Computing Kazhdan-Lusztig polynomials for real Lie groups

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1. Setup

Let $\mathbf{G}$ be a connected reductive complex algebraic group, defined over $\mathbf{R}$, and let G be a subgroup of $\mathbf{G}$ such that $\mathbf{G}(\mathbf{R})^{\circ} \subset G \subset \mathbf{G}(\mathbf{R})$. Let θ be a Cartan involution of G, and let $K = G^{\theta}$ be the corresponding maximal compact subgroup of G. Using lowercase German letters for the corresponding Lie algebras, we have a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k}$ ($\mathfrak{p}$) is the $+1$ (-1) eigenspace of θ on $\mathfrak{g}$.

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra of $\mathfrak{g}$, and fix a character χ of $Z(\mathfrak{g})$. Let $\mathcal{HC}$ be the category of Harish-Chandra modules for G (i.e. finitely generated $(\mathfrak{g}, K)$–modules), and let $\mathcal{HC}_{\chi}$ be the full subcategory of $\mathcal{HC}$ of the modules with generalized central character χ. We are interested in the distribution characters, say, of the irreducible modules in $\mathcal{HC}_{\chi}$.

These are of course a basis of the Grothendieck group of $\mathcal{HC}_{\chi}$. Now there is another basis $\mathcal{D}$ of this Grothendieck group, which we may consider known, made up by the characters of standard representations (parabolically induced from discrete series on cuspidal parabolic subgroups.) To each $\gamma \in \mathcal{D}$, Langlands assigns a specific irreducible subquotient $\overline{\gamma}$; this establishes a bijection from $\mathcal{D}$ to the irreducibles in $\mathcal{HC}_{\chi}$. If we denote $m(\overline{\gamma}, \delta)$ the multiplicity of $\overline{\gamma}$ in δ, we can write

$$\delta = \sum_{\gamma \in \mathcal{D}} m(\overline{\gamma}, \delta)\overline{\gamma}$$

in the Grothendieck group (or as distribution characters). In a suitable ordering of $\mathcal{D}$, the matrix $m(\overline{\gamma}, \delta)$ is unipotent upper triangular; so we can invert it and write

$$\overline{\delta} = \sum_{\gamma \in \mathcal{D}} M(\gamma, \overline{\delta})\gamma$$

The problem is to compute the integers $M(\gamma, \overline{\delta})$.

In the sequel, we will assume for simplicity that χ is the infinetisimal character of the trivial representation of G. Also we fix a Cartan subalgebra $\mathfrak{h}^{a}$ of $\mathfrak{g}$, and a Borel sublagebra $\mathfrak{b}^{a}$ containing $\mathfrak{h}^{a}$. Denote $W = W(\mathfrak{g}, \mathfrak{h}^{a})$.

2. Kazhdan-Lusztig polynomials

Let's take the problem to a higher level (to understand where this comes from, one needs the geometric interpretation of the category $\mathcal{HC}_\lambda$ in terms of $\mathbf{K}$–equivariant constructible sheaves on the flag manifold of $\mathbf{G}$, where $\mathbf{K}$ is the complexification of K.) Let $\mathcal{M}$ be the free $\mathbf{Z}[q^{1/2}, q^{-1/2}]$–module with basis $\mathcal{D}$.

Let $\mathcal{H} = \mathcal{H}(W)$ be the Hecke algebra of W; it is an algebra over the same ring $\mathbf{Z}[q^{1/2}, q^{-1/2}]$. The main ingredient in all that follows will be the definition of an $\mathcal{H}$–module structure on $\mathcal{M}$, which we shall describe later; also we shall define a certain graded partial order relation $\leq$ on $\mathcal{D}$, with associated length function l. It will turn out that there is a unique $\mathbf{Z}[q^{1/2}, q^{-1/2}]$–antilinear involution D on $\mathcal{M}$ such that

(a) $D(hm) = D(h)D(m)$ for all $h \in \mathcal{H}$, $m \in \mathcal{M}$, where we denote also by D the usual Kazhdan–Lusztig involution on $\mathcal{H}$.

(b) $D(t_\delta) = t_\delta + \sum_{\gamma < \delta} r(\gamma, \delta) t_\gamma$, $r(\gamma\delta) \in \mathbf{Z}[q^{1/2}, q^{-1/2}]$, where we have put $t_\gamma = q^{-l(\gamma)/2}\gamma$ for each $\gamma \in \mathcal{D}$.

Then, from general principles, there is a unique basis c_δ of $\mathcal{M}$ (the Kazhdan-Lusztig basis), such that

(a) $c_\delta = t_\delta + \sum_{\gamma < \delta} p(\gamma, \delta) t_\gamma \qquad p(\gamma, \delta) \in q^{-1/2}\mathbf{Z}[q^{-1/2}]$

(b) $D(c_\delta) = c_\delta$

It will turn out that $P_{\gamma,\delta} = q^{(l(\delta)-l(\gamma))/2} p(\gamma, \delta)$ belongs to $\mathbf{Z}[q]$, and that its evaluation at $q = 1$ yields the $M(\gamma, \overline{\delta})$ up to sign :

$$M(\gamma, \overline{\delta}) = (-1)^{l(\delta)-l(\gamma)} P_{\gamma,\delta}(1)$$

The computation of the basis c_δ is a purely formal exercise once the involution D is known. In practice, however, making use of the $\mathcal{H}$–action we can get direct recursion formulas which are probably quite a bit more efficient than the formal approach.

The $P_{\gamma,\delta}$ above are the *Kazhdan-Lusztig polynomials* for G.

3. Parametrization of the irreducible representations

If we want to do anything at all explicitly of course our first task should be to make the set $\mathcal{D}$ explicit. Here is how this goes. The set $\mathcal{D}$ is the set of K–conjugacy classes of triples (H, λ, Φ), where H is a θ–stable Cartan subgroup of G, $\lambda \in \mathfrak{h}^*$ is in the $W(\mathfrak{g}, \mathfrak{h})$– orbit corresponding to χ under the Harish-Chandra isomorphism $Z(\mathfrak{g}) \to S(\mathfrak{h})^W$, and Φ is a character of the compact part T of H, whose differential $d\Phi$ is known from λ. The precise definition is this : let $H = TA$ be the Cartan decomposition of H, and let L be the centralizer of A in G. Then we have a decomposition $L = MA$, and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{m}$. Since λ is a regular weight, no coroot vanishes on λ. Let

$$\rho_\lambda^I = \frac{1}{2} \sum_{\substack{\langle \alpha^\vee, \lambda \rangle > 0 \\ \alpha \in \Delta(\mathfrak{m}, \mathfrak{t})}} \alpha \qquad\qquad \rho_\lambda^{I,c} = \frac{1}{2} \sum_{\substack{\langle \alpha^\vee, \lambda \rangle > 0 \\ \alpha \in \Delta(\mathfrak{t} \cap \mathfrak{m}, \mathfrak{t})}} \alpha$$

be the half-sums of roots of $\mathfrak{t}$ in $\mathfrak{m}$ (resp. $\mathfrak{k} \cap \mathfrak{m}$) which are positive w.r.t. λ. Then we ask that

$$d\Phi = \lambda|_{\mathfrak{t}} + \rho_\lambda^I - 2\rho_\lambda^{I,c}$$

Since the component group of T is a commutative 2–group, there are as many such Φ as there are connected components in T, and this number is a power of 2.

The K–conjugacy classes of θ–stable Cartan subgroups are in $(1,1)$ correspondence with the G–conjugacy classes of Cartan subgroups in G; there are finitely many. If we choose a set of representatives $\{H_i\}_{i \in I}$ of θ–stable Cartans, we see that for each $i \in I$, (λ, Φ) has to be considered up to the action of the normalizer of H_i in K, divided by the centralizer; this is also the Weyl group $W(G, H_i)$.

Let $\Lambda_i \subset \mathfrak{h}_i^*$ be the $W(\mathfrak{g}, \mathfrak{h}_i)$–orbit corresponding to our chosen infinitesimal character χ. Then for each $\lambda \in \Lambda_i$ there is a unique isomorphism $i_\lambda : \mathfrak{h}^a \to \mathfrak{h}_i$ which takes the positive chamber defined by $\mathfrak{b}^a$ to the positive Weyl chamber in $\mathfrak{h}_i$ defined by λ, and which is induced by conjugation with an element of $\mathrm{Ad}(\mathfrak{g})$. So Λ_i is in canonical bijection with a subset of $\mathrm{Hom}(\mathfrak{h}^a, \mathfrak{h}_i)$, and we have a canonical right action of W on Λ_i by composition on the right; this commutes with the left action of $W(\mathfrak{g}, \mathfrak{h}_i)$ and in particular with the left action of $W(G, H_i)$.

Let $\mathcal{D}_i$ be the part of $\mathcal{D}$ corresponding to H_i. The upshot is that we have a decomposition

$$\mathcal{D} = \coprod_{i \in I} \mathcal{D}_i$$

where each $\mathcal{D}_i$ is fibered over a the set of $W(G, H_i)$–orbits in Λ_i, and each fiber is provided with a simply transitive action of the group $(H_i/H_i^\circ)^\wedge$.

4. The cross action and the Cayley transforms

The Hecke algebra action on $\mathcal{M}$ lifts the coherent continuation action of the group W on the Grothendieck group of $\mathcal{HC}_\lambda$. It is convenient to first introduce an action of W on the set $\mathcal{D}$, the so-called the *cross-action*. For this action, the group W acts in fact separately on each $\mathcal{D}_i$.

The action is just a lift of the action of W on Λ_i, defined in the previous section. To make it a left action, set $w \times \lambda = \lambda w^{-1}$. We may also write this as $w \times \lambda = w_\lambda^{-1}(\lambda)$, with $w_\lambda = i_\lambda w i_\lambda^{-1} \in W(\mathfrak{g}, \mathfrak{h}_i)$. Clearly $w \times \lambda - \lambda$ is an integral linear combination of roots. Also, $\rho_\lambda^I - \rho_{w \times \lambda}^I$ and $\rho_\lambda^{I,c} - \rho_{w \times \lambda}^{I,c}$ are integral linear combinations of roots in $\Delta(\mathfrak{m}_i, \mathfrak{t}_i)$. Now it turns out that the differential is a bijection from the lattice in the space $X(H_i)$ of characters of H_i generated by the root characters, to the root lattice in $\mathfrak{h}_i$. So there is a unique element in the "root lattice" of $X(H_i)$ corresponding to

$$(w \times \lambda + \rho_{w \times \lambda}^I - 2\rho_{w \times \lambda}^{I,c}) - (\lambda + \rho_\lambda^I - 2\rho_\lambda^{I,c})$$

This element we take as $w \times \Phi - \Phi$. Now if $\gamma = (H_i, \lambda, \Phi)$, we set $w \times \gamma = (H_i, w \times \lambda, w \times \Phi)$. Of course it should be checked that this is indeed an action.

As a further preparation for the definition of the Hecke algebra action on $\mathcal{M}$, we need the so-called Cayley transforms. These will establish a link between representations associated to various Cartans.

Let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)$ be a simple root; and fix $\gamma = (H_i, \lambda, \Phi) \in \mathcal{D}$. Set $\alpha_\lambda = i_\lambda(\alpha) \in \Delta(\mathfrak{g}, \mathfrak{h}_i)$. Then the Cayley transform $c^\alpha(\gamma)$ is defined whenever α_λ is noncompact imaginary; the idea is that it will take γ to an element associated with a Cartan where α is *real*. Let $\mathfrak{h}_i = \mathfrak{t}_i \oplus \mathfrak{a}_i$ be the Cartan decomposition of $\mathfrak{h}_i$. Let $\mathfrak{l}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}_i$ and the root vectors $\pm\alpha_\lambda$. Then $\mathfrak{l}$ is θ–stable, defined over $\mathbf{R}$, and has exactly two θ–stable conjugacy classes of Cartan subalgebras; $\mathfrak{h}_i$ represents the maximally compact one, and the other, say conjugate to $\mathfrak{h}_j$ in $\mathfrak{g}$, the maximally split one. In fact, $\mathfrak{h}_i \cap \mathfrak{h}_j$ is of codimension one in each of them if we take the representative of $\mathfrak{h}_j$ to lie inside $\mathfrak{l}$. Then there are just two $\lambda' \in \Lambda_j$ such that $\lambda'|_{\mathrm{Ker}\alpha} = \lambda|_{\mathrm{Ker}\alpha}$; moreover these two λ' are conjugate under the reflection by α, which now lies in $W(G, H_j)$. So (H_j, λ') is well-defined up to K–conjugacy.

It remains to see what happens with the Φ–part of γ. What we want is to consider all $\gamma' = (H_j, \lambda', \Phi')$ such that $\Phi = \Phi'$ on $T_i \cap T_j$. There are two cases : (a) $T_j \subset T_i$, in which case Φ' is entirely determined, and we set $c^\alpha(\gamma) = \{\gamma'\}$ (b) $T_j \not\subset T_i$; then $T_i \cap T_j$ is of index two in T_j, and it is easy to see that in this case $T_j \simeq (T_i \cap T_j) \times \mathbf{Z}/2\mathbf{Z}$; hence there are exactly two possibilities for Φ', traditionally labelled γ_+^α and γ_-^α, although this notation is a bit unfortunate because really the two are indistinguishable at this point. We say that α is *type I* in the first case, *type II* in the second case.

The inverse Cayley transform goes in the other direction, from the more split Cartan to the more compact one. It turns out that there is a nice duality : when α is type one for γ, then it is also for $s \times \gamma$, we have $s \times \gamma \neq \gamma$, $c^\alpha(s \times \gamma) = c^\alpha(\gamma)$, and those are the only two elements $\delta \in \mathcal{D}$ such that $\gamma^\alpha \in c^\alpha(\delta)$; we will write $c_\alpha(\gamma^\alpha) = \{\gamma, s \times \gamma\}$. When α is type II for γ, we have $s \times \gamma = \gamma$, $s \times \gamma_+^\alpha = \gamma_-^\alpha$, $s \times \gamma_-^\alpha = \gamma_+^\alpha$, and γ is the only element $\delta \in \mathcal{D}$ such that $c^\alpha(\delta) \cap \{\gamma_+^\alpha, \gamma_-^\alpha\}$ is non-empty; we write $c_\alpha(\gamma_+^\alpha) = c_\alpha(\gamma_-^\alpha) = \gamma$. In short, the domain of c_α is the range of c^α, and the image of γ under c_α is its preimage under c^α; when $c_\alpha(\gamma)$ has two elements, we also denote it as $\{\gamma_\alpha^+, \gamma_\alpha^-\}$. An element $\gamma \in \mathcal{D}$ can be in the domain of c_α only if α_λ is real; if it is in the domain, we say that α is type I real if it comes from a type I noncompact imaginary root, so that $c_\alpha(\gamma)$ has two elements; and type II real if it comes from a type II noncompact imaginary root, so that $c_\alpha(\gamma)$ has one element.

5. The Hecke algebra action and the Bruhat order

Now we are ready for the definition of the Hecke operators. Given $\gamma \in \mathcal{D}$ and a Coxeter generator $s \in W$, corresponding to a simple root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)$, we define $T_s\gamma \in \mathcal{M}$ as follows.

(a) if α_λ is compact imaginary, then $T_s\gamma = q\gamma$;

(b) if α_λ is noncompact type I imaginary, then $T_s\gamma = s \times \gamma + \gamma^\alpha$;

(c) if α_λ is noncompact type II imaginary, then $T_s\gamma = \gamma + \gamma_+^\alpha + \gamma_-^\alpha$;

(d) if α_λ is complex, and $\theta_\lambda\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}^a)$, then $T_s\gamma = s \times \gamma$;

(e) if α_λ is complex, and $\theta_\lambda\alpha \in \Delta^-(\mathfrak{g}, \mathfrak{h}^a)$, then $T_s\gamma = q\, s \times \gamma + (q-1)\gamma$;

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(f) if α is type I real for γ, then $T_s\gamma = (q-2)\gamma + (q-1)(\gamma_\alpha^+ + \gamma_\alpha^-)$;

(g) if α is type II real for γ, then $T_s\gamma = (q-1)\gamma - s \times \gamma + (q-1)\gamma_\alpha$;

(h) if α is real and γ is not in the domain of c_α, then $T_s\gamma = -\gamma$.

Also, we can now define the *length function* on $\mathcal{D}$. For each $\gamma = (H_i, \lambda\Phi) \in \mathcal{D}$, transfer the involution θ on $\mathfrak{h}_i$ to $\mathfrak{h}^a$ by setting $\theta_\lambda = i_\lambda^{-1}\theta i_\lambda$. Then θ_λ is an involution of $\mathfrak{h}^a$ which preserves the root system; in particular it takes Weyl chambers to Weyl chambers. So we may define

$$L(\gamma) = \frac{1}{2}\# \left\{ \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}^a) \mid \theta_\lambda(\alpha) \in \Delta^-(\mathfrak{g}, \mathfrak{h}^a) \right\} + \frac{1}{2}\dim(-1 \text{ eigenspace of } \theta_\lambda)$$

This is a half-integer, but length-differences between comparable elements in the Bruhat ordering (which we are about to define) are always integers.

Let $s \in S$ be given, and $\gamma, \gamma' \in \mathcal{D}$. We write $\gamma \xrightarrow{s} \gamma'$ if $L(\gamma) = L(\gamma') + 1$, and γ' appears in $T_s\gamma$. It is equivalent to say that either (a) α_λ is complex, $\theta_\lambda(\alpha) \notin \Delta^+(\mathfrak{g}, \mathfrak{h}^a)$, and $\gamma' = s \times \gamma$, or (b) α_λ is real, γ is in the domain of c_α, and γ' occurs in $c_\alpha(\gamma)$. For a given γ, the set of s for which there is an arrow $\gamma \xrightarrow{s} \gamma'$ is denoted by τ, and called the abstract τ–invariant (or the descent set) of γ.

The *Bruhat ordering* on $\mathcal{D}$ is defined as follows. It is the weakest partial order $\leq$ with the property that for each $\gamma, \gamma' \in \mathcal{D}$ and $s \in S$ such that $\gamma \xrightarrow{s} \gamma'$ we have $\delta \leq \gamma$ if and only if either $\delta \leq \gamma'$, or there are $\delta, \delta' \in \mathcal{D}$ such that $\delta' \leq \gamma'$ and $\delta \xrightarrow{s} \delta'$. Note that contrary to the group case, it is necessary to take into account *all* the $s \in \tau(\gamma)$, and not justo one of them.

This completes the definition of all the data occurring in the Kazhdan-Lusztig algorithm.

6. Recursion formulas

As we have mentioned in sect. 2, there are two possible approaches for the computation of the Kazhdan-Lusztig polynomials. Either one determines first the involution D, and deduces the c_γ purely formally, or one sets up direct recursion formulas for the c_γ. In practice, the second approach is more efficient; the difficulties in making the recursion effective are the same anyway, whether one wants to determine D or the c_γ.

Let us review how things work out for the Hecke algebra. It is better to replace the basis T_w by $t_w = q^{-l(w)/2}T_w$. The involution on the Hecke algebra is an antilinear algebra involution, given on t_w by $D(t_w) = t_{w^{-1}}^{-1}$. It is easy to see that $t_s^{-1} = t_s + \alpha$, where $\alpha = q^{-1/2} - q^{1/2}$, and that the element c_s of the Kazhdan-Lustig basis of $\mathcal{H}$ is given by $c_s = t_s + q^{-1/2}$.

Then to compute the c_w by induction on the length of w, we assume that $l(w) > 0$, we pick $s \in S$ such that $sw < w$, and we look at $c_s c_{sw} \in \mathcal{H}$. This is certainly a selfdual element, of the form

$$c_s c_{sw} = t_w + \sum_{z < w} a_z t_z$$

It is easy to see that $a_z \in \mathbf{Z}[q^{-1/2}]$; they might, however, have constant terms. In fact, it is easy to see that a constant term occurs in a_z if and only if $z < ws$, $sz < z$, and the element $p(z, ws)$ has a term in degree $-1/2$. If we denote the coefficient of degree $-1/2$ in $p(x, y)$ by $\mu(x, y)$, we then see that

$$c_s c_{sw} - \sum_{\substack{z < sw \\ sz < z}} \mu(z, sw) c_z$$

satisfies all the conditions required of c_w. When we write the corresponding formulas for the coefficients of c_w, we get the familiar recursion formulas for Kazhdan-Lusztig polynomials.

In the current situation, one could try something similar. Let us denote l the absolute length function on $\mathcal{D}$, *i.e.* the one where minimal elements have length 0, and for non-minimal γ, the length of γ is one more than the length of the maximal elements $< \gamma$. Define t_γ as in section 2.

Clearly for γ minimal, we have $c_\gamma = t_\gamma$. Now if γ is not minimal, there is at least one γ', and an element $s \in S$, such that $\gamma \xrightarrow{s} \gamma'$. Then we may consider the element $c_s c_{\gamma'} \in \mathcal{M}$. Again, it is clearly selfdual. Let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)$ be the simple root corresponding to s. Now there are two cases :

(a) $\alpha_{\lambda'}$ is complex, or noncompact type I imaginary. Then $c_s c_{\gamma'}$ is of the form

$$t_\gamma + \sum_{\delta < \gamma} a_\delta t_\delta$$

and we have $a_\delta \in \mathbf{Z}[q^{-1/2}]$ just as in the Hecke algebra case. Defining $\mu(\delta, \gamma)$ exactly in the same way, we may subtract an integral linear combination of c_δ for $\delta < \gamma$, to obtain the required basis element c_γ.

(b) $\alpha_{\lambda'}$ is noncompact type II imaginary. Then $c_s c_{\gamma'}$ is of the form :

$$t_\gamma + t_{s \times \gamma} + \sum_{\delta < \gamma} a_\delta t_\delta$$

and the same kind of thing will lead to a formula for $c_\gamma + c_{s \times \gamma}$.

It is not so easy to get around the difficulty in case *(b)*. Denote $\sim$ the smallest equivalence relation on $\mathcal{D}$ such that γ and $s \times \gamma$ are equivalent whenever α_λ is type II real. Clearly the formulas in (b) will allow to compute the c_γ for all γ in a given equivalence class, once one of them is known. So we are fine if at least one $\delta \sim \gamma$ is such that $\tau(\delta)$ is not made un entirely of type II real roots. Notice also that if there is a loop in the graph defined on the class by the edges $\gamma, s \times \gamma$, $s \in \tau(\gamma)$, where are again fine. So the case where we seem to be really stuck is when in addition this graph is a tree; the simplest example of this arises in $\mathbf{GL}(2, \mathbf{R})$ already.

Nevertheless it may be shown that there is always a way out, which will allow the computation of $p(\delta, \gamma)$ using induction on $l(\gamma) - l(\delta)$ — I'll not go into this as it is a little bit technical and I don't feel I have understood it well enough yet.

7. Computational assessment

It seems to me that the difficulties in the computation lie mostly within the realm of structure theory. I think one should take everything back to our fixed Cartan algebra $\mathfrak{h}^a$. Then at least the set of λ corresponding to the central character χ will be fixed.

In fact, it is probably best to work within a fundamental Cartan subgroup $H = TA$. It is known that conjugacy classes of Cartan subgroups are in $(1, 1)$ correspondence with $W(K, T)$–conjugacy classes of sets of strongly orthogonal noncompact imaginary roots.

To each class H_i will correspond to an involution θ_i on $\mathfrak{h}^a$. Determining the component group of H_i from θ_i is not hard. One of the delicate things will be to determine the Weyl group $W(G, H_i)$, but there are algorithms for that in Vogan's papers. Another delicate aspect is how to describe the component group of $\mathbf{G}(\mathbf{R})$, and how to keep track of the difference between G and $\mathbf{G}(\mathbf{R})$ in the computations. In fact, it might be reasonable as a first approach to assume $G = \mathbf{G}(\mathbf{R})$.

The cross-action of W should be no problem, although it involves the determination of the action of W on $W/W(G, H_i)$, which is not so easy, as $W(G, H_i)$ is not a reflection subgroup in general. This will probably require some version of Todd-Coxeter coset enumeration.

What looks a bit more delicate is the Cayley transform, but perhaps not that much : after all, the new θ_j should basically correspond to changing α from a $+1$–eigenvector into a -1–eigenvector, and the new $+1$–eigenspace is just $\mathrm{Ker}(\alpha)$. (check this!)

Once all these are defined, we should be in good shape. The only thing left, then, would be a reasonably efficient implementation of the recursion process; this may involve some fairly complicated parts for those rare cases where one gets stuck in the situation described in sect. 6, but still should not be all that different from the Hecke algebra case.

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