Generic unitary spherical parameters

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July 23, 2003

► Let $(X, R, X^\vee, R^\vee)$ be the root datum of a split quasi-simple algebraic group G over a non-archimedean local field. Let W be the Weyl group. The affine walls (defined by equations $b = n$, $b \in R^\vee$, $n \in \mathbb{Z}$) divide $V = X \otimes \mathbb{R}$ to *alcoves*. The affine Weyl group W_{a}, the group generated by affine reflections associated to the affine walls, acts simply transitively on the set of alcoves.

If G is adjoint, then the affine Weyl group W_{a} is simply $W \ltimes X$.

► Fix a simple system Δ of $R^\vee$. For simplicity, we assume that w_0, the longest element of W, is equal to -1 (that is, the type is not A_n, D_{odd}, E_6). Write an reduced expression for w_0:

$$w_0 = s_N s_{N-1} \cdots s_2 s_1,$$

where s_i are simple reflections across the hyperplane $b_i = 0$, $b_i \in R^\vee$, and $N = l(w_0) = \#R/2$. Let $v \in V$. We want to consider the element

$$A(v) = \big(1 + b_N(v_N)s_N\big)\big(1 + b_{N-1}(v_{N-1})s_{N-1}\big) \cdots \big(1 + b_2(v_2)s_2\big)\big(1 + b_1(v_1)(s_1)\big) \in \mathbb{R}[W],$$

where $v_i = s_{i-1} \cdots s_2 s_1(v)$.

Fact. The element $A(v)$ depends only on Δ, not on the reduced expression $s_N \cdots s_1$ for w_0. For any unitary representation (σ, E) of W, $\sigma(A(v)) \in \mathrm{End}(E)$ is hermitian.

► We are interested in the signature of $\sigma(A(v))$ when v lies in the dominant chamber C defined by $b(v) \geq 0$ for all $b \in \Delta$. It is clear that $\sigma(A(v))$ is invertible unless $b_i(s_{i-1} \cdots s_2 s_1(v)) = \pm 1$ for some i.

Fact. Let $p_i = b_i \circ (s_{i-1} \cdots s_2 s_1) \in R^\vee$. Then the set $\{p_1, \ldots, p_N\}$ is precisely the set $R^\vee_+$ of positive co-roots (w.r.t. the simple system Δ).

Thus

$$A = \big(1 + p_N(v)s_N\big) \cdots \big(1 + p_1(v)s_1\big).$$

Since $v \in C$, $p_i(v) \geq 0$ and $\sigma(A(v))$ is invertible unless $p_i(v) = 1$ for some i.

Let $\mathcal{C}$ be the set of connected components of $C^\circ \setminus \bigcup_i \{v : p_i(v) = 0\}$. Elements of $\mathcal{C}$ are called (generic or top-dimensional) *cells*. The property "$\sigma(A(v))$ is positive definite for all $\sigma \in W^\wedge$" depends only on the cell containing v. If all points in a cell have this property, we say that that the cell is *positive*.

► The positive cells are of great interests in representation theory.

Theorem. (Barbasch-Moy) Let $v \in C$ and let π_v be the spherical representation of G with Satake parameter v. Then π_v is unitary if and only if $\sigma(A(v))$ is positive semi-definite for all $\sigma \in W^\wedge$.

► Consider a 5-tuple (U, p, U', p', U'') such that

- $p, p' \in R^\vee_+$, and U,U',U'' are cells.
- $\partial U \cap \partial U'$ contains a non-empty open set of the hyperplane $p = 1$; and $\partial U' \cap \partial U''$ contains a non-empty open set of the hyperplane $p' = 1$.

– $p(U) < 1$, $p'(U') < 1$; hence $p(U') > 1$, $p'(U'') > 1$.
– $\langle p, (p')^\vee \rangle = 1 = \langle p', p^\vee \rangle$, where $p^\vee$ is the root associated to the co-root p, etc.

The goal of this note is to prove:

Theorem. Under the above conditions, U is positive $\iff$ U'' is positive.

The result can be verified directly when the type is G_2. In the following we assume that the type is different from G_2.

Lemma 1. There is a simple system Δ' of $R^\vee$ such that $-p, p' \in \Delta'$.

I checked this case by case, partly with the help of a computer. There should be a better proof.

Lemma 2. We can choose the reduced expression $w_0 = s_N \cdots s_1$ such that there exists i satisfying $p_i = p$, $p_{i+1} = p'$.

PROOF. Let Δ' be a simple system containing $-p, p'$. Then $\Delta' = \Delta \circ w$ for some $w \in W$. Therefore, $p = b \circ w$, $p' = b' \circ w$ for some $b, b' \in \Delta$.

By a well-known property of Weyl group, $b \circ w < 0$ implies $l(s_b w) = l(w) - 1$, where s_b is the simple reflection associated to b. Similarly, $b' \circ w > 0$ implies $l(s_{b'} w) = l(w) + 1$. Write $s_b w$ as a reduced expression:

$$s_b w = s_{i-1} \cdots s_1.$$

Put $s_i = s_b$, $s_{i+1} = s_{b'}$, and choose $s_{i+2}, \cdots, s_N$ so that

$$w_0 = s_N s_{N-1} \cdots s_2 s_1$$

is reduced. Then we have $p_i = p$, $p_{i+1} = p'$ as required. ∎

▶ In the following, we maintain the assumption of the theorem and the notations of the preceding lemmas. Let σ be an (irreducible) unitary representation of W of dimension d. Let $\operatorname{Tr} \sigma(s_i) = (d - n) \cdot 1 + n \cdot (-1)$. That is, we assume that the eigenvalues of $\sigma(s_i)$ are 1 occurring $(d - n)$ times, and -1 occurring n times.

Lemma 3. Assume that $\sigma(A(v))$ has signature (p, q) for $v \in U$. Then there exists $n', n'' \geq 0$ such that $n = n' + n''$, and $\sigma(A(v))$ has signaure $(p - n' + n'', q + n' - n'')$ for $v \in U'$.

Remark. We interpret the result as: when going across the wall, n' positive eigenvalues turn negative and n'' negative eigenvalues turn positive. Therefore, exactly n eigenvalues change signs. This is what we prove below.

PROOF. Consider a small line segment $\{v(t) : -\epsilon < t < \epsilon\}$ going from U to U' passing through the wall $p = 1$ transversally at $t = 0$. Clearly, $\det \sigma(A(v(t)))$ vanishes exactly n times at $t = 0$. The coefficients of the characteristic polynomial of $\sigma(A(v(t)))$ are polynomial functions in t. The eigenvalues of $\sigma(A(v(t)))$ can be regarded as (multi-valued) algebraic functions of t, which now is regarded as a *complex* variable. By restricting to a suitable simply connected region in $\mathbb{C} \setminus \{0\}$, they become single-valued algebraic functions $\lambda_1(t), \ldots, \lambda_d(t)$. We may and do assume that $\lambda_1(t), \ldots, \lambda_n(t)$ vanishes at $t = 0$.

Each $\lambda_i(t)$ ($i = 1, \dots, n$) has a Puiseux expansion:

$$\lambda_i(t) = c_i \cdot t^{e_i} + \text{ higher terms}$$

where $c_i \neq 0$ and $e_i > 0$ is a rational number. We claim that $e_i = 1$ for all i. Indeed, if the denominator of e_i is greater than 1, then $\lambda_i(t)$ won't take real value for real t. So all the e_i's are integers. Since these n positive integers add up to n, each of them is equal to 1.

Now it is clear that $\lambda_i(t)$ changes sign as t changes sign, for $n = 1, \dots, n$. The lemma is proved. ∎

If v lies on the co-dimension 1 facet common to U, U', then it is clear that $\operatorname{rank} A(v) = d - n$.

Lemma 4. If v lies on the co-dimension 2 facet common to U, U', U'', then $\operatorname{rank} A(v) = d - n$.

PROOF. From the definition of $A(v)$, it suffices to show that $\operatorname{rank} B(v) = d - n$, where $B(v) = \big(1 + p_{i+1}(v)s_{i+1}\big)\big(1 + p_i(v)s_i\big)$. Our assumption on v means that $p_i(v) = p_{i+1}(v) = 1$. So $B(v) = (1 + s_{i+1})(1 + s_i)$.

Now it is clear that in order to show that $\operatorname{rank} B(v) = \operatorname{rank}(1 + s_i)$, it suffices to decompose (σ, E) into the direct sum of irreducible subrepresentations $E_1, \dots, E_m$ of $H = \langle s_i, s_{i+1}\rangle \simeq S_3$, and check $\operatorname{rank}(B(v)|E_j) = \operatorname{rank}((1 + s_i)|E_j)$ for all j. There are three possible cases.

Case 1. $E_j \simeq \mathbf{1}$. Then $\operatorname{rank}(B(v)|E_j) = 1 = \operatorname{rank}((1 + s_i)|E_j)$.

Case 2. $E_j \simeq$ sgn, the sign representation of S_3. Then $\operatorname{rank}(B(v)|E_j) = 0 = \operatorname{rank}((1 + s_i)|E_j)$.

Case 3. $E_j \simeq$ the 2-dimensional irreducible representation of S_3. Then $\operatorname{rank}(B(v)|E_j) = 1 = \operatorname{rank}((1 + s_i)|E_j)$.

The lemma is proved completely. ∎

► We are now ready to prove the theorem. Assume that U is a positive cell. Then $\sigma(A(v))$ has signature $(d - n, n)$ for $v \in U'$ by Lemma 3. We will denote the d eigenvalues of $\sigma(A(v))$ as $\lambda_1(v), \dots, \lambda_d(v)$, indexed so that $\lambda_1(v) \leq \cdots \leq \lambda_d(v)$.

Let v_0 be a point on the facet of co-dimension 2, common to U, U', U''. Then $\sigma(A(v_0))$ has 0 as an eigenvalue of multiplicity n by Lemma 4. The rest of the eigenvalues are positive since v_0 lies on the closure of U. It follows that there is a $\delta > 0$, and an open ball N_0 around v_0 such that the following is true: *for any $v \in N_0$, there are exactly n eigenvalues of $\sigma(A(v))$ of absolue values $< \delta$; moreover, the other eigenvalues are* $> \delta$. The n eigenvalues singled out this way are of course $\lambda_1(v), \dots, \lambda_n(v)$.

Consider a small line segment $\{v(t) : -\epsilon < t < \epsilon\}$ lying inside N_0, going from U to U' passing through the wall $b = 1$ transversally at $t = 0$. By Lemma 3, as t varying from ϵ toward $-\epsilon$, exactly n eigenvalues shrinks to 0, then turns negative. We may and do assume that these eigenvalues are $< \delta$ in absolute value by choosing ϵ small enough. Then it is clear that this collection of sign-changing eigenvalues are simply $\lambda_1(v(t)), \dots, \lambda_n(v(t))$.

For $v \in N_0 \cap U'$, $\lambda_i(v)$ varies continuously with v. In particular, its sign doesn't change. Since $\lambda_i(v(t)) < 0$ for $t < 0$, $i = 1, \dots, n$, we conclude that $\lambda_i(v) < 0$ for all $i = 1, \dots, n$ and all $v \in N_0 \cap U'$.

Consider another small line segment $\{u(t) : -\epsilon < t < \epsilon\}$ lying in N_0, going from U' to U'' passing through the wall $b' = 1$ transversally at $t = 0$. By the same argument, there are n sign-changing eigenvalues of $\sigma(A(u(t))$, and they are simply $\lambda_1(u(t)), \dots, \lambda_n(u(t))$. Since $\lambda_i(u(t)) < 0$ for $t > 0$, we have $\lambda_i(u(t)) > 0$ for $t < 0$. This shows that $\sigma(A(v))$ is positive definite for at least one $v \in U''$, hence for all $v \in U''$.

We have completed the prove of $\Longrightarrow$. The proof of $\Longleftarrow$ is similar. ∎

► Let U_0 be the cell defined by $b(v) > 0$ for all $b \in \Delta$, and $b_0(v) < 1$, where $-b_0$ is the highest root. This cell is obviously positive.

Consider the set $\mathcal{C}_0$ of cells U with the following property: there exist cells $U_0, U_1, \ldots, U_{2k} = U$, positive roots $p_0, \ldots, p_{2k-1}$ such that the 5-tuple $(U_{2i}, p_{2i}, U_{2i+1}, p_{2i+1}, U_{2i+2})$ satisfies the assumption of the theorem, for $i = 0, \ldots, k-1$.

Then every cells in $\mathcal{C}_0$ is positive by the theorem. There are a few remarkable experimental facts about $\mathcal{C}_0$.

Fact 1. $\mathcal{C}_0$ is the set of all positive cells.

Fact 2. Each cell in $\mathcal{C}_0$ is an alcove.

Fact 3. $\#\mathcal{C}_0$ is a power of 2. [1 for B_n, 2^{n-1} for C_{2n-1}, C_{2n}, D_{2n}, 8 for E_7, 16 for E_8, 2 for F_4, G_2]

Fact 4. There is a symmetry structure on $\mathcal{C}_0$. [More explanations later].

Remark. All these are now established facts, via case by case verification using classification. Fact 1 is most difficult and surprising. I checked it using Barbasch's work on classical groups and computer computations with Adams for exceptional groups, except the case of E_8. In this workshop, we learned that Barbasch/Ciubotaru now know that there are 16 positive cells for E_8. Since $\#\mathcal{C}_0 = 16$, Fact 1 must be true for E_8 as well.

I emphasize that Facts 2, 3, and 4 are also verified using classification. I do not have any conceptual understanding of these facts.

► **Examples.** Below I will show some diagrams displaying the structure of $\mathcal{C}_0$ in several examples. Each $\bullet$ represents an element of $\mathcal{C}_0$. The top $\bullet$ represents U_0. Two $\bullet$ are joined by a line segment if there are two reflections relating the two cells in the way prescribed by the theorem.

Each edge is labelled with 2 numbers as

$$\begin{array}{c}\bullet\\ i \,\Big|\, j\\ \bullet\end{array}$$

It means that the lower $\bullet$ is obtained from the upper $\bullet$ by performing a reflection of type i, followed by a reflection of type j. Here, we notice that for any alcove, each vertex as a type, indexed by the extended Dynkin diagram of $R^\vee$; and by *the reflection of type* i, we mean the reflection across the facet opposite to the vertex of type i.

Remark. For each group, the diagram encodes the complete set of generic parameters very effectively, in a coordinate-free way.

Remark. It is very easy to write a computer program to generate the diagram from the Cartan matrix. It only takes a few seconds to finish E_8. It is also fairly easy to compute the diagram by hand.

Example. We claim that for type B_n, $\mathcal{C}_0 = \{U_0\}$ is a singleton.

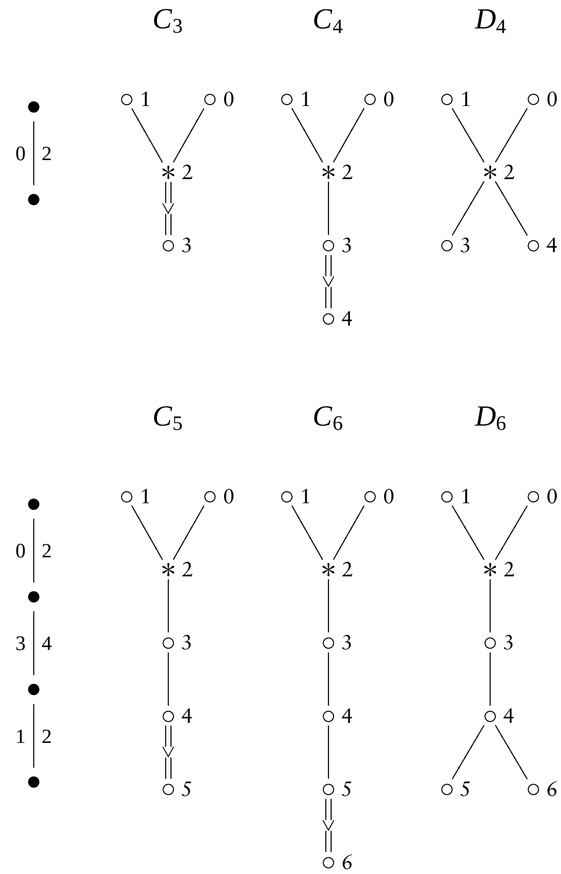
Indeed, U_0 is given by

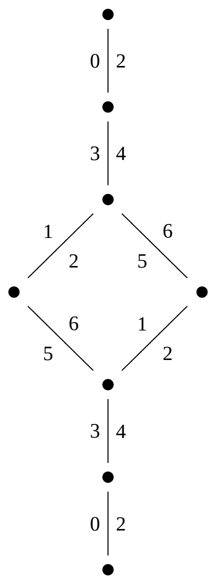
$$\frac{1}{2} > x_1 > \cdots > x_n > 0$$

The first reflection r can only be the one acrossing the wall $2x_1 = 1$ (otherwise we are going out of the dominant chamber C). Next, we need to find a wall $p = 1$ for $r(C)$ such that $\langle 2x_1, p^\vee \rangle = 1$. But this is impossible since $\langle 2x_1, p^\vee \rangle$ is either 0 or ± 2. Therefore the algorithm stops and we have $\mathcal{C}_0 = \{U_0\}$.

The symmetry. We now explain the symmetry mentioned in Fact 4: the diagram remains unchanged if you turn it upside down. Even the labellings of the edges remain the same up to applying an involution of the extended Dynkin diagram (needed only in cases of C_{4n+1}, C_{4n+2}, D_{4n+2}, and E_7).

Since the diagram always has a "head" U_0, the symmetry means that it has a tail U_∞, "the unitary alcove fartherest from the origin". The type 0 (specieal) vertex of U_∞ may also have some significance. It turns out that whenever $\#\mathcal{C}_0 > 1$, this vertex is always a fundamental weight. Its associated coroot is marked by a black dot $*$ in the diagrams below.

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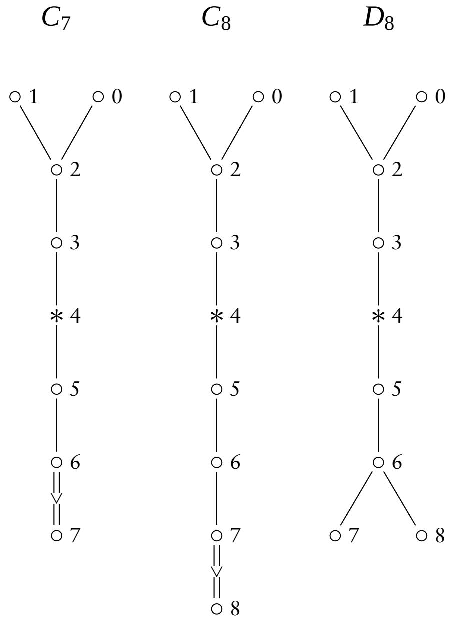
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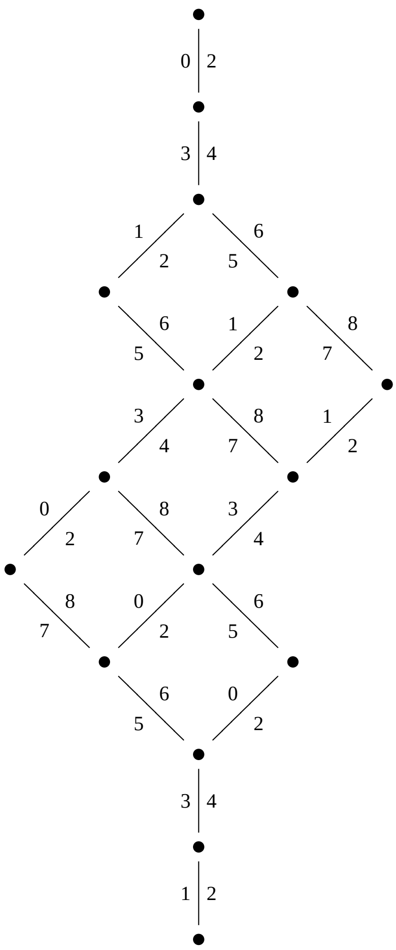
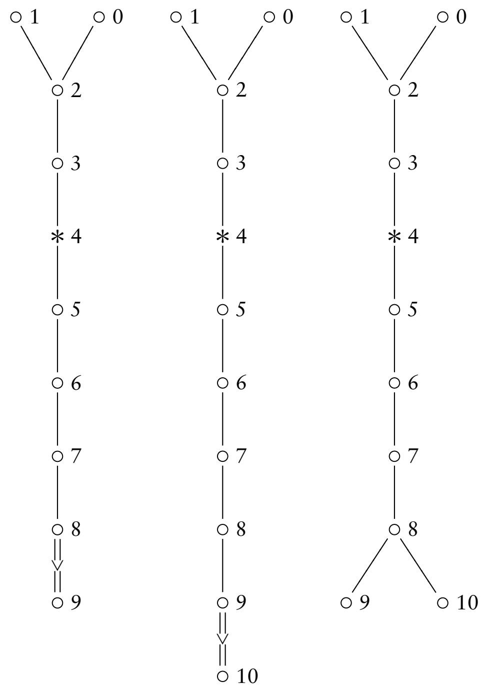
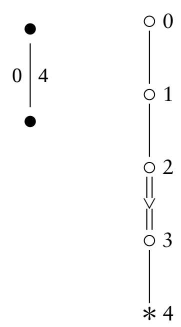


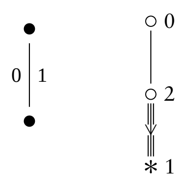
C_9 C_{10} D_{10}



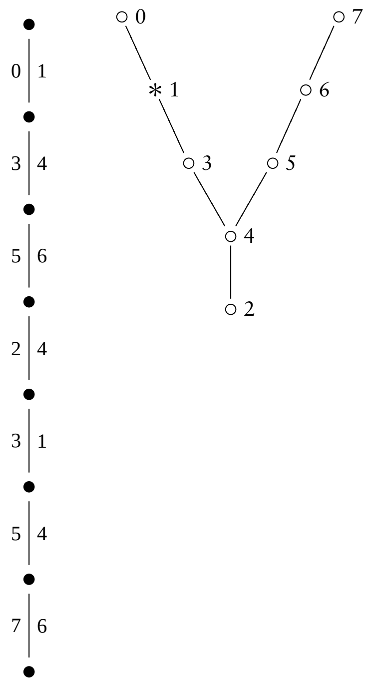
F_4



G_2



E_7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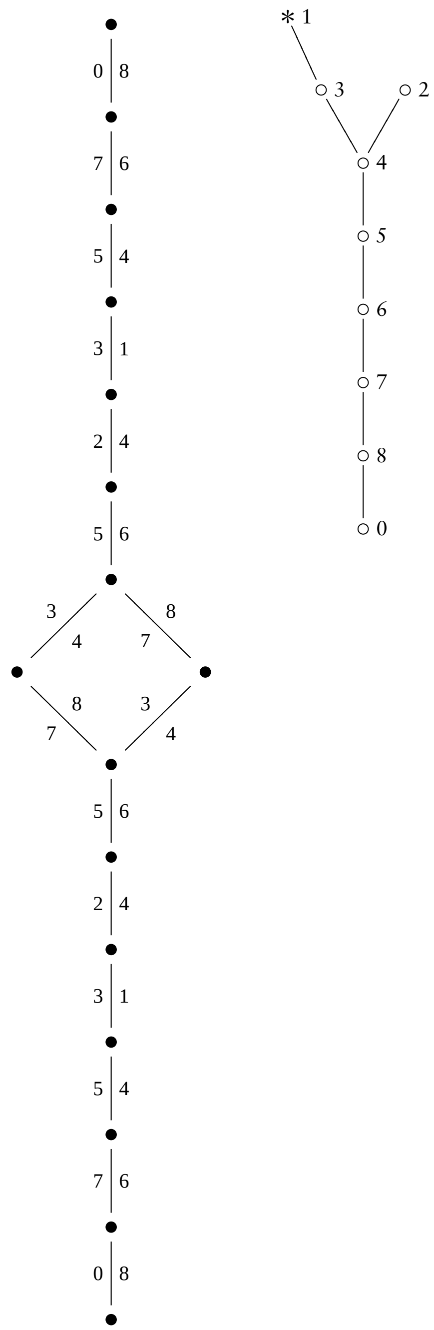
E_8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