UNITARY DUAL FOR P-ADIC GROUPS OF TYPE $F_4$

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1. INTRODUCTION

Finding the Iwahori-spherical unitary dual for p-adic groups can be reduced to the classification of unitary representations for the associated graded Hecke algebras with real parameters.

Let $G$ denote a split reductive p-adic group, $\hat{G}$ is its complex dual with maximal torus $\hat{A}$ and $\hat{g}$, $\hat{a}$ the corresponding Lie algebras, $W$ the Weyl group. $\hat{A}$ is the symmetric algebra over $\hat{a}^\mathbb{C}$ and $\mathbb{H} = CW \otimes C \hat{A}$ the graded Hecke algebra associated to the group $G$.

1.1. Standard modules and Langlands quotients. I recall the classification of irreducible modules of $\mathbb{H}$ (Kazhdan-Lusztig, Lusztig).

Theorem 1.1 (KL, L). The irreducible modules for $\mathbb{H}$ are parametrized by $\hat{G}$-conjugacy classes $(s, \mathcal{O}, \psi)$, where $s \in \hat{g}$ is semisimple, $\mathcal{O} \subset \hat{g}$ is a nilpotent orbit such that $[s, e] = e$ for $e$ a representative in $\mathcal{O}$ and $\psi \in \hat{A}(s, e)$ is an irreducible representation of the component group of the centralizer of $s$ and $e$.

To $(s, \mathcal{O}, \psi)$, one associates an $\mathbb{H}$-module, $X(s, \mathcal{O}, \psi)$ called standard module. Every irreducible module of $\mathbb{H}$ can be realized as the unique quotient (Langlands quotient) $L(s, \mathcal{O}, \psi)$ of some standard module $X$. Moreover, the factors in $X(s, \mathcal{O}, \psi)$ correspond to parameters $(s, \mathcal{O}', \psi')$ with $\mathcal{O} < \mathcal{O}'$.

Date: July 23, 2003.
If \( \{e, h, f\} \) is a standard triple for \( e \in \mathcal{O} \), it is possible to write \( s = \frac{1}{2} h + \tilde{\nu} \), where \( \tilde{\nu} \) centralizes the triple. If \( \tilde{\nu} = 0 \), the parameter is called \textit{tempered}. If, in addition, \( \mathcal{O} \) doesn’t meet any Levi component of \( \tilde{\mathfrak{g}} \), it is called \textit{discrete series}. These are the basic examples of unitary representations of \( \mathbb{H} \).

When \( \mathcal{O} \) meets some Levi component \( \tilde{\mathfrak{m}} \subset \tilde{\mathfrak{g}} \),
\[
X(s, \mathcal{O}, \psi) = \mathbb{H} \otimes_{\mathbb{H}_{\mathcal{M}}} (V \otimes 1_\nu),
\]
for some tempered representation \( V \) of \( \mathbb{H}_{\mathcal{M}} \) and character \( \nu \). In this parametrization, \( X(M, V, \nu) \) denotes the standard module and \( L(M, V, \nu) \) its Langlands quotient.

1.2. Intertwining operators and K-types. Barbasch-Moy showed that a Langlands quotient \( L(M, V, \nu) \) (with \( \nu \) real) is Hermitian if and only if there exists a Weyl group element \( w \) which conjugates the triple into \( (M, V, -\nu) \). Corresponding to this Weyl element \( w \), they considered the intertwining operator \( I(w, \nu) : X(M, V, \nu) \to X(M, V, -\nu) \) defined as follows. If \( w = s_1 \cdots s_k \) is a reduced decomposition, \( I(w, \nu) = \prod I(s_j, \nu) \), where for each simple root \( \alpha \), put \( r_\alpha = (t_\alpha \alpha - 1)(\alpha - 1)^{-1} \) and then
\[
I(s_\alpha, \nu) : X(M, V, \nu) \to X(M, V, s_\alpha \nu), \quad x \otimes 1_\nu \mapsto x r_\alpha \otimes 1_{s_\alpha \nu}.
\]

Of great importance for the actual classification is the \( CW \)-structure of the standard modules. Recall the decomposition \( CW = \sum_{\sigma \in W} V_{\sigma} \otimes V^*_{\sigma}, \) \( (\sigma, V_{\sigma}) \) denoting the irreducible representations of the Weyl group, which, by analogy with the real groups, are called \textit{K-types}. The Weyl group representations for type \( F_4 \) were classified by Kondo.

Because \( r_\alpha \) acts on the right, by Frobenius reciprocity, \( I(s_\alpha, \nu) \) gives rise to an operator \( r_\sigma (s_\alpha, \nu) : (V_{\sigma}^*)^{W(M)} \to (V^*_{\sigma})^{W(M)}. \) If the module were unitary, all the operators arising by the restrictions to K-types would be positive semidefinite. One of the main tools for showing modules are not unitary is to compute the signature of these operators.

Among the Weyl group representations associated to a nilpotent orbit, there are some special ones called \textit{lowest K-types}, coming from the Springer correspondence, which appear with multiplicity one in the standard module and which parametrize the Langlands quotient.

The spherical modules are precisely those which contain the trivial K-type and for this case it is enough to consider the long intertwining operator. The spherical Langlands parameters are uniquely determined by the infinitesimal character. \( : \) fix \( s \) a semisimple element in \( \tilde{\mathfrak{g}} \) corresponding to an infinitesimal character. Define \( \tilde{\mathfrak{g}}_1 \):
\[
\tilde{\mathfrak{g}}_1 = \{ x \in \tilde{\mathfrak{g}} : [s, x] = x \}.
\]
Then there exists a unique largest nilpotent orbit \( \mathcal{O} \) in \( \tilde{\mathfrak{g}} \) which meets \( \tilde{\mathfrak{g}}_1 \) and one can write \( s = \frac{1}{2} h + \tilde{\nu} \) for a triple \( (e, h, f) \) of \( \mathcal{O} \) and \( \tilde{\nu} \in Z(\mathcal{O}) \).

If \( \mathcal{O} \) meets a Levi component \( \tilde{\mathfrak{m}} = \tilde{\mathfrak{g}}_0 \times \mathfrak{gl}(k_1) \times \cdots \times \mathfrak{gl}(k_l) \), the intersection is \( \mathcal{O}_0 \times (k_1) \times \cdots \times (k_l) \) and the Langlands quotient \( L(s) \) is the spherical
quotient of $\text{Ind}_M^G(L(s_0) \otimes \nu_1 \otimes \cdots \otimes \nu_n)$, with $s_0$ half the middle element of $O_0$.

**Theorem 1.2** (Barbasch). If $\nu$ dominant (i.e. $\langle \nu, \alpha \rangle \geq 0$ for all positive roots $\alpha$) and if $w_0$ denotes as before the long Weyl element, $X(\nu) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_{\nu}$ ($\nu$ real), then $\text{Im} I(w_0, \nu) = L(\nu)$, where $L(\nu)$ denotes the irreducible spherical module parametrized by $\nu$.

1.3. **Parameters for $F_4$.**

The root system for $F_4$ is

**Diagram:** $\alpha_1 - \alpha_2 = \alpha_3 - \alpha_4$

The reflections $s_1$, $s_2$ correspond to the long simple roots.

**Simple Roots and Weights:**

\[
\begin{align*}
\alpha_1 &= \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 & \omega_1 &= \epsilon_1 \\
\alpha_2 &= 2\epsilon_4 & \omega_2 &= \frac{3}{2} \epsilon_1 + \frac{1}{2} \epsilon_2 + \frac{1}{2} \epsilon_3 + \frac{1}{2} \epsilon_4 \\
\alpha_3 &= \epsilon_3 - \epsilon_4 & \omega_3 &= 2\epsilon_1 + \epsilon_2 + \epsilon_3 \\
\alpha_4 &= \epsilon_2 - \epsilon_3 & \omega_4 &= \epsilon_1 + \epsilon_2
\end{align*}
\]

2. **Spherical unitary dual**

Spherical unitary dual is partitioned by the nilpotent orbits. To each nilpotent orbit $O$, one attaches a set of unitary parameters, called complementary series. In most cases, the complementary series match the spherical unitary parameters attached to the trivial nilpotent in the centralizer of $O$. Exception: $O = A_1 + \tilde{A}_1$.

2.1. **Maximal parabolic cases.** The starting case is that of the nilpotent orbits coming from maximal parabolics. They allow a single parameter. The calculations are done entirely using the relevant $K$-types, which are a minimal set of Weyl group representations that are sufficient for determining the unitarity. The payoff is that one can match these Weyl representations with $K$-types in the real $F_4$, so that the spherical real unitary would follow.

**Lemma 2.1.** The $K$-types $1_1$, $2_3$, $4_2$, $8_1$ and $9_1$ are a minimal set for determining the unitarity of the spherical parameters in the maximal parabolic cases.

**Example:**

**B3:** Induced from the trivial representation on $B_3$ has parameter

\[
(\nu + \frac{3}{2}, \nu - \frac{3}{2}, \frac{3}{2}, \frac{1}{2}).
\]
The $K$-types are:

$$(1) \quad 1_1 + 4_2 + 9_1 + 2_1 + 8_1.$$ 

Reducibility occurs at $\nu = 1$, 2, 4 corresponding to $F_4(a_2)$, $F_4(a_1)$, $F_4$.
 Relevant values of the intertwining operator are:

$$
\begin{align*}
2_1: & \quad \frac{(\nu-2)(\nu-4)}{(\nu+2)(\nu+4)} \text{ if } \nu \neq 1 \text{ and } 0 \text{ if } \nu = 1 \\
4_2: & \quad \frac{2 \nu}{1+\nu} \\
9_1: & \quad \frac{(\nu-2)(\nu-4)}{(\nu+2)(\nu+4)} \\
8_1: & \quad \frac{(1-\nu)(2-\nu)(4-\nu)}{(\nu+1)(\nu+2)(\nu+4)}
\end{align*}
$$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$1_1$</th>
<th>$4_2$</th>
<th>$9_1$</th>
<th>$2_1$</th>
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There is a complementary series $0 \leq \nu < 1$.

2.2. **Induction step.** For a nilpotent $\mathcal{O}$, one determines all hyperplanes of reducibility of the standard module. On each hyperplane, the spherical parameter is parametrized by an orbit $\mathcal{O}' > \mathcal{O}$, so one knows which such parameters are unitary. Keep only the regions bounded by unitary walls; for those show they are unitarily induced from unitary parameters of some subgroup.

**Example a.**

$\tilde{A}_2$:

Infinitesimal character $(\nu_1 + \frac{\nu_2}{2}, 1 + \frac{\nu_2}{2}, \frac{\nu_2}{2}, -1 + \frac{\nu_2}{2})$, $\nu_1 \geq \nu_2 \geq 0$.

Reducibility lines: $2\nu_1 + \nu_2 = 1$, $\nu_1 + 2\nu_2 = 1$, $\nu_1 - \nu_2 = 1$ from $A_1 + \tilde{A}_2$, $\nu_1 + \nu_2 = 1$, $\nu_1 = 1$ and $\nu_2 = 1$ from $C_3(a_1)$, $\nu_1 + \nu_2 = 3$, $\nu_1 = 3$ and $\nu_2 = 3$ from $C_3$.

The complementary series is $\{0 \leq \nu_2 \leq \nu_1 < \frac{1-\nu_2}{2}\}$ and $\{1 - 2\nu_2 < \nu_1 < 1 - \nu_2\}$.

On the line $\nu_1 = \nu_2$, for $0 \leq \nu_2 < \frac{1}{2}$, the parameter is unitarily induced irreducible from a complementary series associated to the nilpotent (33) in $C_3$.

As seen from the picture, the spherical complementary series is identical with the one of the centralizer $G_2$.

**Example b.**

$A_1 + \tilde{A}_1$:
Infinitesimal character $(\nu_1, \frac{1}{2}, \frac{1}{2} + \nu_2, -\frac{1}{2} + \nu_2, \frac{1}{2})$, $\nu_1 \geq 0$, $\nu_2 \geq 0$.

Reducibility lines: $\nu_1 = \frac{1}{2}$ from $A_2$, $\nu_1 = \frac{3}{2}$ from $B_2$, $\nu_1 = \frac{5}{2}$ from $B_3$, $\nu_2 = 1$ from $C_3(a_1)$, $\nu_2 = 2$ from $C_3$, $\nu_1 - 2\nu_2 = -\frac{3}{2}$, $\nu_1 + 2\nu_2 = \frac{3}{2}$ and $\nu_1 - 2\nu_2 = \frac{3}{2}$ from $A_1 + \tilde{A}_2$.

The complementary series is $\{\nu_1 + 2\nu_2 < \frac{3}{2}, \nu_1 < \frac{1}{2}\}$ and $\{2\nu_2 - \nu_1 > \frac{3}{2}, \nu_2 < 1\}$.

On the line $\nu_1 = 0$, $0 \leq \nu_2 < \frac{3}{2}$ and $\frac{3}{2} < \nu_2 < 1$, the standard module is unitarily induced irreducible from a complementary series associated to the nilpotent $(222)$ in $C_3$.

This is an interesting example: the complementary series does not match that of the centralizer $A_1 + A_1$.

2.3. Generic case.

**Theorem 2.2.** Assume the graded Hecke algebra is of type $F_4$. 
(1) The spherical complementary series associated to a nilpotent orbit \( O \) coincides with that attached to the trivial orbit in its centralizer, except in the case \( O = A_1 + \tilde{A}_1 \). In this one case, the complementary series is strictly included in the one for the centralizer.

(2) The complementary series associated to the trivial nilpotent with dominant infinitesimal character \((\nu_1, \nu_2, \nu_3, \nu_4)\), \( \nu_1 \geq \nu_2 \geq \nu_3 \geq \nu_4 \geq 0 \) and \( \nu_1 - \nu_2 - \nu_3 - \nu_4 \geq 0 \) are:

(a) \( \{ \nu_1 < \frac{1}{2} \} \);

(b) \( \{ \nu_1 + \nu_2 + \nu_3 + \nu_4 > 1, \nu_1 + \nu_2 + \nu_3 - \nu_4 < 1 \} \).

The first observation is that any region bounded by a wall on which a short root is 1 is not unitary. On any such wall, there is a factor coming from \( A_1 \) which can’t be unitary at all points since the complementary series for \( \tilde{A}_1 \) is two-dimensional.

Therefore, one can look only at regions bounded by long roots and finds the hyperplanes on which the parameter can be unitary. Then, in each region determined by such hyperplanes, the parameter can be deformed to
a wall where a simple root equals to zero and the parameter is unitarily induced from $B_3$ or $C_3$.

One can reformulate the classification of the spherical dual in an analogous way to the results for classical groups of Barbasch-Moy:

**Theorem 2.3.** A spherical parameter for $F_4$ is unitary if and only if it is a complementary series from a unitarily induced module from the Iwahori-Matsumoto dual of a tempered representation tensored with a GL-complementary series.

### 3. UNITARY DUAL

The method is to look at the lowest K-types for each nilpotent orbit $O$ and compare the intertwining operator on them to the intertwining operator on lowest K-types of nilpotents $O'$ immediately above $O$ in the partial order.

This is very advantageous in some cases, for example in the case of maximal parabolics.

#### 3.1. Maximal parabolics

For this special case, the following argument (Barbasch-Moy) applies: for the modules parametrized by such a nilpotent orbit $O$ and lowest K-type $\mu$, the next bigger nilpotent $O'$ has the property that a factor with lowest K-type $\mu'$ attached to $O'$ appears at the first point of reducibility. Beyond this point, $\mu$ and $\mu'$ stay in the same factor and they have opposite signatures at $\infty$. Two such K-types have opposite signatures at $\infty$ if and only if their respective lowest harmonic degrees have different parity.

**Example:**

**$B_3$:** The infinitesimal character is $(\frac{3}{2} + \nu, -\frac{3}{2} + \nu, \frac{3}{2}, \frac{1}{2})$, Centralizer $A_1$, LKT $\hat{8}_2$.

The first reducibility point is at $\nu = 1$, where there are factors with LKT $9_4$ and $2_2$ coming from $F_4(a_2)$. For $\nu > 1$, these K-types will stay in the same factor with $\hat{8}_2$. $\hat{8}_2$ and $\hat{9}_4$, or $\hat{8}_2$ and $\hat{2}_2$, have opposite signs at $\infty$, ruling out $\nu > 1$.

Moreover, when normalized by the value on the LKT, the intertwining operators give

$$2_2: \frac{1 - \nu}{1 + \nu} \text{ and } 9_4: \frac{1 - \nu}{1 + \nu}$$

#### 3.2. Matching of intertwining operators

**Example a.**

$\hat{A}_2$: The infinitesimal character is $(\nu_1 + \frac{\nu_2}{2}, 1 + \frac{\nu_2}{2}, \frac{\nu_2}{2}, -1 + \frac{\nu_2}{2})$ with $0 \leq \nu_2 \leq \nu_1$, centralizer $G_2$, LKT $\hat{8}_1$.

The intertwining operator decomposes into a product of intertwining operator for subgroups and maximal parabolics: $(\hat{s}_1 \circ \hat{s}_2)^3$, where $\hat{s}_1$ is the operator corresponding to $\text{Ind}_{\hat{A}_2}^{A_1 + \hat{A}_2}$ and $\hat{s}_2$ to $\text{Ind}_{\hat{A}_2}^{C_3}$.
One can match the calculations with those in the centralizer $G_2$.

- $8_1$ matches $I_1$ in $G_2$
- $6_1$ matches $I_2$ in $G_2$
- $16_1$ matches $2_1$ in $G_2$
- $6_2$ matches $1_3$ in $G_2$

**Example b.**

$A_1 + \tilde{A}_1$: The infinitesimal character is $(\nu_1, \frac{1}{2} + \nu_2, -\frac{1}{2} + \nu_2, \frac{1}{2})$, centralizer $A_1 + A_1$, LKT $9_1$.

One tries to match the dual with that of $A_1 + A_1$. The intertwining operator gives:

- $9_1$: $+1$
- $8_3$: $\frac{\pm \nu_1}{1 + \nu_1}$
- $8_1$: $\frac{\pm \nu_2}{1 + \nu_2}$

This implies that the unitary dual is included in $0 \leq \nu_2 \leq 1$, $0 \leq \nu_1 \leq \frac{1}{2}$. However, there are two lines that cut through this region: $\nu_1 + 2\nu_2 = \frac{3}{2}$ and $-\nu_1 + 2\nu_2 = \frac{3}{2}$ and one also needs the operator on $4_4$ to finish this case.

**Example c.**

$A_1$: The infinitesimal character is $(\nu_1, \nu_2, \nu_3, \frac{1}{2})$ with $0 \leq \nu_3 \leq \nu_2 \leq \nu_1$, centralizer $C_3$, LKT $2_3$.

The K-types that match the intertwining operator in $C_3$ are:

- $2_3$ with $3 \times 0$
- $4_2$ with $0 \times 3$
- $8_1$ with $0 \times 12$
- $9_1$ with $1 \times 2$
- $4_3$ with $0 \times 1^3$

One cannot match $12 \times 0$ and $2 \times 1$ which are relevant for $C_3$. Although the generic unitary parameters associated to this nilpotent are the same as for $C_3$, there are other unitary parameters which do not correspond to $C_3$.

4. The Other Exceptional Groups

The same methods can be applied to study the unitary dual for the other exceptional groups. There are examples in $E_7$ and $E_8$ of spherical complementary series which do not match those of the corresponding centralizer. However they occur very rarely, e.g. only one such example in $E_7$, and it is unclear how to predict them.

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