UNITARY DUAL FOR P-ADIC GROUPS OF TYPE F_4

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Outline.

1. Introduction

Finding the Iwahori-spherical unitary dual for p-adic groups can be reduced to the classification of unitary representations for the associated graded Hecke algebras with real parameters.

Let G denote a split reductive p-adic group, $\check{G}$ is its complex dual with maximal torus $\check{A}$ and $\check{\mathfrak{g}}$, $\check{\mathfrak{a}}$ the corresponding Lie algebras, W the Weyl group. $\mathbb{A}$ is the symmetric algebra over $\check{\mathfrak{a}}^*$ and $\mathbb{H} = \mathbb{C}W \otimes_{\mathbb{C}} \mathbb{A}$ the graded Hecke algebra associated to the group G.

1.1. **Standard modules and Langlands quotients.** I recall the classification of irreducible modules of $\mathbb{H}$ (Kazhdan-Lusztig, Lusztig).

Theorem 1.1 (KL, L)**.** *The irreducible modules for $\mathbb{H}$ are parametrized by $\check{G}$-conjugacy classes $(s, \mathcal{O}, \psi)$, where $s \in \check{\mathfrak{g}}$ is semisimple, $\mathcal{O} \subset \check{\mathfrak{g}}$ is a nilpotent orbit such that $[s, e] = e$ for e a representative in $\mathcal{O}$ and $\psi \in \widehat{A}(s, e)$ is an irreducible representation of the component group of the centralizer of s and e.*

To $(s, \mathcal{O}, \psi)$, one associates an $\mathbb{H}$-module, $X(s, \mathcal{O}, \psi)$ called *standard module.* Every irreducible module of $\mathbb{H}$ can be realized as the unique quotient (*Langlands quotient*) $L(s, \mathcal{O}, \psi)$ of some standard module X. Moreover, the factors in $X(s, \mathcal{O}, \psi)$ correspond to parameters $(s, \mathcal{O}', \psi')$ with $\mathcal{O} < \mathcal{O}'$.

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If $\{e,h,f\}$ is a standard triple for $e\in\mathcal{O}$, it is possible to write $s=\frac{1}{2}h+\bar{\nu}$, where $\bar{\nu}$ centralizes the triple. If $\bar{\nu}=0$, the parameter is called *tempered.* If, in addition, $\mathcal{O}$ doesn't meet any Levi component of $\check{\mathfrak{g}}$, it is called *discrete series.* These are the basic examples of unitary representations of $\mathbb{H}$.

When $\mathcal{O}$ meets some Levi component $\check{\mathfrak{m}}\subset\check{\mathfrak{g}}$,

$$X(s,\mathcal{O},\psi)=\mathbb{H}\otimes_{\mathbb{H}_M}(V\otimes 1_\nu),$$

for some tempered representation V of $\mathbb{H}_M$ and character ν. In this parametrization, $X(M,V,\nu)$ denotes the standard module and $L(M,V,\nu)$ its Langlands quotient.

1.2. **Intertwining operators and K-types.** Barbasch-Moy showed that a Langlands quotient $L(M,V,\nu)$ (with ν real) is Hermitian if and only if there exists a Weyl group element w which conjugates the triple into $(M,V,-\nu)$. Corresponding to this Weyl element w, they considered the intertwining operator $I(w,\nu):X(M,V,\nu)\to X(M,V,-\nu)$ defined as follows. If $w=s_1\dots s_k$ is a reduced decomposition, $I(w,\nu)=\prod I(s_j,\nu)$, where for each simple root α, put $r_\alpha=(t_\alpha\alpha-1)(\alpha-1)^{-1}$ and then

$$I(s_\alpha,\nu):X(M,V,\nu)\to X(M,V,s_\alpha\nu),\quad x\otimes 1_\nu\to xr_\alpha\otimes 1_{s_\alpha\nu}.$$

Of great importance for the actual classification is the $\mathbb{C}W$-structure of the standard modules. Recall the decomposition $\mathbb{C}W=\sum_{\sigma\in\widehat{W}}V_\sigma\otimes V_\sigma^*$, (σ,V_σ) denoting the irreducible representations of the Weyl group, which, by analogy with the real groups, are called *K-types.* The Weyl group representations for type F_4 were classified by Kondo.

Because r_α acts on the right, by Frobenius reciprocity, $I(s_\alpha,\nu)$ gives rise to an operator $r_\sigma(s_\alpha,\nu):(V_\sigma^*)^{W(M)}\to(V_\sigma^*)^{W(M)}$. If the module were unitary, all the operators arising by the restrictions to K-types would be positive semidefinite. One of the main tools for showing modules are *not* unitary is to compute the signature of these operators.

Among the Weyl group representations associated to a nilpotent orbit, there are some special ones called *lowest K-types*, coming from the Springer correspondence, which appear with multiplicity one in the standard module and which parametrize the Langlands quotient.

The spherical modules are precisely those which contain the trivial K-type and for this case it is enough to consider the *long intertwining operator.* The spherical Langlands parameters are uniquely determined by the infinitesimal character. : fix s a semisimple element in $\check{\mathfrak{g}}$ corresponding to an infinitesimal character. Define $\check{\mathfrak{g}}_1$:

$$\check{\mathfrak{g}}_1=\{x\in\check{\mathfrak{g}}:\ [s,x]=x\}.$$

Then there exists a unique largest nilpotent orbit $\mathcal{O}$ in $\check{\mathfrak{g}}$ which meets $\check{\mathfrak{g}}_1$ and one can write $s=\frac{1}{2}h+\bar{\nu}$ for a triple (e,h,f) of $\mathcal{O}$ and $\bar{\nu}\in\mathfrak{Z}(\mathcal{O})$.

If $\mathcal{O}$ meets a Levi component $\check{\mathfrak{m}}=\check{\mathfrak{g}}_0\times\mathfrak{gl}(k_1)\times\cdots\mathfrak{gl}(k_l)$, the intersection is $\mathcal{O}_0\times(k_1)\times\cdots\times(k_l)$ and the Langlands quotient $L(s)$ is the spherical quotient of $\mathrm{Ind}_M^G(L(s_0) \otimes \nu_1 \otimes \cdots \otimes \nu_k)$, with s_0 half the middle element of $\mathcal{O}_0$.

Theorem 1.2 (Barbasch). *If ν dominant (i.e. $\langle \nu, \alpha \rangle \geq 0$ for all positive roots α) and if w_0 denotes as before the long Weyl element, $X(\nu) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu$ (ν real), then Im $I(w_0, \nu) = L(\nu)$, where $L(\nu)$ denotes the irreducible spherical module parametrized by ν.*

1.3. **Parameters for F_4.**
The root system for F_4 is

Diagram: $\alpha_1 - \alpha_2 = \alpha_3 - \alpha_4$

The reflections s_1, s_2 correspond to the long simple roots.

Simple Roots and Weights:

$\alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 \qquad \omega_1 = \epsilon_1$

$\alpha_2 = 2\epsilon_4 \qquad \omega_2 = \frac{3}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 + \frac{1}{2}\epsilon_3 + \frac{1}{2}\epsilon_4$

$\alpha_3 = \epsilon_3 - \epsilon_4 \qquad \omega_3 = 2\epsilon_1 + \epsilon_2 + \epsilon_3$

$\alpha_4 = \epsilon_2 - \epsilon_3 \qquad \omega_4 = \epsilon_1 + \epsilon_2$

2. Spherical unitary dual

Spherical unitary dual is partitioned by the nilpotent orbits. To each nilpotent orbit $\mathcal{O}$, one attaches a set of unitary parameters, called *complementary series*. In most cases, the complementary series match the spherical unitary parameters attached to the trivial nilpotent in the centralizer of $\mathcal{O}$. Exception: $\mathcal{O} = A_1 + \widetilde{A}_1$.

2.1. **Maximal parabolic cases.** The starting case is that of the nilpotent orbits coming from maximal parabolics. They allow a single parameter. The calculations are done entirely using the *relevant K-types*, which are a minimal set of Weyl group representations that are sufficient for determining the unitarity. The payoff is that one can match these Weyl representations with K-types in the real F_4, so that the spherical real unitary would follow.

Lemma 2.1. *The K-types 1_1, 2_3, 4_2, 8_1 and 9_1 are a minimal set for determining the unitarity of the spherical parameters in the maximal parabolic cases.*

Example:

B3: Induced from the trivial representation on B_3 has parameter

$$(\nu + \frac{3}{2}, \nu - \frac{3}{2}, \frac{3}{2}, \frac{1}{2}).$$

The K-types are:

$$1_1 + 4_2 + 9_1 + 2_1 + 8_1. \quad (1)$$

Reducibility occurs at $\nu = 1,\ 2,\ 4$ corresponding to $F_4(a_2),\ F_4(a_1),\ F_4$.

Relevant values of the intertwining operator are:

2_1: $\frac{(\nu-2)(\nu-4)}{(\nu+2)(\nu+4)}$ if $\nu \neq 1$ and 0 if $\nu = 1$

4_2: $\frac{4-\nu}{4+\nu}$

9_1: $\frac{(\nu-2)(\nu-4)}{(\nu+2)(\nu+4)}$

8_1: $\frac{(1-\nu)(2-\nu)(4-\nu)}{(\nu+1)(\nu+2)(\nu+4)}$

ν	1_1	4_2	9_1	2_1	8_1
	$+$	$+$	$+$	$+$	$+$
1	$+$	$+$	$+$	0	0
	$+$	$+$	$+$	$+$	$-$
2	$+$	$+$	0	0	0
	$+$	$+$	$-$	$-$	$+$
4	$+$	0	0	0	0
	$+$	$-$	$+$	$+$	$-$

There is a complementary series $0 \leq \nu < 1$.

2.2. **Induction step.** For a nilpotent $\mathcal{O}$, one determines all hyperplanes of reducibility of the standard module. On each hyperplane, the spherical parameter is parametrized by an orbit $\mathcal{O}' > \mathcal{O}$, so one knows which such parameters are unitary. Keep only the regions bounded by unitary walls; for these show they are unitarily induced from unitary parameters of some subgroup.

Example a.

$\widetilde{\mathbf{A}}_{\mathbf{2}}$:

Infinitesimal character $(\nu_1 + \frac{\nu_2}{2}, 1 + \frac{\nu_2}{2}, \frac{\nu_2}{2}, -1 + \frac{\nu_2}{2})$, $\nu_1 \geq \nu_2 \geq 0$.

Reducibility lines: $2\nu_1 + \nu_2 = 1$, $\nu_1 + 2\nu_2 = 1$, $\nu_1 - \nu_2 = 1$ from $A_1 + \widetilde{A}_2$, $\nu_1 + \nu_2 = 1$, $\nu_1 = 1$ and $\nu_2 = 1$ from $C_3(a_1)$, $\nu_1 + \nu_2 = 3$, $\nu_1 = 3$ and $\nu_2 = 3$ from C_3.

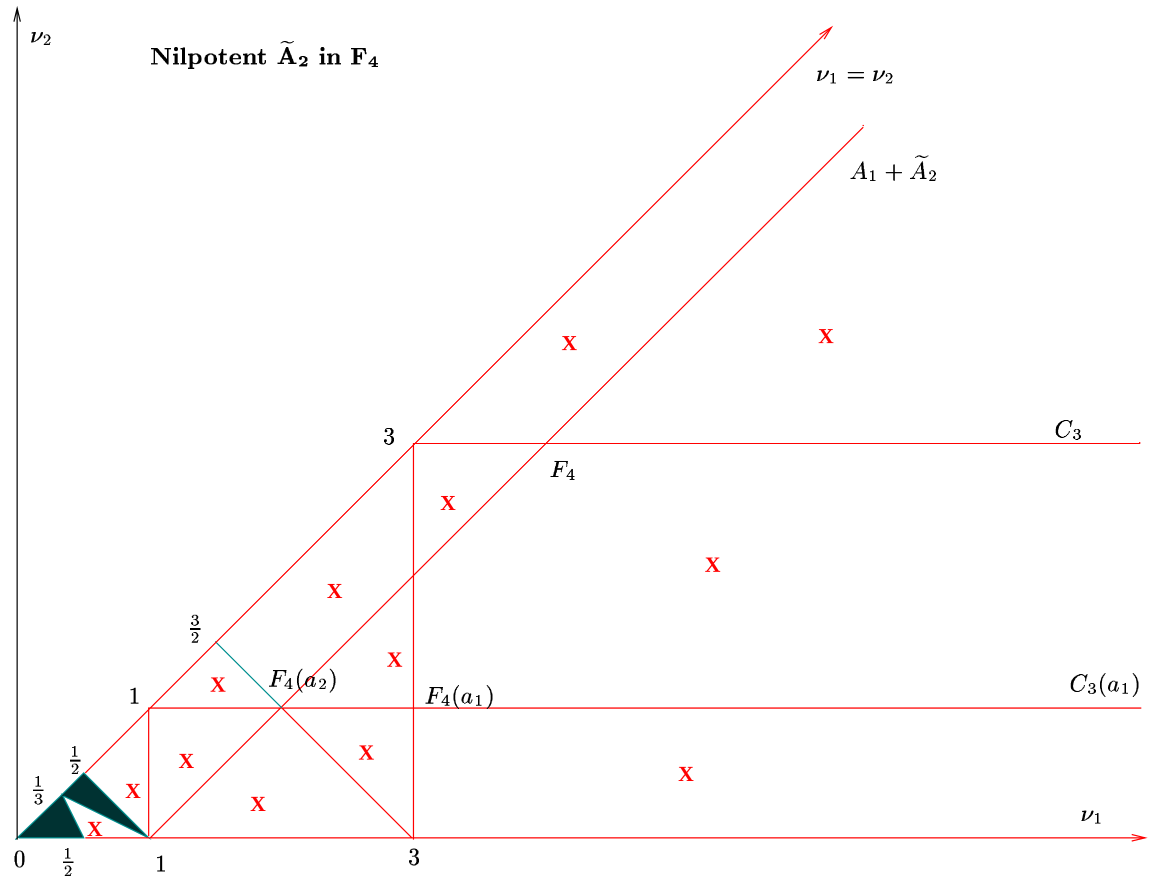
The complementary series is $\{0 \leq \nu_2 \leq \nu_1 < \frac{1-\nu_2}{2}\}$ and $\{1 - 2\nu_2 < \nu_1 < 1 - \nu_2\}$.

On the line $\nu_1 = \nu_2$, for $0 \leq \nu_2 < \frac{1}{2}$, the parameter is unitarily induced irreducible from a complementary series associated to the nilpotent (33) in C_3.

As seen from the picture, the spherical complementary series is identical with the one of the centralizer G_2.

Example b.

$\mathbf{A_1} + \widetilde{\mathbf{A}}_{\mathbf{1}}$:



Infinitesimal character $(\nu_1, \frac{1}{2} + \nu_2, -\frac{1}{2} + \nu_2, \frac{1}{2})$, $\nu_1 \geq 0$, $\nu_2 \geq 0$.

Reducibility lines: $\nu_1 = \frac{1}{2}$ from A_2, $\nu_1 = \frac{3}{2}$ from B_2, $\nu_1 = \frac{5}{2}$ from B_3, $\nu_2 = 1$ from $C_3(a_1)$, $\nu_2 = 2$ from C_3, $\nu_1 - 2\nu_2 = -\frac{3}{2}$, $\nu_1 + 2\nu_2 = \frac{3}{2}$ and $\nu_1 - 2\nu_2 = \frac{3}{2}$ from $\widetilde{A}_1 + A_2$, $\nu_1 - \nu_2 = -\frac{3}{2}$, $\nu_1 + \nu_2 = \frac{3}{2}$ and $\nu_1 - \nu_2 = \frac{3}{2}$ from $A_1 + \widetilde{A}_2$.

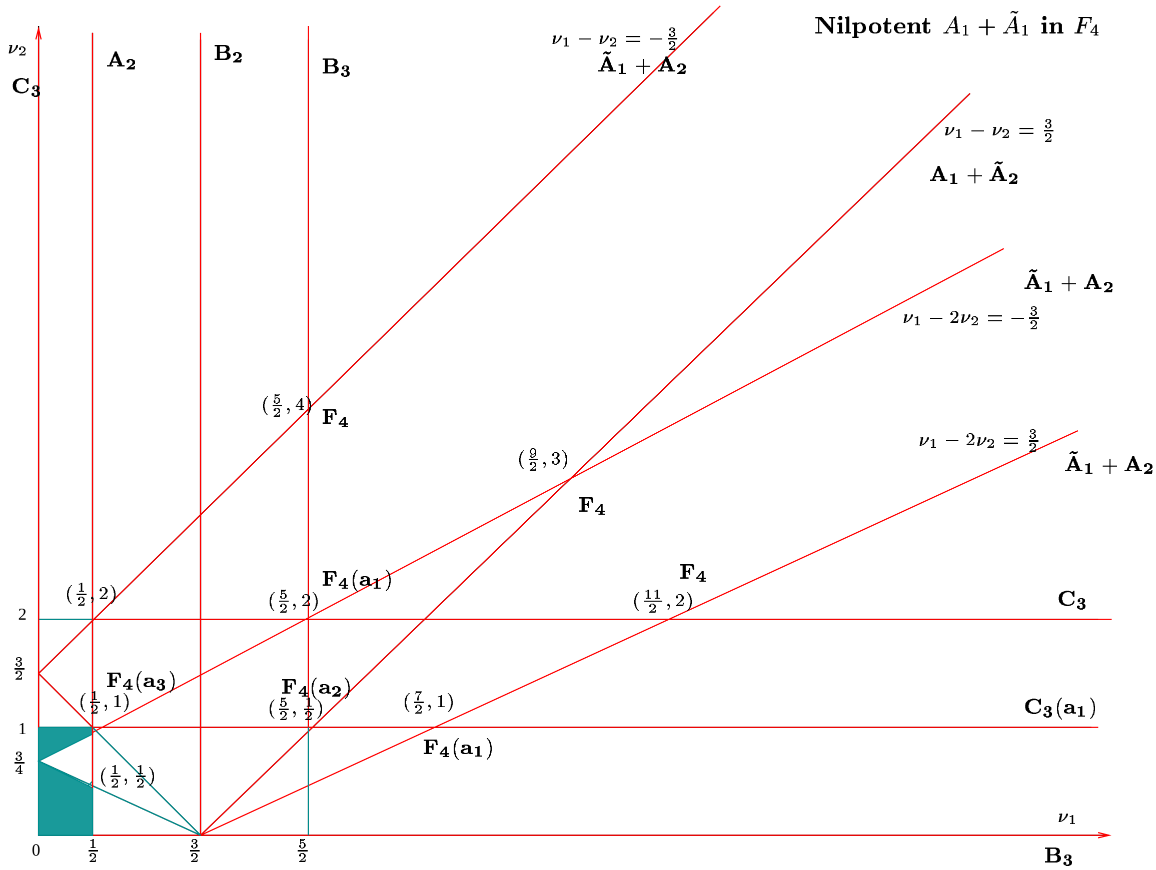
The complementary series is $\{\nu_1 + 2\nu_2 < \frac{3}{2},\ \nu_1 < \frac{1}{2}\}$ and $\{2\nu_2 - \nu_1 > \frac{3}{2},\ \nu_2 < 1\}$.

On the line $\nu_1 = 0$, $0 \leq \nu_2 < \frac{3}{4}$ and $\frac{3}{4} < \nu_2 < 1$, the standard module is unitarily induced irreducible from a complementary series associated to the nilpotent (222) in C_3.

This is an interesting example: the complementary series does not match that of the centralizer $A_1 + A_1$.

2.3. **Generic case.**

Theorem 2.2. *Assume the graded Hecke algebra is of type F_4.*



(1) *The spherical complementary series associated to a nilpotent orbit $\mathcal{O}$ coincides with that attached to the trivial orbit in its centralizer, except in the case $\mathcal{O}=A_1+\widetilde{A}_1$. In this one case, the complementary series is strictly included in the one for the centralizer.*

(2) *The complementary series associated to the trivial nilpotent with dominant infinitesimal character $(\nu_1,\nu_2,\nu_3,\nu_4)$, $\nu_1\ge\nu_2\ge\nu_3\ge\nu_4\ge 0$, $\nu_1-\nu_2-\nu_3-\nu_4\ge 0$ are:*
  (a) $\{\nu_1<\frac{1}{2}\}$;
  (b) $\{\nu_1+\nu_2+\nu_3+\nu_4>1,\ \nu_1+\nu_2+\nu_3-\nu_4<1\}$.

The first observation is that any region bounded by a wall on which a short root is 1 is *not* unitary. On any such wall, there is a factor coming from $\widetilde{A}_1$ which can't be unitary at all points since the complementary series for $\widetilde{A}_1$ is two-dimensional.

Therefore, one can look only at regions bounded by long roots and finds the hyperplanes on which the parameter can be unitary. Then, in each region determined by such hyperplanes, the parameter can be deformed to a wall where a simple root equals to zero and the parameter is unitarily induced from B_3 or C_3.

One can reformulate the classification of the spherical dual in an analogous way to the results for classical groups of Barbasch-Moy:

Theorem 2.3. *A spherical parameter for F_4 is unitary if and only if it is a complementary series from a unitarily induced module from the Iwahori-Matsumoto dual of a tempered representation tensored with a GL-complementary series.*

3. Unitary dual

The method is to look at the lowest K-types for each nilpotent orbit $\mathcal{O}$ and compare the intertwining operator on them to the intertwining operator on lowest K-types of nilpotents $\mathcal{O}'$ immediately above $\mathcal{O}$ in the partial order.

This is very advantageous in some cases, for example in the case of maximal parabolics.

3.1. **Maximal parabolics.** For this special case, the following argument (Barbasch-Moy) applies : for the modules parametrized by such a nilpotent orbit $\mathcal{O}$ and lowest K-type μ, the next bigger nilpotent $\mathcal{O}'$ has the property that a factor with lowest K-type μ' attached to $\mathcal{O}'$ appears at the first point of reducibility. Beyond this point, μ and μ' stay in the same factor and they have opposite signatures at ∞. Two such K-types have opposite signatures at ∞ if and only if their respective lowest harmonic degrees have different parity.

Example:

$\mathbf{B_3}$: The infinitesimal character is $(\frac{3}{2}+\nu, -\frac{3}{2}+\nu, \frac{3}{2}, \frac{1}{2})$, Centralizer A_1, LKT 8_2.

The first reducibility point is at $\nu = 1$, where there are factors with LKT 9_4 and 2_2 coming from $F_4(a_2)$. For $\nu > 1$, these K-types will stay in the same factor with 8_2. 8_2 and 9_4, or 8_2 and 2_2, have opposite signs at ∞, ruling out $\nu > 1$.

Moreover, when normalized by the value on the LKT, the intertwining operators give

$$2_2 : \frac{1-\nu}{1+\nu} \text{ and } 9_4 : \frac{1-\nu}{1+\nu}$$

3.2. **Matching of intertwining operators.**
Example a.
$\widetilde{\mathbf{A}}_{\mathbf{2}}$: The infinitesimal character is $(\nu_1 + \frac{\nu_2}{2}, 1 + \frac{\nu_2}{2}, \frac{\nu_2}{2}, -1 + \frac{\nu_2}{2})$ with $0 \leq \nu_2 \leq \nu_1$, centralizer G_2, LKT 8_1.

The intertwining operator decomposes into a product of intertwining operator for subgroups and maximal parabolics: $(\bar{s}_1 \circ \bar{s}_2)^3$, where $\bar{s}_1$ is the operator corresponding to $\mathrm{Ind}_{\widetilde{A}_2}^{A_1+\widetilde{A}_2}$ and $\bar{s}_2$ to $\mathrm{Ind}_{\widetilde{A}_2}^{C_3}$.

One can match the calculations with those in the centralizer G_2.
8_1 matches 1_1 in G_2
6_1 matches 1_2 in G_2
16_1 matches 2_1 in G_2
6_2 matches 1_3 in G_2

Example b.

$\mathbf{A_1} + \widetilde{\mathbf{A}}_\mathbf{1}$: The infinitesimal character is $(\nu_1, \frac{1}{2}+\nu_2, -\frac{1}{2}+\nu_2, \frac{1}{2})$, centralizer $A_1 + A_1$, LKT 9_1.

One tries to match the dual with that of $A_1 + A_1$. The intertwining operator gives:
9_1: $+1$
8_3: $\frac{\frac{1}{2}-\nu_1}{\frac{1}{2}+\nu_1}$
8_1: $\frac{1-\nu_2}{1+\nu_2}$

This implies that the unitary dual in included in $0 \le \nu_2 \le 1$, $0 \le \nu_1 \le \frac{1}{2}$. However, there are two lines that cut through this region: $\nu_1 + 2\nu_2 = \frac{3}{2}$ and $-\nu_1 + 2\nu_2 = \frac{3}{2}$ and one also needs the operator on 4_4 to finish this case.

Example c.

$\mathbf{A_1}$: The infinitesimal character is $(\nu_1, \nu_2, \nu_3, \frac{1}{2})$ with $0 \le \nu_3 \le \nu_2 \le \nu_1$, Centralizer C_3, LKT 2_3.

The K-types that match the intertwining operator in C_3 are:
2_3 with 3×0
4_2 with 0×3
8_1 with 0×12
9_1 with 1×2
4_3 with 0×1^3

One cannot match 12×0 and 2×1 which are relevant for C_3. Although the generic unitary parameters associated to this nilpotent are the same as for C_3, there are other unitary parameters which do not correspond to C_3.

4. The other exceptional groups

The same methods can be applied to study the unitary dual for the other exceptional groups. There are examples in E_7 and E_8 of spherical complementary series which do not match those of the corresponding centralizer. However they occur very rarely, e.g. only one such example in E_7, and it is unclear how to predict them.

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