factor trivial: that is, change $\bar{\gamma}$ to $\bar{\gamma}' := \bar{\gamma} - \nu$, let $\Gamma' := \Gamma \otimes e^{-\nu}$, but leave Ψ unchanged. The resulting γ' is still a limit pseudocharacter which satisfies the first final condition (F1). Unfortunately since we have changed the infinitesimal character, (F2) can fail. If (F2) does hold, then $\gamma' \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}$, and the lowest K-type of X is simply the one parametrized (according to Corollary 11) by γ'; in particular the lowest K type is unique. In the case that (F2) fails, we may apply the procedure of Remark 13 to arrive at $\gamma_1', \ldots, \gamma_k' \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}$. Then the lowest K types of X are precisely those parametrized by $\gamma_1', \ldots, \gamma_k'$, i.e. the lowest K types of the standard modules $X(\gamma_i')$. This completes the calculation of the lowest K types of X in parametrization of Corollary 11.

Interlude: What K-types matter?

Retain the setting above. Recall our eventual goal: to compute signatures of invariant Hermitian forms on irreducible $(\mathfrak{g}, K)$ modules that possess them. This computation can of course be handled K-type by K-type, so it's evident that it is essential to be able to compute K-isotypic components of irreducible $(\mathfrak{g}, K)$ modules. The terminology "computing isotypic components" is vague, but in the very least it requires computing multiplicities of K-types in irreducible modules. The Kazhdan-Lustzig-Vogan algorithm (which appears reasonably close to being efficiently implementable in Fokko's software) makes the problem of computing K-types in irreducibles equivalent to computing those in standard modules. This is discussed more carefully in Part II below.

But before turning to Part II, we recall that for applications to unitarity one need not be concerned with all K-types, but only certain "small ones." The modifier "small" may be quantified in the following definition.

Definition 15 (Salamanca-Vogan [SV]) Fix G. Recall the notation introduced in the proof of Theorem 1. We say that a K type μ is unitarily small (for G) if $\lambda(\mu)$ is contained in the convex hull of the Weyl group orbit of ρ_G.

Here is the reason for the definition.

Theorem 16 (Salamanca-Vogan [SV]) *Suppose that X is an irreducible Hermitian representation of G which does not contain a unitarily small K-type. Then there exists a proper θ-stable parabolic subgroup $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ and an irreducible Hermitian representation X^L of L containing a unitarily small $L \cap K$ type so that*

$$X \text{ is unitary if and only if } X^L \text{ is unitary.}$$

Hence the classification of the unitary dual is reduced (by induction on rank) to the classification of unitary representations of G which contain a unitarily small K-type.

The above theorem is a consequence of a kind of bottom layer argument. In particular examples, that argument yields far more information (and hence a more effective induction) than provided for by the theorem.

Here is how to compute unitarity of representations of G which contain a unitarily small K-type.

Conjecture 17 (Salamanca-Vogan [SV]) *Suppose X is an irreducible $(\mathfrak{g}, K)$ module that possesses an invariant Hermitian form $\langle \cdot, \cdot \rangle$. Suppose in addition that X contains a unitarily small K type. Write $\langle \cdot, \cdot \rangle_\mu$ for Hermitian form obtained by restricting $\langle \cdot, \cdot \rangle$ to the μ-isotypic component of X. Then $\langle \cdot, \cdot \rangle$ is positive definite if and only if $\langle \cdot, \cdot \rangle_\mu$ is positive definite for all unitarily small K types μ. (By our hypthesis on X this condition is not empty.)*

We recall evidence for the conjecture taken from [Vunit]. Define the *length of* μ, denoted $\| \mu \|$, to be the length of $\lambda(\mu)$ computed using a fixed invariant bilinear form. Then [Vunit] proves (in the setting of Conjecture 17) that there exists a explicitly computable constant N so that $\langle \cdot, \cdot \rangle$ is positive definite if and only if $\langle \cdot, \cdot \rangle_\mu$ is positive definite for each μ such that $\| \lambda(\mu) \| \leq N$. The number of μ such that $\| \lambda(\mu) \| \leq N$, while finite, is significantly larger than the number of unitarily small K-types.

In practice, the number of K-types that ones needs to check is *much* less than those described in the conjecture. In any any event, to determine the unitarity of an irreducible Hermitian $(\mathfrak{g}, K)$ module for which Theorem 16 provides no reduction, one need only to test that a finite number of the forms $\langle \cdot, \cdot \rangle_\mu$ are indeed positive.

Part II. Computing K-type multiplicities in standard modules.

As described at the beginning of the interlude, what we really want to do is compute the multiplicity of sufficiently "small" K-types (in the sense of Definition 15 for instance) in irreducible (Hermitian) Harish-Chandra modules. To describe a strategy to do so, we need some additional notation. Given a Harish-Chandra module, let $[X]$ denote its class in the Grothendieck group of Harish-Chandra modules. Let $\mathbb{Z}[[\widehat{K}]]$ denote the ring of formal integral linear combinations of K-types,

$$\mathbb{Z}[[\widehat{K}]] = \left\{ \sum_{\mu \in \widehat{K}} n_\mu [\mu] \;\middle|\; \mu \in \widehat{K} \right\}.$$

Given any (virtual) class $[Z]$, there is an obvious notion of restriction to K that gives rise to an element of $\mathbb{Z}[[\widehat{K}]]$. Write $[Z]|_K$ for this restriction.

Given a final limit pseudocharacter γ consider the class $[X(\gamma)]$ of the corresponding standard module. Recall that we didn't really define $X(\gamma)$ in general in Part I: we failed to specify the nilradical N precisely. But in the Grothendieck group, $[X(\gamma)]$ is independent of the choice of N.

Fix an irreducible X. Let λ denote the infinitesimal character of X. Let $\mathcal{P}^\lambda_{\text{fin}}$ denote the set of final limit pseudocharacters whose corresponding standard modules have infinitesimal character λ. Then $\mathcal{P}^\lambda_{\text{fin}}$ is finite. In the Grothendieck group, it is well-known that we may write

$$[X] = \sum_{\gamma \in \mathcal{P}^\lambda_{\text{fin}}} m_\gamma [X(\gamma)],$$

where the sum is of course finite. The proof of the Kazhdan-Lusztig conjecture (due to Vogan) implies that the coefficients m_γ are explicitly computable for linear groups in Harish-Chandra's class whose Cartans are all abelian. Restricting to K, we may write

$$[X]|_K = \sum_{\gamma \in \mathcal{P}^\lambda_{\text{fin}}} m_\gamma [X(\lambda)]|_K,$$

in $\mathbb{Z}[[\widehat{K}]]$. Thus computing the coefficient of $[\mu]$ in $[X]|_K$ is equivalent (given the Kazhdan-Lusztig algorithm) to computing the coefficient of $[\mu]$ in each standard module $[X(\gamma)]|_K$. According to the definition of these modules, Frobenius reciprocity implies that this amounts to branching from K to $K \cap M$ (together with the Blattner formula for M). Since M and $K \cap M$ are disconnected, this is a difficult branching problem. The purpose of this section is an alternative approach that takes advantage of the fact that K is a maximal compact subgroup of M.

Vogan proposes to solve an inverse problem: instead of computing the coefficient of $[\mu]$ in a standard module, he suggests that, roughly speaking, one ought to try instead to write $[\mu]$ as a linear combination of standard modules in $\mathbb{Z}[[\widehat{K}]]$. That is, he suggests that one ought to look for expressions of the form

$$[\mu] = \sum_{\gamma \in \mathcal{P}_{\text{fin}}} [X(\gamma)]|_K$$

for each sufficient small μ (of which there are only finitely many), and then invert these expressions to compute the coefficient of $[\mu]$ is a particular standard module $[X(\gamma)]|_K$.

To make this precise, we need to work within the framework of Part I. Recall that Corollary 11 defines a bijection between $\widehat{K}$ and standard modules of the form $X(\gamma)$ with $\gamma \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}$. The length of the infinitesimal character of $X(\gamma)$ defines a partial order on the latter set (and using the bijection on $\widehat{K}$). Using this bijection and partial order, we may define a matrix whose rows are indexed by $\widehat{K}$ and whose columns are indexed by the standard modules with real infinitesimal character corresponding to tempered representations with entries gives as follows: the entry in the $X(\gamma)$ column and in the $[\mu]$ row is the coefficient of $[\mu]$ in $[X(\gamma)]|_K$. Here is a strengthened version of Corollary 11 taken again from [Vunit].

Theorem 18 *The (infinite) matrix, say $\mathbb{M}$, defined in the previous paragraph relating standard tempered modules with real infinitesimal character to $\widehat{K}$ in $\mathbb{Z}[[\widehat{K}]]$ is upper-triangular with 1's on the diagonal. Hence it is invertible.*

According to the Interlude, we are interested in only sufficiently small μ. Said differently, we are only interested in some (finite) upper left corner of $\mathbb{M}$. More precisely, if for all μ so that $\parallel \mu \parallel \leq N$, we find expressions of the form

$$[\mu] = \sum_{\gamma \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}} m_\gamma [X(\gamma)]|_K \tag{19}$$

then (after inverting a finite matrix) we obtain, for all μ and $\gamma \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}$ with $\parallel \bar{\gamma} \parallel \leq N$ and $\parallel \mu \parallel \leq N$, the multiplicity of μ in $[X(\gamma)]|_K$.

There is one last minor point to make. Initially we were concerned with the multiplicity of sufficiently small μ in an arbitrary standard representation $X(\gamma)$ with $\gamma \in \mathcal{P}_{\text{fin}}$. The previous paragraph describes a strategy to solve that problem for $\gamma \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}$. But the former reduces to the latter as follows. Given arbitrary $\gamma \in \mathcal{P}_{\text{fin}}$, recall from Part I that

$$X(\gamma) = \text{ind}_{MAN}(X^M \otimes \nu \otimes \mathbb{1}).$$

It is obvious (from Frobenius reciprocity) that

$$[X(\gamma)]|_K = [\text{ind}_{MAN}(X^M \otimes \mathbb{1} \otimes \mathbb{1})]|_K.$$

So let γ' be the pseudocharacter by taking $\nu = 0$ in γ, i.e. so that the right hand side of the previous displayed equation is the standard module corresponding to γ'. Just as in Remark

14, it is easy to check that γ' is a limit pseudocharacter satisfying (L1)–(L3); but γ' is no longer final: (F1) holds, but (F2) need not. Using Remark 13, one may write

$$[X(\gamma')] = \sum_i [X(\gamma_i')],$$

with each $\gamma_i' \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}$. Since we remarked above that $[X(\gamma)]|_K = [X(\gamma')]|_K$ it is indeed enough to determine expressions of the form of (19) for all sufficiently small μ. Summarizing, the main problem is

Problem A. For all sufficiently small μ, obtain explicit expressions of the form (19) expressing $[\mu]$ as a linear combination of standard tempered modules with real infinitesimal character.

Example 20 Let $\mathcal{B}_{\mathbb{1}} \subset \mathcal{P}_{\text{fin}}$ denote the subset of pseudocharacters with infinitesimal character ρ in the block of the trivial representation $\mathbb{1}_G$ of G. (For terminology relating to blocks see [Vgr, Chapter 9].) Then one version of the Zuckerman character formula is the following identity in the Grothendieck group of Harish-Chandra modules,

$$[\mathbb{1}_G] = \sum_{\gamma \in \mathcal{B}_{\mathbb{1}}} \epsilon_\gamma [X(\gamma)];$$

here ϵ is zero if the character Γ of H (which is part of the data of γ) is not trivial on the component group of H, and otherwise ϵ is ± 1. (The precise sign — either plus or minus — is easy to determine.) Restricting to K, we the following identity in $\mathbb{Z}[[\widehat{K}]]$,

$$[\mathbb{1}_K] = \sum_{\gamma \in \mathcal{B}_{\mathbb{1}}} \epsilon_\gamma [X(\gamma)]|_K.$$

As in the discussion preceding Problem A, we can take the ν parameter of each standard module on the right-hand side to be zero (without affecting the formula in $\mathbb{Z}[[\widehat{K}]]$), and then express the right-hand side as a sum of standard tempered modules. So this solves Problem A for $\mu = \mathbb{1}_K$.

We now describe Vogan's strategy to solve Problem A in more detail. He suggest a method to find expressions for μ as a sum of various "continuations" of standards modules. These standard modules need not be of the form $X(\gamma)$ for $\gamma \in \mathcal{P}_{\text{fin}}^{\text{temp},\circ}$, but as in Example 20, they can be converted into standard tempered modules (in $\mathbb{Z}[[\widehat{K}]]$) and hence lead to a solution of Problem A.

We first define the relevant continued standard modules. First we assume that G possesses a compact Cartan subgroup. Then the relevant continued standard modules are simply coherent continuations of discrete series, as originally considered by Schmid.

In more detail, Let T denote a (generally disconnected) θ-stable compact subgroup of G. Fix a choice of positive roots Ψ_c for T in $\mathfrak{k}$, and a choose positive roots Ψ (containing

Ψ_c) for T in $\mathfrak{g}$. (Ψ_c will be fixed once and for all, but Ψ will vary in the discussion below.) Write ρ_c for the half-sum of the elements of Ψ_c and likewise for ρ.

Write $\Lambda(T)$ for the character lattice of T. Recall that $2\rho \in (i\mathfrak{t}_\circ)^*$ exponentiates to T, but ρ need not. We will be somewhat sloppy and write $2\rho \in \Lambda(T)$ for both the exponentiated character (in $\Lambda(T)$) and its differential in $i(\mathfrak{t}_\circ)^*$. Let $\widetilde{T}$ denote the (abelian) double cover of T defined by the squareroot of the 2ρ character and write $\Lambda(\widetilde{T})$ for its lattice of genuine characters. Recall that $\widetilde{T}$ comes equipped with a genuine character whose differential is ρ. Again we will be sloppy and write $\rho \in \Lambda(\widetilde{T})$ for both this character and its differential. By tensoring with ρ, we may canonically identify $\Lambda(\widetilde{T})$ with $\Lambda(T) \otimes \rho$, but we avoid doing this at the moment.

Fix $\Phi \in \Lambda(\widetilde{T})$. We claim that the character Φ together with the choice of Ψ (containing the fixed Ψ_c) uniquely specify a unique K conjugacy class, say $K \cdot (\Gamma, \Psi, \bar{\gamma})$ which is almost in $\mathcal{P}_{\text{fin}}^{\text{temp},\circ}$ (except possibly for the failure of (F1) and (L2) as explained below). Here Ψ in the pseudocharacter is the Ψ we have already specified, and we now explain how to define $\bar{\gamma}$ and Γ. Let $\mathfrak{b}_\Psi = \mathfrak{t} \oplus \mathfrak{u}$ denote the Borel subalgebra of G corresponding to Ψ. Then $\wedge^{\text{top}}(\bar{\mathfrak{u}} \cap \mathfrak{k})$ defines a character of T with differntial $-2\rho_c$. Meanwhile $\Phi \otimes \rho$ also defines a character of T. Define

$$\Gamma = (\Phi \otimes \rho) \otimes \wedge^{\text{top}}(\bar{\mathfrak{u}} \cap \mathfrak{k}) \in \Lambda(T),$$

a character of T. The differential of Γ is

$$d\Gamma = d\Phi + \rho - 2\rho_c \in (i\mathfrak{t}_\circ)^*.$$

Define

$$\bar{\gamma} = d\Phi.$$

Since we have fixed T, the K-conjugacy class of $(\Gamma, \Psi, \bar{\gamma})$ is defined only up to the Weyl group of K. But we have also fixed Ψ_c. So the K-conjugacy class of the triple $(\Gamma, \Psi, \bar{\gamma})$ is uniquely specified by $\Psi_c \subset \Psi$ and $\Phi \in \Lambda(\widetilde{T})$.

It remains to determine whether $\gamma = (\Gamma, \Psi, \bar{\gamma})$ so-defined actually defines a final limit pseudocharacter. Clearly (L1) and (L3) hold and (F2) is empty (since there are no real roots). So it remains to investigate (L2) and (F1). From the definitions, it is easy to see that (L2) is equivalent to requiring that the differential $d\Phi$ to be weakly dominant for Ψ, and unfortunately this may not always hold. Meanwhile, (F1) is equivalent to requiring that if $\alpha \in \Psi$ is a simple root for which $\langle d\Phi, \alpha \rangle = 0$, then α is not in Ψ_c (i.e. is noncompact). In this case,

$$\overline{X}(\gamma) \text{ is a nonzero limit of discrete series with Harish-Chandra parameter } d\Phi,$$

and if $d\Phi$ is actually Ψ dominant (and hence automatically Ψ_c dominant),

$$\overline{X}(\gamma) \text{ is a discrete series with Harish-Chandra parameter } d\Phi.$$

But in general, γ need not satisfy (L2) or (F1).

Even though (F1) and (L2) may fail for γ, we now define a virtual Harish Chandra module $\Theta(\Psi, \Phi)$ which we may regard as the image of a continued standard module attached to the pseudocharacter γ. We follow [Vgr, Definition 7.2.8]. Let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ be the θ-stable Borel subgroup of $\mathfrak{g}$ defined by Ψ. Recall the derived Zuckerman functors $\mathcal{R}_{\mathfrak{b}}^j$ that (on objects) take representations of T to Harish-Charish modules for G. Here the normalization is arranged so that $\mathcal{R}_{\mathfrak{b}}^j$ *preserves* infinitesimal character. (This is the normalization of [Vorange] and differs from the one in [KV].) Let V_Φ denote the representation space for Φ (which, recall, need not be one-dimensional). Then $\mathcal{R}_{\mathfrak{b}}^j(V_\Phi)$ has infinitesimal character $d\Phi = \bar{\gamma}$. We define

$$\Theta(\Psi, \Phi) = \text{Euler characteristic of } \mathcal{R}_{\mathfrak{b}}^{\bullet}(V_\Phi), \tag{21}$$

which is a virtual Harish-Chandra module (since $\mathcal{R}_{\mathfrak{b}}^j$ has finite cohomological dimension). It has infinitesimal character $\bar{\gamma}$ and it is zero if (F1) fails.

If $d\Phi$ is sufficiently dominant in the sense explained above, then $\Theta(\Psi, \Phi)$ is simply the class of a discrete series (or limit). Corollary 7.2.10 of [Vgr] explains the sense in which $\Theta(\Psi, \Phi)$ is a continuation of a discrete series. (That reference assumes T is a abelian, so matters are slightly more complicated than indicated there.)

Example 22 Suppose now that G is connected. So T is connected and abelian and any element of $\Lambda(T)$ is determined by its differential. We will be sloppy and blur the distinction between a character and its differential. For instance, the canonical identification of $\Lambda(\widetilde{T})$ with $\Lambda(T) \otimes \rho$ may now be written as

$$\Lambda(\widetilde{T}) = \Lambda + \rho,$$

where we will just write Λ for $\Lambda(T)$. Fix Ψ_c as above and choose Ψ containing Ψ_c and $\lambda \in \Lambda + \rho$ (viewed as a character of $\widetilde{T}$). Let γ denote the pseudocharacter defined above and recall the virtual Harish-Chandra modules $\Theta(\Psi, \lambda)$. Note that the fixed system Ψ_c defines a holomorphic structure on K/T and hence on the line bundle,

$$\mathcal{L}_\phi := K \times_T \mathbb{C}_\phi,$$

where $\mathbb{C}_\phi$ denotes the (one-dimensional) representation space corresponding to $\phi \in \Lambda$.

As explained below, the following is a consequence of the Blattner formula.

Theorem 23 *Assume G is connected with compact Cartan subgroup T. Fix $\Psi_c \subset \Psi$. Let $\rho = \rho(\Psi)$ and use Ψ_c to define a holomorphic structure on K/T. Let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ denote the Borel subalgebra corresponding to Ψ and write $\mathfrak{n}_n$ for the span of the noncompact roots spaces in $\mathfrak{n}$. Fix $\lambda \in \Lambda + \rho$. Since $2\rho \in \Lambda$, $\lambda + \rho$ defines a character of T and hence we can consider $\mathcal{L}_{\lambda+\rho}$, a holomorphic line bundle on K/T. Let $\wedge^i$ denote the set of T-weights on $\wedge^i \mathfrak{n}_n$. Then in $\mathbb{Z}[[\widehat{K}]]$,*

$$\sum_j (-1)^j \mathrm{H}^{\dim(K/T)-j}(K/T, \mathcal{L}_{\lambda+\rho}) = \sum_i \sum_{\gamma \in \wedge^i} (-1)^i \Theta(\Psi, \lambda+\gamma)|_K. \tag{24}$$

Since the left-hand side is (up to sign) an irreducible representation of K, and since the right-hand side may be expressed as a sum of standard tempered modules (see Remark 30), Equation (24) provides a solution to Problem A if G is connected and equal rank.

Sketch. There are three ingredients to the proof: a Blattner-type formula, a formal identity in $\mathbb{Z}[[\widehat{K}]]$, and a fact about tensoring. We start with the latter. A basic fact about cohomological induction (cf. [Vgr, Lemma 7.2.9(b)]) implies that if

$$\lambda \otimes \mu|_T = \sum_i m_i \lambda_i,$$

then

$$[\mu] \otimes \Theta(\Psi, \lambda)|_K = \sum_i m_i \Theta(\psi, \lambda_i)|_K \tag{25}$$

in $\mathbb{Z}[[\widehat{K}]]$ (where the obvious notion of tensor product is used in $\mathbb{Z}[[\widehat{K}]]$). Next we discuss the formal identity we need in $\mathbb{Z}[[\widehat{K}]]$. Suppose V is a representation of K. Let $\mathrm{sym}(V) \in \mathbb{Z}[[\widehat{K}]]$ denote the symmetric algebra of V. Let

$$\wedge_{\pm}(V) = \sum_i (-1)^i \wedge^i (V) \quad \in \quad \mathbb{Z}[[\widehat{K}]],$$

the signed exterior algebra of V. The formal identity we need is

$$\mathrm{sym}(V) \cdot \wedge_{\pm}(V) = [\mathbb{1}] \text{ in } \mathbb{Z}[[\widehat{K}]]. \tag{26}$$

This is obvious for a T-weight λ; in $\mathbb{Z}[[\widehat{T}]]$, we may compute directly

$$\mathrm{sym}(\lambda) \cdot \wedge_{\pm}(\lambda) = ([\mathbb{1}_T] + [\lambda] + [2\lambda] \cdots) \otimes ([\mathbb{1}_T] - [\lambda]) = [\mathbb{1}_T]. \tag{27}$$

Because

$$\mathrm{sym}(V \oplus W) = \mathrm{sym}(V) \otimes \mathrm{sym}(W),$$

and

$$\lambda_{\pm}(V \oplus W) = \lambda_{\pm}(V) \otimes \lambda_{\pm}(W),$$

it follows that (27) implies that the restriction of $\mathrm{sym}(V) \cdot \wedge_{\pm}(V)$ from $\mathbb{Z}[[\widehat{K}]]$ to $\mathbb{Z}[[\widehat{T}]]$ is simply $[\mathbb{1}_T]$. Hence (26) follows.

Finally we turn to the the Blattner-type formula we need. Let S denote the multiset of highest weights of irreducible representations of K appearing in $\mathrm{sym}(\mathfrak{n}_n)$. Then

$$\Theta(\Psi, \lambda)|_K = \sum_{\tau \in S} \sum_j (-1)^j \mathrm{H}^j(K/T, \mathcal{L}_{\lambda+\tau}). \tag{28}$$

This is the usual Blattner formula "coherently continued" in case that λ is not suitably dominant.

To finish the sketch, we take Equation (28) and multiply both sides by $\wedge_{\pm}(\mathfrak{n}_n)$ in $\mathbb{Z}[[\widehat{K}]]$. Combining the first two facts gives conclusion of the theorem. $\square$

To continue with the example, we remark that something very close to Equation (24) holds if G (i.e. K) is disconnected. But notice that the left-hand side of Equation (24) need not be an irreducible representation of K if K is disconnected. This is the obstacle to extending the solution of Problem A in the connected case to the disconnected case. For instance, in GL(2), one finds that the span of the various $\Theta(\Psi,\lambda)|_K$ in $\mathbb{Z}[[\widehat{K}]]$ is spanned by the set $\{\mathbb{1}_K + \det, \mu_1, \mu_2, \dots\}$; that is, we get formulas (of the form (19)) for each $[\mu_i]$, but we cannot separate $\mathbb{1}_K$ from det. So something new is needed. But Example 20 provides a formula of the form (19) for $\mathbb{1}_K$, and tensoring it with det gives a formula for $\det_K$. This kind of tensoring will be an ingredient in Vogan's general solution to Problem A.

This completes the example. $\square$

We return to the general disconnected case and modify our notation slightly to more closely resemble that in the connected case. In our initial discussion we were careful to distinguish $\Phi \in \Lambda(\widetilde{T})$ from its differential $d\Phi$. Now we will be sloppy and write $\Lambda + \rho$ for $\Lambda(\widetilde{T})$ and write $\lambda \in \Lambda + \rho$ for a both character in $\Lambda(\widetilde{T})$ and its differential. This imprecision is customary and causes no confusion in practice.

Now we introduce more general "continued" standard modules. Let $H = TA$ be a θ-stable Cartan subgroup of G and write MA for the Levi factor of the corresponding cuspidal parabolic. Let Ψ denote a choice of positive imaginary roots of T in $\mathfrak{m}$ containing a fixed compact system Ψ_c. Fix a character $\lambda \in \Lambda+\rho$ (recalling the new notational convention of the previous paragraph). The discussion above defines a pseudocharacter $\gamma_M = (\Gamma_M, \Psi, \bar{\gamma}_M)$ for M for which (L2) and (F1) may fail. Nonetheless, as above, we may consider the continued standard module $\Theta_M(\Psi,\lambda)$, a virtual representation of M. If (F1) fails, it is zero; if (F1) holds, it is nonzero. Define

$$\Theta(\Psi,\lambda) = [\mathrm{ind}_{MAN}^{G}(\Theta_M(\Psi,\lambda) \otimes \mathbb{1} \otimes \mathbb{1})], \tag{29}$$

a virtual Harish-Chandra module for G. (Here, as usual, the choice of N is irrelevant.) Just as we defined the pseudocharacter γ_M, we may also define a pseduocharacter $\gamma = (\Gamma, \Psi, \bar{\gamma})$ for the Cartan $H = TA$ of G: Ψ is the fixed choice of positive imaginary roots, Γ is defined (uniquely) by requiring its retriction to A to be trivial and its restriction to T to be Γ_M, and $\bar{\gamma}$ is determined by (L3). One again γ need not satisfy (L2) or (F1) and $\Theta(\Psi,\lambda)$ need not be of the form $X(\eta)$ for $\eta \in \mathcal{P}_{\mathrm{fin}}^{\mathrm{temp},\circ}$. Here is how to remedy that.

Remark 30 Fix (Ψ,λ) as in the previous paragraph and let γ denote the corresponding pseudocharacter. So γ is a pseudocharacter for an arbitrary θ-stable Cartan subgroup $H = TA$ satisfying all of the conditions to be a final limit pseudocharacter except for (L2) and (F1). (This is the situation we encountered in the previous paragraph and also just before Example 22.) If (F1) fails, then we have already remarked that $\Theta(\Psi,\lambda)$ is zero. So suppose (F1) holds and suppose $\alpha \in \Psi$ is an imaginary root for which $\bar{\gamma}$ is not dominant. If α is compact, then we may write

$$\Theta(\Psi, \lambda) = -\Theta(\Psi', \lambda');$$

this fact reduces to one about continued discrete series where it is due to Hecht-Schmid (see [Vgr, Proposition 8.4.3] for instance). Let γ' be the pseudocharacter attached to (Ψ', λ'). The payoff here is that the number of roots in Ψ' for which (L2) fails for γ' is strictly smaller than the corresponding number for γ. Next suppose that α is noncompact. Then there is a Hecht-Schmid identity of the form

$$\Theta(\Psi, \lambda) = \Theta(\Psi', \lambda') + \Theta'';$$

see [Vgr, Proposition 8.4.5][1]. In this expression, the number of roots in Ψ' for which (L2) fails for γ' (the pseudocharacter attached to (Ψ', λ') as above) is strictly smaller than the corresponding number for γ. Meanwhile Θ'' is an effectively computible sum of modules of the form $\Theta(\Psi'', \lambda'')$ whose corresponding pseduocharacters γ'' are attached to a Cartan that is *less compact* than H. Thus Ψ'' is strictly smaller in cardinality that Ψ, and thus the number of of roots in Ψ'' for which (L2) fails for γ'' is strictly smaller than the corresponding number for γ.

We may thus proceed by induction on the number of roots in Ψ for which (L2) fails to conclude that that there is an effective algorithm to express $\Theta(\Psi, \lambda)$ as a sum of standard tempered modules with real infinitesimal character. (The number of iterations that are needed in this algorithm is bounded by the number of roots in Ψ for which (L2) fails.) We conclude that in order to solve Problem A, we may instead solve

Problem B. Find expression in $\mathbb{Z}[[\widehat{K}]]$ of the form

$$[\mu] = \sum m_{\Psi,\lambda} \Theta(\Psi, \lambda)|_K \tag{31}$$

for each sufficiently small μ.

This concludes the remark. □

Now we explore the tensoring idea in more generality. Suppose that μ is an arbitrary K-type. Consider a continued standard module $\Theta(\Psi, \lambda)$ induced from a continued discrete series for M arising from a θ-stable parabolic TA. Since $T \subset K$, we may restrict μ to T and then pullback to $\rho(\Psi)$ cover $\widetilde{T}$ of T. Hence we can consider μ_T as a (non-genuine) representation of $\widetilde{T}$. Since λ is a genuine representation of $\widetilde{T}$, the tensor product is also genuine, and we can decompose it as

$$\mu|_T \otimes \lambda = \sum m_i \lambda_i \in \Lambda(\widetilde{T}).$$

Combined with Equation (25), we immediately have the following conclusion.

[1] If G is nonlinear nonlinear, [Vgr, Proposition 8.4.5] doesn't cover all the identities that are needed in this remark.

Proposition 32 *Suppose there is an effective algorithm to compute the restriction of* $\mu \in \widehat{K}$ *to the compact part of an arbitrary* θ*-stable Cartan for* G*. Then by tensoring Zuckerman's formula for* $[\mathbb{1}_K]$ *(Example 20) with* μ*, we obtain an effectively computable expression for* μ *of the form (31) (and hence (19)).*

The issue is of course disconnectedness of the compact part of an arbitrary Cartan. This can be overcome for fine K-types of quasisplit groups. Then the machinery of cohomological induction and coherent continuation will ultimately reduce to that case. This is Vogan's proposed solution to Problem A. We begin with fine K-types.

Suppose G is quasisplit. Let $H_s = T_s A_s$ denote a maximally split θ-stable Cartan. Since G is quasisplit, $T_s = M_s$, the centralizer of A_s in G. Our starting point is that we assume that the characters of M_s are effectively computable. This is part of what du Cloux has already done. Now fix a fine K-type μ. Write

$$\mu|_M = \delta_1 \oplus \cdots \oplus \delta_k.$$

Since μ is fine, this decomposition amounts to computing the R-group of (say) δ_1. We assume this decomposition is effectively computable (and again du Cloux has essentially already implemented this). In short, our starting point is that we assume that the restriction of μ to the maximally split Cartan is effectively computable. Now we turn to general Cartans.

So let $H = TA$ denote an arbitrary θ-stable Cartan subgroup of G. We recall how T is built. Let $\{\alpha_1, \ldots, \alpha_r\}$ denote a system of strongly orthogonal roots which (via inverse Cayley transforms) take M_s to H. Then the Lie algebra of T is given by the span

$$\mathfrak{t}_\circ = \langle Z_1, \ldots, Z_r \rangle,$$

where

$$Z_i = d\phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;;$$

here $d\phi_i$ is the usual inclusion of $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{g}_\circ$. Set

$$\sigma_i = \exp\left(\frac{\pi}{2} Z_i\right),$$

and $m_i = \sigma_i^2$. Let Φ_i be the exponentiated be the exponentiation of ϕ_i. If G is linear, $\Phi_i \,:\, \mathrm{SL}(2,\mathbb{R}) \to M$, each $m_i \in M$ is necessarily of order 2, and the eigenvalues of $\Phi_i(m_i)$ are ± 1. (In the nonlinear case, the domain of Φ_i will be a cover of SL(2), and the eigenvalues of $\Phi_i(m_i)$ will be roots of unity.) Define

$$M' = \{m_i \mid \Phi_i(m_i) = \mathrm{Id}\}.$$

Then

$$T = T_\circ \cdot M'.$$

This equality (and the relations between $T_\circ$ and M') is algorithmically understood in du Cloux's software. The restriction of μ to $T_\circ$ is effectively computable (by say a version of the Kostant multiplicity formula and the description of $\mathfrak{t}_\circ$ above). Meanwhile, since the restriction of μ to M is effectively computable, and since M' is an explicitly defined subgroup of M, the restriction of μ to M' is effectively computable. As the example of $\mathrm{SL}(2,R)$ indicates, this is not quite enough to determine the restriction to T. Nonetheless, the remaining ambiguity is tractable. We have:

Proposition 33 *Suppose G is quasisplit, μ is fine, and $H = TA$ is an arbitrary θ-stable Cartan subgroup of G. Then the restriction of μ to T is effectively computable. Hence, by Proposition 32, we obtain an expression for μ of the form (19).*

Finally we must reduce to the case that G is quasisplit and μ is fine. Suppose G is arbitrary and (changing notation slightly) let E denote an arbitrary K-type. Let T denote a maximal torus in K. Let μ denote a (not necessarily unique) highest weight of E; this is an irreducible representation of T. Then [Vgr, Section 5.3] attaches a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ to μ. The analytic subgroup L of G corresponding to $\mathfrak{l}_\circ = \mathfrak{q} \cap \bar{\mathfrak{q}}$ is quasisplit. Moreover the μ weight space of E generates an irreducible representation of $L \cap K$, say $E'_{L\cap K}$. Set

$$E_{L\cap K} = E'_{L\cap K} \otimes \wedge^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{k}).$$

Then $E_{L\cap K}$ is fine.

As usual, $\mathfrak{u} \cap \mathfrak{k}$ gives a holomorphic structure to $K/L\cap K$, and so from $E_{L\cap K}$ we may form a holomorphic vector bundle $\mathcal{E}_{L\cap K} = K \times_{L\cap K} E_{L\cap K}$.

Proposition 34 *We have that*

$$E = \mathrm{H}^{\dim(K/L\cap K)}(K/L\cap K, \mathcal{E}_{L\cap K});$$

in particular the right-hand side is irreducible.

Sketch. The proposition amounts to the irreducibility assertion. It is more or less obvious that the indicated cohomology, say E', is irreducible for $K^\# := K_\circ \cdot (L \cap K)$. The tricky point is showing that the induction from $K^\#$ to K is irreducible. This amounts to showing that $K/K^\#$ acts on the irreducible representation E' of $K^\#$ with no isotropy. □

The next ingredient we need is a version of the Blattner formula for cohomological induction from $\mathfrak{q}$, together with the way that coherent families behave under cohomological induction.

Proposition 35 *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. Let L denote the analytic subgroup of G with Lie algebra $\mathfrak{q} \cap \bar{\mathfrak{q}}$. Suppose $H = T_L A_L \subset L$ is a θ-stable Cartan subgroup of L and that*

$$\Theta_L(\Psi_L, \lambda_L)$$

is a continued standard module for L induced from the cuspidal parabolic subgroup corresponding to H. Define

$$\Psi_G = \Psi_L \cup \{\textit{the imaginary roots of } H \textit{ in } \mathfrak{u}\}.$$

Recall that λ_L is a character of the ρ_L cover of H; hence $\lambda_L \otimes \rho_L^{-1}$ is a character of H and

$$\lambda_L \otimes \rho_L^{-1} \otimes \rho_G = \lambda_L \otimes \rho(\mathfrak{u})$$

is a genuine character of the ρ_G cover of H. Set

$$\lambda_G = \lambda_L \otimes \rho(\mathfrak{u}) \otimes \wedge^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{p}), \tag{36}$$

a genuine character of the ρ_G cover of H. Then, in $\mathbb{Z}[[\widehat{K}]]$,

$$\Theta_G(\Psi_G, \lambda_G)|_K = \textit{Euler characteristic of } \mathrm{H}^\bullet\left(K/L \cap K, \Theta_L(\Psi_L, \lambda_L)|_{L\cap K} \otimes \mathrm{sym}(\mathfrak{u} \cap \mathfrak{p})\right).$$

Sketch. Recall the cohomological induction functors $\mathcal{R}_{\mathfrak{q}}^j$ that appear around Equation (21). We claim that (up to a sign)

$$\Theta_G(\Psi_G, \lambda_G) = \text{Euler characteristic of } \mathcal{R}_{\mathfrak{q}}^\bullet\left(\Theta_L(\Psi_L, \lambda_L)\right); \tag{37}$$

then the theorem follows from the Blattner formula for cohomological induction from $\mathfrak{q}$ (for instance, page 376 of [KV]). The Euler characteristic of the functors $R_{\mathfrak{q}}^j$ take coherent families for L to coherent families for G; this is the content of [Vgr, Corollary 7.2.10]. So to verify the equality in Equation (37), we need only verify it for λ_L (and hence λ_G) sufficiently dominant. In this case $\Theta_G(\Psi_G, \lambda_G)$ and $\Theta_L(\Psi_L, \lambda_L)$ are (Langlands) standard modules. So the assertion of (37) follows from understanding how cohomological induction behaves with respect to standard modules. This is the subject of [KV, Section XI.10] and, in particular, [KV, Theorem 11.255]. □

Recall that $\wedge_\pm(\mathfrak{u}\cap\mathfrak{p})$ is the "inverse" of the symmetric algebra in $\mathbb{Z}[[\widehat{K}]]$ (Equation (26)). After commuting tensoring with induction, Proposition 35 then implies that

$$\Theta_G\left(\Psi_G, \lambda_G \otimes \wedge_\pm(\mathfrak{u} \cap \mathfrak{p})|_T\right) = \text{Euler characteristic of } \mathrm{H}^\bullet\left(K/L \cap K, \Theta_L(\Psi_L, \lambda_L)|_{L\cap K}\right).$$

Recall that $E_{L\cap K}$ is fine and L is quasisplit. So Proposition 33 given an expression (reverting to the notation $\mu_{L\cap K}$ for $E_{L\cap K}$) of the form

$$[\mu_{L\cap K}] = \sum m_{\Psi,\lambda} \Theta_L(\Psi_L, \lambda_L)|_K$$

Now apply the Euler characteristic of holomorphic induction from $L \cap K$ to K to both sides of the above displayed equation. Propositions 34 and 35 then apply to conclude (again reverting to the notation μ for E) that

$$[\mu] = \sum \pm m_{\Psi,\lambda} \Theta_G\left(\Psi_G, \lambda_G \otimes \wedge_\pm(\mathfrak{u} \cap \mathfrak{p})\right)|_K.$$

This is the desired expression for μ of the form given in (31).

This completes the outline of Vogan's solution to Problem A.

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