“Honest” Arthur Packets *

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Peter talked about a procedure to attach to a complex even nilpotent orbit $\mathcal{O}^\vee$ for $G^\vee$ a set of "special unipotent representations"; these are unions of the form

$$\bigcup_{\mathcal{O}_1^\vee} \prod(\mathcal{O}_1^\vee).$$

Here $\prod(\mathcal{O}_1^\vee)$ is a set of representations attached to a real form $\mathcal{O}_1^\vee$ of $\mathcal{O}^\vee$; this orbit $\mathcal{O}_1^\vee$ is a nilpotent $K^\vee$ orbit on $(\mathfrak{g}^\vee/\mathfrak{k}^\vee)^*$ for a “real form” $K^\vee$ of $G^\vee$. These sets $\prod(\mathcal{O}_1^\vee)$ are what Jeff calls “honest” Arthur packets.

Question: How to compute these subsets $\prod(\mathcal{O}_1^\vee)$?

This is an aspect of the more general problem: How to compute the associated variety of a Harish-Chandra module?

Back to the G side...

Atlas point of view: Choose an inner class of real forms

$$G^\Gamma = G \rtimes \Gamma.$$

We have fixed $G \supset B \supset T$ (Γ-stable). A “strong real form” of G is an element x in the nonidentity coset such that $x^2 \in Z(G)$. This gives us $K = Cent_G(x)$.

Start with a complex nilpotent orbit $\mathcal{O}$ for G. We want to describe all real forms (for all K's in the inner class). One way to do this:

Jacobson-Morozov:

$$\Psi : SL(2) \longrightarrow G$$

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$$d\Psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{O}$$

We can line things up so that

$$\Psi(\text{diagonal elements}) \subseteq T \quad \text{and } \Psi \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq B.$$

If G^Ψ is the centralizer of $im(\Psi)$ in G, let

$$G^\Gamma_\Psi = G^\Psi \cup \left\{ z \in G^\Gamma \backslash G : Ad(z) \text{ acts on } im(\Psi) \text{ by inverse transpose} \right\};$$

i.e.,

$$Ad(z)\left(\Psi(y)\right) = \Psi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \right).$$

Lemma 1 *The second component is non-empty $\Longleftrightarrow$ $Cent(im\Psi)$ meets the second component of G^Γ $\Longleftrightarrow$ the Dynkin diagram of $\mathcal{O}$ is stable by the diagram automorphism corresponding to G^Γ.*

Theorem 2 (Kostant-Rallis) *There is a 1-1 correspondence between real forms of $\mathcal{O}$ (i.e., G-conjugacy classes of pairs $(x, \mathcal{O}_1)$, where x is a strong real form, and $\mathcal{O}_1$ a nilpotent K_x-orbit on $(\mathfrak{g}/\mathfrak{k}_x)^*$) and G^Ψ orbits on $\left\{x \text{ in the second component of } G^\Gamma_\Psi : x^2 \in Z(G)\right\}$.*

Corollary 3 (Reformulation) *There is a bijection*

$$\{\text{pairs } (x, \mathcal{O}_1)\} / G\text{-conj.} \longleftrightarrow \left\{ y \in Cent(\Psi) \text{ in } G^\Gamma \backslash G : y^2 \in \Psi \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot Z(G) \right\}.$$

The twist was chosen to get a nicer condition. The Cartan involution is now given by $y \cdot \Psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

0.1 Example: $G = Sp(2n, \mathbb{C})$

Consider the split inner class $G^\Gamma = G \times \Gamma$, and an orbit given by a partition into even parts

$$\mathcal{O} \longleftrightarrow 2n = (2m_1)^{a_1} + ... + (2m_r)^{a_r},\ m_1 > m_2 > ... > m_r > 0.$$

This corresponds to

$$\Psi : SL(2) \longrightarrow Sp(2n)$$

given by

$$\bigoplus_{i=1}^{r} \left(a_i \text{ copies of the } m_i\text{-dimensional representation of } SL(2)\right).$$

We have

$$G^{\Psi} = O(a_1, \mathbb{C}) \times O(a_2, \mathbb{C}) \times ... \times O(a_r, \mathbb{C})$$

(this is explained in Collingwood/McGovern). The component group is

$$A(\mathcal{O}) = (\mathbb{Z}/2\mathbb{Z})^r, \text{ since } O(a_i, \mathbb{C})/\text{ident.comp.} = \mathbb{Z}/2\mathbb{Z}.$$

We have

$$\Psi \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \in Sp(2n)$$

and

$$(G \rtimes \Gamma)^{\Psi} = Cent(\Psi) \text{ in } G^{\Gamma} \text{ (without the twist).}$$

The corollary says that real forms of $\mathcal{O}$ are in bijection with

$\{O(a_1) \times ... \times O(a_r) \text{ conj. classes of elts } z \in [O(a_1) \times ... \times O(a_r)] \cdot (1, \sigma) \text{ s.t. } z^2 = \pm 1\}$

(We may ignore the factor $(1, \sigma)$.)

Case 1: $z^2 = 1$; the eigenvalues are ± 1. Choose $p_1, ..., p_r$, such that $0 \leq p_i \leq a_i$ (giving the number of -1 eigenvalues in $O(a_i)$). These r-tuples correspond to real forms of $\mathcal{O}$ in $Sp(2n, \mathbb{R})$. This is because these give

$$x = \Psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z, \text{ so } x^2 = \Psi \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1,$$

and the split real form $Sp(2n, \mathbb{R})$ corresponds to $x^2 = -1$.

Case 2: $z^2 = -1$.

Lemma 4 *-1 is a square in $O(a)$ $\iff$ a is even; in this case the square root is one conjugacy class.*

If all a_i are even then we get one more real form

$$x \longleftrightarrow Sp\left(\frac{n}{2}, \frac{n}{2}\right) \supseteq GL\left(\frac{n}{2}, \mathbb{H}\right);$$

this is characterized by:

the nilpotent orbit over $\mathbb{R}$ meets $GL\left(\frac{n}{2}, \mathbb{H}\right)$; this corresponds to the partition

$$\frac{n}{2} = m_1 \frac{a_1}{2} + ... + m_r \frac{a_r}{2}.$$

Note: In the equal rank case, the real form of $\mathcal{O}$ is given by this element $z \in G^{\Psi}$ satisfying $z^2 = \Psi \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

0.2 Recall Peter/Dan's Example F_4

Self-dual orbit $\mathcal{O}$:

$$G^\Psi \simeq S_4 \ \ \text{(the identity component is trivial);}$$
$$\Psi\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \in Z(F_4) \ \ \text{(since } \mathcal{O} \text{ is even).}$$

Note: $Z(F_4) = \{1\}$.

Kostant/Rallis says: real forms correspond to elements of order 1 or 2 in S_4 (modulo conjugacy). These are

$$1 \rightsquigarrow S_4$$
$$(12) \rightsquigarrow S_2 \times S_2$$
$$(12)(34) \rightsquigarrow D_4$$

0.3 Computing associated varieties

Fix a special nilpotent orbit $\mathcal{O}$ for G. Because of the things Birne/Steve/Alfred explained, we can identify cells with

$$AV(\text{annihilators for cell}) \ = \overline{\mathcal{O}}.$$

List the real forms of $\mathcal{O}$: $\mathcal{O}_1$,...,$\mathcal{O}_r$

0.3.1 Aside:

If $\mathcal{O}$ is even (corresponding to some parabolic Q), use Peter's list:

- list the minimal kgb elements "in each $W(L)$ orbit"
- list the ones corresponding to a θ-stable Q
- list the corresponding block elements X_1,..., X_s

This is easy to do.

Peter's procedure then says to check whether $G \cdot (\mathfrak{u} \cap \mathfrak{p}) = G \cdot \mathfrak{u}$.

This is hard to compute!

Instead: Check whether $cell(X_i)$ is attached to $\mathcal{O}$. This is equivalent, but now computable! We get "yes" for a list $X_{i_1}, ..., X_{i_r}$. Conclusion:

$$av(cell(X_{i_j})) = \mathcal{O}_j$$

Real forms of $\mathcal{O}$ are in $1-1$ correspondence with nice cells for X_{i_j}.

0.3.2 Back to the general case ($\mathcal{O}$ not necessarily even)

Suppose X is irreducible, in a cell corresponding to $\mathcal{O}$. The module X has an associated cycle

$$ac(X) = \sum m_i(X)\mathcal{O}_i$$

where the $m_i(X)$ are nonnegative integers, and the $\mathcal{O}_i$ are "real forms" of $\mathcal{O}$. By definition,

$$av(X) = \{\mathcal{O}_i : m_i(X) \neq 0\}\,.$$

Computing honest Arthur packets is equivalent to computing $av(X)$ when $\mathcal{O}$ is even nilpotent.

We would like to know $ac(X)$ ...What are the difficulties?

First problem: $m_i(X)$ is not constant on translation families.

Given X, we have $Ann(X)$ (a two-sided ideal in $\mathcal{U}(\mathfrak{g})$), which has rank $rk(Ann(X))$ (the Goldierank of the annihilator).

Joseph: As X varies in a translation family, $rk(Ann(X))$ is a homogeneous polynomial on the translation parameters. The degree of this polynomial is the number of positive roots minus $\frac{1}{2}\dim(\mathcal{O})$.

Proposition 5 *The numbers $m_i(X)$ are given by*

$$m_i(X) = c_i(X)rk(Ann(X)),$$

for some rational numbers $c_i(X)$ which are constant on translation families.

This suggests to try to compute the numbers $c_i(X)$.

A natural question: Is $c_i(X)$ constant on cells?

Guess: Probably not. We need to modify the question to make the answer "yes".

atlas gives character formulas (functions on CSG's) which allow to compute something like $c_i(X)$ (not quite). It is a horrible computation. We may look at how to clear this up...

Another problem: No one knows how to compute $rk(Ann(X))$ - just how to compute the polynomials up to a constant.

0.4 Question/Desideratum

(Think on the dual side $G^\vee$; for simplicity of notation we make the statements for G.) Choose $G \supset K$ and a block $\mathcal{B}$ at regular integral infinitesimal character.

$$\mathbb{Z}\mathcal{B}(\text{virtual rep'ns}) \stackrel{\sim}{\rightsquigarrow} \boxed{\begin{array}{c}\text{virtual characters}\\ (\text{distributions on } G(\mathbb{R}))\end{array}} \supseteq \boxed{\begin{array}{c}\text{characters vanishing}\\ \text{near } 1\end{array}} \simeq \mathbb{Z}\mathcal{B}_{van}$$

Recall that

$\mathbb{Z}\mathcal{B}_{van} = \text{span}\left\{I(\gamma) - I(\gamma') : I(\gamma), I(\gamma') \text{ stand. chars supp. on same } K\text{-orbit on } G/B\right\}.$

There is a W-action on $\mathbb{Z}\mathcal{B}$, and the sublattice $\mathbb{Z}\mathcal{B}_{van}$ is W-invariant. Consider

$$\mathbb{Z}\mathcal{B}/\mathbb{Z}\mathcal{B}_{van}$$

which carries a W-action and is dual to stable characters on the other side.

Question: Relate this to the cell filtration of $\mathbb{Z}\mathcal{B}$ (as W-representations).

Beilinson-Bernstein:
$\mathbb{Z}\mathcal{B}$ = Grothendieck group of some category of K-equivt. $\mathcal{D}$-modules on G/B. We have a the characteristic cycle map

$$ch : \mathbb{Z}\mathcal{B} \longrightarrow \begin{array}{c}\mathbb{Z}\text{-span of conormal}\\ \text{bundles of } K\text{-orbits}\end{array}$$

Proposition 6 *Ker ch* $= \mathbb{Z}\mathcal{B}_{van}$.

Remark 7 *W acts naturally on the right-hand side. Peter: Outside of type A, we do NOT get the KL W-graphs.*

What does this have to do with honest Arthur packets?

Fix a K-orbit on G/B. This gives $KB_x \simeq K/K \cap B_x$ and the conormal bundle

$$K \times_{K \cap B_x} (\mathfrak{g}/\mathfrak{k} + \mathfrak{b}_x)^* \subseteq T^*(G/B).$$

We have the moment map (Grothendieck-Springer resolution)

$$\mu : T^*(G/B) \longrightarrow \mathcal{N}$$

(here $\mathcal{N}$= the nilpotent elements in $\mathfrak{g}^*$) such that

$$\mu|_{\substack{\text{conormal}\\ \text{to } K\text{-orbits}}} \longrightarrow \mathcal{N} \cap (\mathfrak{g}/\mathfrak{k})^* \text{ (the nilpotent cone in } \mathfrak{p}).$$

Proposition 8

$$\begin{aligned}\mu\left(K \times_{K\cap B_x} \left(\mathfrak{g}/\mathfrak{k} + \mathfrak{b}_x\right)^*\right) &= K \cdot \left(\mathfrak{g}/\mathfrak{k} + \mathfrak{b}_x\right)^* \\ &= \textit{the closure of a single } K\textit{-orbit } \mathcal{O}_x \subseteq \mathcal{N} \cap \left(\mathfrak{g}/\mathfrak{k}\right)^* .\end{aligned}$$

Moreover, we get all K-orbit closures (special and non-special) this way.

See P. Trapa: *Leading term cycles of Harish-Chandra modules and partial orders on components of the Springer fiber,* Compositio Mathematica 2007.

This says: To compute $av(X)$, it would be enough to compute

- $ch(X)$ —– which is REALLY HARD, and
- μ —- which is just hard.

This should tell you how many elements are in cells related to a given nilpotent, modulo $\mathbb{Z}\mathcal{B}_{van}$.