

# Algorithms for structure theory

Fokko du Cloux

July 13, 2004

These are draft notes for the first part of the Atlas of reductive Lie groups project, dealing with the structure theory part of the program. Computationally, most of this should be very light, and certainly much or all of it could be implemented in a high-level environment such as `Maple` or `GAP`. However, since the structure theory things lie at the very heart of the program, it is necessary to have them available at the C++ level as well.

## 1 Defining the group

**1.1.** Let  $\mathbf{G}$  be a connected complex reductive Lie group, defined over  $\mathbf{R}$ ; let  $\mathbf{G}(\mathbf{R})$  be the full group of real points,  $\mathbf{G}(\mathbf{R})^\circ$  its identity component. We will be interested in Lie groups  $G$  such that  $\mathbf{G}(\mathbf{R})^\circ \subset G \subset \mathbf{G}(\mathbf{R})$ ; such groups we call *real reductive Lie groups* in these notes. Note that later on one might wish to consider finite (or even infinite ?) covers of such groups  $G$ , but for now we refrain from that. Denote  $\mathfrak{g}$  the Lie algebra of  $\mathbf{G}$ , also the complexified Lie algebra of  $G$ .

**1.2.** The datum of  $\mathbf{G}$  is equivalent to that of a *root datum* : this is a quadruple  $(X, R, X^\vee, R^\vee)$  where  $X$  and  $X^\vee$  are two lattices in duality, and  $R \subset X$  and  $R^\vee \subset X^\vee$  are two finite subsets, for which we are given a bijection denoted  $\alpha \rightarrow \alpha^\vee$  such that (a)  $\langle \alpha, \alpha^\vee \rangle = 2$  for all  $\alpha \in \mathbf{R}$ , and (b) for each  $\alpha \in R$  the reflection  $s_\alpha : \lambda \rightarrow \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$  takes  $R$  into  $R$ , and similarly  $s_{\alpha^\vee}$  takes  $R^\vee$  into  $R^\vee$ . In particular, this means that  $R$  is a root system within the real vector space it spans in  $X \otimes_{\mathbf{Z}} \mathbf{R}$ .

To set up the root datum, choose a maximal torus  $\mathbf{T}$  in  $\mathbf{G}$ , and let  $X$  be the character group of  $\mathbf{T}$ ,  $X^\vee$  the cocharacter group (*i.e.*, the group of homomorphisms from  $\mathbf{C}^\times$  to  $\mathbf{T}$ ), let  $R$  be the set of roots of  $\mathbf{T}$  in  $\mathfrak{g}$ ,  $R^\vee$  the set of coroots. We will also say that  $X$  is the *weight lattice* of  $\mathbf{G}$ .

**1.3. Entering a group into the system : Lie type.** Entering a group into the system requires quite a bit of user interaction. The group  $\mathbf{G}$  always has a finite cover  $\tilde{\mathbf{G}}$  which is the direct product of a simply connected complex semisimple group  $\tilde{\mathbf{G}}'$  and a torus  $R(\tilde{\mathbf{G}})$ . Taking into account the real structure, we may split up  $\tilde{\mathbf{G}}'$  into a product of groups of the following forms (where “simple” really means “quasi-simple”): (a) simply connected simple groups defined over  $\mathbf{R}$  (b) simply connected simple complex groups, viewed as real groups (such a group  $G_1$  is a real form of  $G_1 \times G_1$ ). Similarly, the radical  $R(\tilde{\mathbf{G}})$  is a product of tori defined over  $\mathbf{R}$  which are either anisotropic (or, as we prefer to say, compact, *i.e.*, such that  $\mathbf{T}(\mathbf{R}) \simeq \mathbf{U}(1)^n$  is compact), split ( $T(\mathbf{R}) \simeq \mathbf{R}^{\times n}$ ), or complex ( $T(\mathbf{R}) \simeq \mathbf{C}^{\times n}$ .)

The user enters a sequence of symbols specifying the factors of  $\tilde{\mathbf{G}}$ , and for each factor specifies a real form (complex factors actually use up two consecutive isomorphic factors of  $\tilde{\mathbf{G}}$ .) From this, the program knows enough to construct the root datum for  $\tilde{\mathbf{G}}$ , and its corresponding involution. For the description of real forms, we use for now the tables from Helgason [1] Table VI in Chapter X.

**1.4. Entering a group into the system : character lattice.** The practical description of the lattice  $X$  may be done as follows. For each simple factor of  $\tilde{G}$ , we carry out a Smith normal form process to find a basis of the weight lattice such that suitable multiples of the basis vectors generate the root lattice. This yields an explicit description of the quotient as a product of finite groups (in fact it is always cyclic, except for type D in even rank, where it is a group  $\mathbf{Z}_2 \times \mathbf{Z}_2$ .) Adding the torus factor, we now have an explicit description of the quotient  $X(\tilde{G})/Q$ . Then  $X/Q$  may be any finite index subgroup of  $\tilde{X}/Q$ , stable under involution; one convenient way to describe it is in terms of a set of generators. This is how we do it in the program.

It should be remarked that in this situation,  $X$  is a sublattice of  $X(\tilde{G})$  which contains the roots, and  $X^\vee$  is a superlattice of  $X^\vee(\tilde{G})$ , so it obviously contains the coroots. In this setup, the images of the roots in  $X$  and the coroots in  $X^\vee$  automatically constitute a root datum. So it is not hard to get the full root datum of our real reductive group once we have its weight lattice as a sublattice of  $X(\tilde{G})$ .

**1.5.** The real form may be described using either the Galois or the Cartan involution; we will use the Cartan involution  $\theta$ . As usual, we denote  $\mathfrak{k}$  (resp.  $K, \mathbf{K}$ ) the set of fixed points of  $\theta$  in  $\mathfrak{g}$  (resp.  $G, \mathbf{G}$ ), and  $\mathfrak{p}$  the  $-1$ -eigenspace of  $\theta$  in  $\mathfrak{g}$ . Of course,  $K$  is a maximal compact subgroup in  $G$ . The real torus we used in our root datum is assumed to be a maximally split  $\theta$ -stable torus in  $G$ ; any two such are conjugate under  $K$ .

The involution induced by  $\theta$  on the character lattice of  $G$  is just the negative of the Galois involution. It is entirely described by the so-called Satake diagram. To get the involution from the Satake diagram, assume for simplicity that the Lie algebra is absolutely simple (the complex and torus factors are easy.) Then form a basis of the vector space  $X \otimes_{\mathbf{Z}} \mathbf{Q}$  as follows : take a root vector for each black node, and a weight vector for each white node. Interchange and change signs for linked white nodes; change sign for unlinked white nodes; do nothing for black nodes. Then come back to the original basis of the lattice. Actually this happens already in stage 1.3; restricting the involution to a sublattice is linear algebra.

**1.6. Component groups : torus case.** Now we come to what is conceptually the most delicate part, *viz.* the determination of component groups. We start with the torus case.

Let  $\mathbf{T}$  be a torus defined over  $\mathbf{R}$ ,  $T = \mathbf{T}(\mathbf{R})$ ,  $\pi_0(T)$  the component group of  $T$ . We will see that  $T$  decomposes as a direct product of factors  $\mathbf{U}(1)$ ,  $\mathbf{R}^\times$  and  $\mathbf{C}^\times$ ; it follows that  $\pi_0$  is an elementary 2-group of the form  $\mathbf{Z}_2^r$ , where  $r$  is the number of factors  $\mathbf{R}^\times$ . We wish to address the following two problems : (a) describe  $\pi_0(T)$  in terms of the data defining  $\mathbf{T}$  (the character lattice and the involution) (b) describe the effect of any homomorphism  $\varphi : \mathbf{T} \rightarrow \mathbf{T}'$  on  $\pi_0$ .

Let  $X = X(\mathbf{T})$ ,  $X_{\mathbf{Q}} = X \otimes_{\mathbf{Z}} \mathbf{Q}$  and let  $X_+$  and  $X_-$  be the intersections of  $X$  with the  $\pm 1$ -eigenspaces of  $\theta$  in  $X_{\mathbf{Q}}$ . Since any  $\lambda \in X$  may be written in  $X_{\mathbf{Q}}$  as  $\frac{1}{2}(\lambda + \theta(\lambda)) + \frac{1}{2}(\lambda - \theta(\lambda))$ , it is clear that we have the inclusions :

$$2X \subset X_+ \oplus X_- \subset X$$

Both  $X_+$  and  $X_-$  have complements in  $X$ ; hence  $X_s := X/X_+$  and  $X_c := X/X_-$  are again lattices. They correspond to subtori  $\mathbf{T}_s$  and  $\mathbf{T}_c$  of  $\mathbf{T}$ , which we shall call the *split part* and the *compact part* of  $\mathbf{T}$  (because they correspond respectively to split and compact real forms.) The natural map  $\mathbf{T}_c \times \mathbf{T}_s \rightarrow \mathbf{T}$  is dual to the injection  $X \rightarrow X_c \oplus X_s$ ; therefore its kernel (which is the antidiagonal injection of  $\mathbf{T}_c \cap \mathbf{T}_s$  into  $\mathbf{T}_c \times \mathbf{T}_s$ ) is dual to the elementary 2-group  $(X_c \oplus X_s)/X$ .

From this it follows easily that each real torus is isomorphic to a product of split, compact or complex factors. Indeed, denote  $\mathbf{T}(2)$  the subgroup of elements of order two in  $\mathbf{T}$ , and use similar notation for other tori, and for their groups of real points. Then by a suitable integrable base change one may write  $\mathbf{T}_s = \mathbf{T}'_s \times \mathbf{T}''_s$ ,  $\mathbf{T}_c = \mathbf{T}'_c \times \mathbf{T}''_c$ , in such a way that  $\mathbf{T}_s \cap \mathbf{T}_c$  is  $\mathbf{T}''_s(2) = \mathbf{T}''_c(2)$ . Then  $\mathbf{T}'_s$  is a split direct factor in  $\mathbf{T}$ ,  $\mathbf{T}'_c$  a compact one, and  $\mathbf{T}''_s \mathbf{T}''_c = \mathbf{T}''_s \times \mathbf{T}''_c / (\mathbf{T}''_s \cap \mathbf{T}''_c)$  a complex one.

Using this decomposition, it also follows that the canonical surjection  $\mathbf{T}_s \times \mathbf{T}_c \rightarrow \mathbf{T}$  induces a surjection on the groups of real points; perhaps there is a more direct argument but I don't see it right now. Since compact and complex tori have connected groups of real points, the rank of the component group of  $T$  is equal to the rank of the torus  $\mathbf{T}'_s$  above; also the rank of the lattice  $X_-$  minus the number of complex factors.

It is convenient to consider  $\pi_0(T)$  as the quotient group of  $T(2)$  by  $T_c(2)$ . Notice that when there are complex factors, we have  $T(2) \neq \mathbf{T}(2)$ , *i.e.*, not all points of order two in  $\mathbf{T}$  are real. To study this a little bit more precisely, denote  $V = X/2X$ , considered as an  $\mathbf{F}_2$ -vector space. The map induced by  $\theta$  on  $V$  is unipotent; since  $T(2)$  is the kernel of  $\theta - 1$  on  $\mathbf{T}(2)$ , its orthogonal is the image of  $\theta - 1$  in  $V$ . Denote  $V_+$ ,  $V_-$  the images of  $X_+$ ,  $X_-$  in  $V$ . Then it is not hard to show that  $\text{Im}(\theta - 1) = V_+ + V_-$ ,  $\text{Ker}(\theta - 1) = V_{+-} := V_+ \cap V_-$ . So the dual group  $T(2)^\vee$  identifies naturally with  $V/V_{+-}$ . Since the character lattice of  $\mathbf{T}_c$  is by definition  $X/X_-$ , it follows that the orthogonal of  $\mathbf{T}_c(2) = T_c(2)$  in  $V$  is  $V_-$ ; and its orthogonal in  $T(2)^\vee$  is the canonical image of  $V_-$  in  $V/V_{+-}$ . This is our realization of the dual of the component group of  $T$ .

It is now an easy matter (at least in principle—it still requires some programming) to deduce the map induced at the level of dual component groups by any homomorphism  $\mathbf{T} \rightarrow \mathbf{T}'$ .

**1.7. Component groups : general case** Now we explain how to compute the component group of  $G = \mathbf{G}(\mathbf{R})$ . We use the following two facts : (a) when  $\mathbf{G}$  is semisimple and simply connected,  $G$  is connected; (b) for any maximally split  $\mathbf{R}$ -torus  $\mathbf{T}$  in  $\mathbf{G}$ , the group  $T$  meets all the connected components of  $G$  (references ??).

Already this shows that the map from  $\pi_0(T)$  to  $\pi_0(G)$  is surjective, and hence that  $\pi_0(G)$  is an elementary 2-group. We will take advantage from this fact to identify the dual group  $\pi_0(G)^\vee$  with a subgroup of our group  $\pi_0(YT)^\vee$  determined in 1.6.

Now consider a finite covering  $\varphi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$ . It is clear that this induces a surjection  $\tilde{G}^\circ \rightarrow G^\circ$ , and since the kernel of  $\varphi$  is necessarily contained in  $\tilde{T}$ , we have also a surjection from  $\pi_0(\tilde{T} \cap \tilde{G}^\circ)$  to  $\pi_0(T \cap G^\circ)$ . Passing to the orthogonals, which are precisely the component groups of  $\tilde{G}$  and  $\mathbf{G}$ , we see that  $\pi_0^\vee(G)$  is the inverse image in  $\pi_0(T)^\vee$  of  $\pi_0(\tilde{T})^\vee$  by the map  $\pi_0(T)^\vee \rightarrow \pi_0(\tilde{T})^\vee$  induced by  $\varphi$ .

In the case where  $\mathbf{G}$  is semisimple, it follows from this and (a) above that  $\pi_0(G)^\vee$  may be computed by taking  $\tilde{G}$  to be the simply connected cover;  $\pi_0(G)^\vee$  is then simply the kernel of the map from  $\pi_0(T)^\vee$  to  $\pi_0(\tilde{T})^\vee$ . In the

general case, one may always find a finite cover of the form  $\tilde{G} = \mathbf{H} \times \mathbf{S}$ , where  $\mathbf{H}$  is semisimple simply connected, and  $\mathbf{S}$  is a torus. The component and dual component groups of  $\tilde{G}$  are then those of  $\mathbf{S}$ , and the component group of  $\mathbf{G}$  is computed by the inverse image construction above.

**1.8. Example :  $\mathbf{GL}(2, \mathbf{R})$ .** Let  $\mathbf{G}$  be the split  $\mathbf{GL}(2)$ ,  $G = \mathbf{GL}(2, \mathbf{R})$ . Consider the diagonal torus, with its obvious identification to  $(\mathbf{C}^\times)^2$ , yielding an identification of  $X$  with  $\mathbf{Z}^2$ . The involution  $\theta$  is identically  $-1$ , so that of course  $\pi_0(T)$  is  $\mathbf{F}_2^2$ . We may take  $\tilde{G} = \mathbf{SL}(2) \times \mathbf{C}^\times$ . If we identify the torus of  $\mathbf{SL}(2)$  with  $\mathbf{C}^\times$  using the first coordinate, the inclusion map  $X \subset \tilde{X}$  becomes  $(x_1, x_2) \rightarrow (x_1 - x_2, x_1 + x_2)$ . Clearly this is a sublattice of index 2. The group  $\pi_0(\tilde{T})$  is also  $\mathbf{F}_2^2$ , and if we look at the induced map  $\mathbf{F}_2^2 \rightarrow \mathbf{F}_2^2$ , we see that the first coordinate of the image of  $(a_1, a_2)$  is zero if and only if  $a_1 = a_2$ . So we find that  $G$  has two connected components, as expected, and that the components of  $T$  which lie in the identity component of  $G$  are those with equal signs in both coordinates, also as expected.

## 2 The complex Weyl group

*In this section we will explain our approach to the implementation of the complex Weyl group. Here the main issue is to choose the representation of the group elements. Right now my guess is it might be best to come back to a representation in terms of "arrays"; of course going from there to a matrix representation should be available.*

## 3 Maximal compact subgroup

*Determination of the maximal compact subgroup. The determination of the topology ( $\pi_0$  and  $\pi_1$ ) may be somewhat delicate; also, the lifting of the component group as a subgroup, which I believe is always true.*

## 4 Cartan subgroups

*Determination of the conjugacy classes of Cartan subgroups. This will probably be done in terms of strongly orthogonal sets of roots. A large amount of data should be attached to each Cartan : the involution it defines on the root datum, the classification of roots, the corresponding Levi subgroup as a real reductive group, complete with the topological data, ...*

## 5 Weyl groups

*Determination of the real Weyl group attached to a given Cartan subgroup.  
I'm not sure I understand all the details here yet.*

### References

- [1] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*, corrected reprint of the 1978 original, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, R.I., 2001.
- [2] D.A. Vogan Jr. *Representations of Real Reductive Lie Groups*. Birkhäuser, Boston, Mass., 1981.