NILPOTENT ORBITS ASSOCIATED TO COXETER CELLS

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Abstract. The goal of this paper is to is to describe algorithms for identifying the special nilpotent orbit attached to a cell in terms of descent sets appearing in the cell. In principle, these algorithms could be implemented as a package of the *Atlas of Lie groups and representations* software developed by Fokko du Cloux and Marc van Leeuwen[1].

1. Introduction

Let $\mathfrak{g}$ be a complex reductive Lie group with adjoint group G and Weyl group $\mathcal{W}$. This paper describes a simple algorithm by which one can read off the complex nilpotent orbit associated with a cell representation of $\mathcal{W}$—provided that $\mathcal{W}$ is of classical type.

Specifically, given any representation of $\mathcal{W}$, we define the *τ-signature* of V to be the set of all conjugacy classes of parabolic subgroups $\mathcal{P} \subseteq \mathcal{W}$ such that $V|_{\mathcal{P}}$ contains a copy of the sign representation of $\mathcal{P}$. We show that, for $\mathcal{W}$ of classical type, the irreducible representations of $\mathcal{W}$ are determined by their τ-signatures[1], and we give a simple algorithm by which one can use the τ-signature to find the partition or partition-pair indexing a given irreducible representation.

The τ-signature of a cell representation $\mathcal{C}$ coincides with that of its unique special subrepresentation, and also with the collection of all parabolics $\mathcal{P}$ such that the simple roots of $\mathcal{P}$ are contained in some τ-invariant of the cell. Combining this with the Springer correspondence, we obtain a simple method which computes the nilpotent orbit associated with the cell directly from the τ-invariants.

More generally, our algorithm for identifying an irreducible from its τ-signature implies an algorithm for calculating the full isotypic decomposition of any cell representation: implicit in our algorithm is a linear ordering on irreducible $\mathcal{W}$-modules which refines containment ordering on τ-signatures. Since one knows the sign multiplicities of all parabolics from the τ-invariants of the cell, one can identify the highest component of the cell, subtract its sign multiplicities from that of the cell as a whole, and proceed by recursion to find the full isotypic decomposition. None of this requires calculation of the character table of $\mathcal{W}$ or of matrices for the $\mathcal{W}$-action on the cell, so it is significantly less computationally intensive than the obvious approach via character theory.

Key words and phrases.

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[1]Note that this is false in exceptional type: already for $\mathfrak{g} = \mathfrak{g}_2$ we have two irreducible $\mathcal{W}$-modules sharing the same τ-signature. It is still true that *special* irreducibles are determined by their τ-signatures.

In section 2, we describe an identification algorithm for type A, which is due essentially to Young. In sections 3 and 4 we generalize this algorithm to other classical Weyl groups, and in section 5 we give examples using the output of the *Atlas of Lie groups and representations* software.

2. Type A

Let $\mathfrak{g} = \mathfrak{gl}_n\mathbb{C}$ the set of $n \times n$ matrices. Then $\mathcal{W}$ is the symmetric group $\mathcal{S}_n$ and irreducible representations of $\mathcal{W}$ are parametrized by partitions of n. The partition $[n]$ corresponds to the trivial representation while $[1, 1, \ldots 1, 1]$ is associated with the sign representation. Our main objective is to prove that a representation $\mathcal{R}$ of $\mathcal{S}_n$ is determined by the collection of Levi factors for which it contains sign representations. Here is a concrete example of the kind of problems that we aim to solve:

Suppose we are given an irreducible representation of $\mathcal{S}_5$ that admits sign representations for parabolic subgroups of $\mathcal{S}_3 \times \mathcal{S}_1 \times S_1$ and $\mathcal{S}_2 \times \mathcal{S}_2 \times S_1$ only. Determine that representation.

The first step is to understand branching laws for irreducible representations of $\mathcal{S}_n$ when restricted to subgroups of the form

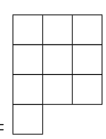
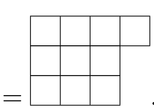
$$H = \mathcal{S}_{p_1} \times \mathcal{S}_{p_2} \times \cdots \times \mathcal{S}_{p_k} \qquad \text{with } \sum_{i=1}^{k} p_i = n$$

More precisely we have the following theorem:

Theorem 2.1. *Maintaining the above notations, let F be partition of n and V_n^F the irreducible representation of S_n associated with F via Schur duality. The multiplicity of the sign representation in V_n^F restricted to H is the number of semi-standard tableaux of shape $F^\perp$ with content $\lambda = (p_1, p_2, \ldots, p_k, 0, 0 \ldots)$. This is the Kostka number $K_{F^\perp,\lambda}$.*

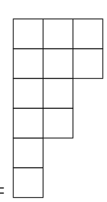
Perhaps it would be useful to give a few examples before the proof.

Let $H = \mathcal{S}_5 \times \mathcal{S}_3 \times \mathcal{S}_2 \subseteq \mathcal{S}_{10}$. Suppose we take F and $F^\perp$ to be

$F =$  $\mathrm{F}^\perp =$  .

We see that the sign representation does not occur because it is impossible for $F^\perp$ to have content $(5, 3, 2)$.

On the other hand if we had

$F =$  $\mathrm{F}^\perp = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array}$

one would see that the desired multiplicity is one for there is only one way to satisfy the above theorem.

First we consider the case where $p+q=n$ and $H=\mathcal{S}_p\times\mathcal{S}_q$. Then we have the following theorem:

Theorem 2.2. *Maintaining the above notations the restriction of V_n^F to H is $\bigoplus\limits_{D,E} c_{D,E}^F(V^D\otimes V^E)$ where $c_{D,E}^F$ is the Littlewood-Richardson coefficient corresponding to the partitions D,E,F, that is the number of Littlewood-Richardson skew tableaux on $F\setminus D$ having content E.*

Moreover we are interested in two special cases:

1. E has content $(q,0,0,0,0\dots)$ (trivial representation). In this case $c_{D,E}^F=1$ if $F\setminus D$ is a horizontal skew strip or is equal to zero otherwise.

2. E has content $(1,1,1,\dots,1,1,1,0,0,\dots)$ (sign representation). In this case $c_{D,E}^F=1$ if $F\setminus D$ is a vertical skew strip or is equal to zero otherwise.

The last two statements are consequences of the Pieri rule. See Fulton's book [3]. We wish to compute $c_{D,E}^F$ when

$$D=\begin{array}{|c|}\hline \\\hline \\\hline \\\hline \\\hline \\\hline \\\hline \\\hline \\\hline \\\hline\end{array}\text{ with depth } p \text{ and } E=\begin{array}{|c|}\hline \\\hline \\\hline \\\hline \\\hline \\\hline \\\hline\end{array}\text{ with depth } q.$$

¿From the previous theorem $c_{D,E}^F=1$ if and only if F is obtained from D by adding q boxes in a vertical skew strip. It is customary to label the p boxes in D with the digit 1 and the added q boxes with the digit 2. Here is an example:

$$F=\begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline 1&2\\\hline 1&2\\\hline 1\\\cline{1-1} 1\\\cline{1-1} 1\\\cline{1-1} 1\\\cline{1-1} 1\\\cline{1-1} 2\\\cline{1-1} 2\\\cline{1-1}\end{array}\qquad F^\perp=\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&2&2\\\hline 2&2&2&2\\\cline{1-4}\end{array}$$

Hence $c_{D,E}^F=1$ exactly when $F^\perp$ has content $(p,q,0,0,\dots)$.

As another example we choose $p=3$ and $q=2$.

One finds that the only represenations of $\mathcal{S}_5$ containing a sign representation of $\mathcal{S}_3\times\mathcal{S}_2$ are the one parametrized by the following partitions:

$$[2,2,1] = \ydiagram{2,2,1} \qquad [2,1,1,1] = \ydiagram{2,1,1,1} \qquad [1,1,1,1,1] = \ydiagram{1,1,1,1,1}$$

which correspond to the following transposes with content (3,2):

$$[2,2,1]^{\perp} = \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \end{ytableau} \qquad [2,1,1,1]^{\perp} = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 \end{ytableau} \qquad [1,1,1,1,1]^{\perp} = \begin{ytableau} 1 & 1 & 1 & 2 & 2 \end{ytableau}$$

Next we give a proof for Theorem 2.1.

Proof. This will be done by induction. The above discussion shows that the theorem is true for $k=2$. Hence

$$V_n^F|_{\mathcal{S}_{p_1+\dots p_{k-1}} \times \mathcal{S}_{p_k}} = \bigoplus c_{D,E}^F V_{p_1\dots p_{k-1}}^D \otimes V_{p_k}^E.$$

By induction we have

$$V_n^F|_{\mathcal{S}_{p_1} \times \dots \times \mathcal{S}_{p_{k-1}} \times \mathcal{S}_{p_k}} = \bigoplus c_{D,E}^F [c_{\emptyset,D_1}^{E_{K_1}} c_{E_1,D_2}^{E_2} \dots c_{E_{k-2},D_{k-1}}^{E_{k-1}} V_{p_1}^{D_1} \otimes \dots \otimes V_{p_{k-1}}^{D_{k-1}}] \otimes V_{p_k}^E.$$

Here $E_{k-1} = D$. Relabel E with D_k to obtain

$$V_n^F|_{\mathcal{S}_{p_1} \times \dots \times \mathcal{S}_{p_{k-1}} \times \mathcal{S}_{p_k}} = \bigoplus c_{\emptyset,D_1}^{E_{K_1}} c_{E_1,D_2}^{E_2} \dots c_{E_{k-2},D_{k-1}}^{E_{k-1}} c_{E_{k-1},D_k}^F V_{p_1}^{D_1} \dots V_{p_{k-1}}^{D_{k-1}} \otimes V_{p_k}^{D_k}.$$

In particular we are looking for sign representations:

$$D_i = \ydiagram{1,1,1,1,1,1,1} \qquad \text{with depth } p_i$$

which has non-zero multiplicity if and only if $E_1 = D_1$ and E_i is obtained form E_{i-1} by adding p_i boxes in a vertical skew strip. More precisely, starting with :

$$E_1 = D_1 = \begin{ytableau} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{ytableau}$$

there are $p_2 + 1$ possibilities of getting tableaux of the form:

$$E_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline 1 & 2 \\ \hline 1 & 2 \\ \hline 1 \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

To obtained a correct E_3 we only need to add 3's as follows:

$$E_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 \\ \cline{1-2} 1 & 3 \\ \cline{1-2} 1 & 3 \\ \cline{1-2} 1 \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

adding p_3 boxes to the previous tableau.

One sees clearly that all the information will be available in E_k labeled accordingly. In other words, we can generated all the possible young tableaux by counting all boxes labeled appropriately with numbers from 1 to k.

The transpose of E_k is a is a semi-standard tableau of shape $F^\perp$ and content $(p_1, p_2, \ldots, p_k, 0, 0 \ldots)$.

Conversely if one starts with a semi-standard tableaux of shape $F^\perp$ and content $(p_1, p_2, \ldots, p_k, 0, 0 \ldots)$, then one can reverse the process to obtain a sequence of diagrams $E_1, \ldots, E_k$ satisfying the above condition. $\square$

Example 2.3. Let $k = n = 4$. Then $p_1 = p_2 = p_3 = p_4 = 1$. We can compute the multiplicity of sign representation in each case using the previous theorem as follows:

$$F = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline \end{array} \qquad F^\perp = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \qquad \text{multiplicity } = 1$$

$$F = \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \end{array} \qquad F^\perp = \{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \} \qquad \text{multiplicity } = 3$$

$$F = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array} \qquad F^{\perp} = \{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array}\} \qquad \text{multiplicity } = 3$$

$$F = \begin{array}{|c|c}\hline \ & \multicolumn{1}{c|}{\ } \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array} \qquad F^{\perp} = \{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}\} \qquad \text{multiplicity } = 3$$

$$F = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \qquad F^{\perp} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array} \qquad \text{multiplicity } = 1$$

2.1. **Algorithm.** The Levi factors in type A are all of the form:

$$L_{\Pi} = \mathcal{S}_{\pi_1} \times \cdots \times \mathcal{S}_{\pi_{k-1}} \times \mathcal{S}_{\pi_k} \text{ with } \sum_i \pi_i = n$$

Let D be partinion of n and $V(D)$ the irreducible representation of $\mathcal{S}_n$ corresponding to D. We shall denote by $m(\Pi, D)$ the multiplicity of $sgn(L_\pi)$ in $V(D)$. Then

$$m(\Pi, D) = K_{D^{\perp}\Pi} \text{ the number of semi-standard Young tableaux of Shape } D^{\perp}$$
$$\text{and contents } \Pi = (\pi_1, \dots, \pi_k.0, 0, \dots)$$

Denote by Ł$(V) = \{\Pi | m(\Pi, D) \neq 0\}$. Ł$(V)$ is called the *Levi set of* V, it is the collection of all Levi sugroups of $\mathcal{W}_n$ admittiing sign representation on V. Given Ł(V) we would like to recover the partition D. This is done using the following algorithm which computes the rows of $D^{\perp}$. Observe that L_{Π} is in Ł(V) if and only if there exists a semi-standard tableau of shape D and content Π.

1. To compute the first row find:
$$[D^{\perp}]_1 = \{\Pi = (\max \pi_1 = d_1, \dots, \dots) : \Pi \in \text{Ł}(V)\}$$
2. To compute the second row find:
$$[D^{\perp}]_2 = \{\Pi = (d_1, d_2 = \max \pi_2, \dots, \dots) : \Pi \in [D^{\perp}]_1\}$$

$\vdots$

$\vdots$

$\vdots$

i. To compute ith row find:
$$[D^{\perp}]_i = \{\Pi = (d_1, \dots, d_i = \max \pi_i, \dots, \dots) : \Pi \in [D^{\perp}]_{i-1}\} \ \vdots$$

$\vdots$

k. To compute last row find:

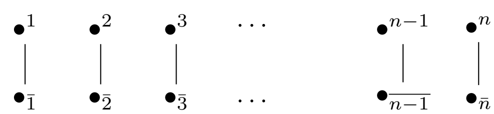
$$[D^{\perp}]_i = \{\Pi = (d_1, \ldots, \ldots d_k = \max \pi_k) : \Pi \in [D^{\perp}]_{k-1}\} \vdots$$

It is clear that the algorithm will terminate and that the orbit desired is the one associate with the partition $D = (d_1, d_2, \ldots,\ d_k)$.

We shall now turn our attention to the types B and C.

3. Types B and C

First, we describe a useful combinatorial representational of the Weyl groups. Consider the following graph:



The automorphism group, $\mathcal{W}_n$, of this graph permutes the vertices while keeping edge incidence unchanged. In other words it permutes edges and vertices on the same edge. The subgroup of $\mathcal{W}_n$ that stabilizes the edges is of course isomorphic to $\mathcal{S}_n$. Moreover the kernel of this action on the edges is a normal subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Finally, we see that $\mathcal{S}_n \cap (\mathbb{Z}/2\mathbb{Z})^n = e$, the identity element in $\mathcal{W}_n$. Since $\mathcal{W}_n \subseteq \mathcal{S}_n \times (\mathbb{Z}/2\mathbb{Z})^n$, we conclude that

$$\mathcal{W}_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathcal{S}_n.$$

Hence $\mathcal{W}_n$ is isomorphic to the Weyl groups of types B and C.

Our next step is to describe the irreducible representations of $\mathcal{W}_n$. We shall use *Proposition 25* in Serre [5]. From now on we let $\mathcal{G} = \mathcal{W}_n$, $\mathcal{H} = \mathcal{S}_n$ and $\mathcal{A} = (\mathbb{Z}/2\mathbb{Z})^n$. Hence:

$$\mathcal{G} = \mathcal{A} \rtimes \mathcal{H}$$

The irreducible representations of $\mathcal{G}$ will be built from those of certain subgroups of $\mathcal{H}$ using the method of *little groups* of Wigner and Makey. Since the character table of $\mathbb{Z}/2\mathbb{Z}$ is given by :

	1	-1
χ_1	1	1
χ_{-1}	1	-1

we conclude that the irreducible characters of $\mathcal{A}$ are :

$$\chi_{\epsilon_1} \otimes \chi_{\epsilon_2} \otimes \chi_{\epsilon_3} \otimes \ldots \chi_{\epsilon_{n-1}} \otimes \chi_{\epsilon_n} \text{ with } \epsilon_i = \pm 1.$$

Since $\mathcal{A}$ is Abelian the irreducible characters of A are of degree 1 and form a group $X = \mathrm{Hom}(\mathcal{A}, \mathbb{C}^*)$. The group $\mathcal{G}$ acts on X by

$$s\chi(a) = \chi(s^{-1}as) \text{ for } s \in \mathcal{G}, \chi \in X, a \in A.$$

Observe that $\mathcal{A}$ acts trivially on X. So we should try to understand the action of $\mathcal{H}$ on X.

Lemma 3.1. *$\mathcal{H}$ acts on X by permuting the ϵ_i's.*

Proof. □

Keeping in mind that $\chi_1 \otimes \chi_{-1} \neq \chi_{-1} \otimes \chi_1$ we wish to describe the orbits of $\mathcal{H}$ on X. We see that every character of $\mathcal{A}$ is conjugate under the action of $\mathcal{H}$ to a unique character of the form

$$\underbrace{\chi_{-1} \otimes \chi_{-1} \otimes \chi_{-1}}_{i} \otimes \cdots \underbrace{\chi_1 \otimes \chi_1 \otimes \chi_1}_{n-i}.$$

This is a system of representatives for the orbits. For each χ_i let $\mathcal{H}_i$ be the centralizer of χ_i in $\mathcal{H}$, that is the set of elements $h \in \mathcal{H}$ such that $h(\chi_i) = \chi_i$. Observe that

$$\mathcal{H}_i \simeq \mathcal{S}_i \times \mathcal{S}_{n-i}.$$

And we define $\mathcal{G}_i = \mathcal{A} \rtimes \mathcal{H}_i \simeq \mathcal{W}_i \times \mathcal{W}_{n-i}$ as the corresponding subgroup of $\mathcal{G}$. Now we extend χ_i to $\mathcal{G}_i$ by setting

$$\chi_i(ah) = \chi_i(a) \text{ for } a \in \mathcal{A}, h \in \mathcal{H}_i$$

Since $h(\chi_i) = \chi_i$ for all $h \in \mathcal{H}_i$ we conclude that χ is a character of G_i. In other words the extended χ_i only looks at edge flips from the i-component $\mathcal{W}_i$ of $\mathcal{G}$.

Let ρ be an irreducible representation of H_i then

$$\rho = \rho_i^D \otimes \rho_{n-i}^E$$

where ρ_i^D is the Specht module $\mathcal{S}_i$ indexed by the partition D. Composing ρ with the canonical projection $\mathcal{G}_i \to H_i$ we obtain $\tilde{\rho}$, an irreducible representation of $\mathcal{G}_i$. Since χ_i is of degree 1, $\chi_i \otimes \tilde{\rho}$ is an irreducible representation of $\mathcal{G}_i$ of degree equal to that of ρ.

Define $\Theta_{i,\rho} = Ind_{\mathcal{G}_i}^{\mathcal{G}} \chi_i \otimes \tilde{\rho} = Ind_{\mathcal{W}_i \times \mathcal{W}_{n-i}}^{\mathcal{W}_n} \chi_i \otimes \tilde{\rho}$. Then

Proposition 3.2. Maintaining the above notations,
(a) $\Theta_{i,\rho}$ is irreducible.
(b) if $\Theta_{i,\rho} \simeq \Theta_{i',\rho'}$ then $i = i'$ and $\rho \simeq \rho'$.
(c) Every irreducible representation of $\mathcal{G}$ is isomorphic to one of the $\Theta_{i,\rho}$.

Proof. See Serre [5] Proposition 25. □

In order to develop a combinatorial theory it is more appropriate to use the parametrization in terms of the partitions D and E since they contain all the needed information to specify ρ and i.

Our next step is to describe a branching law for $\Theta_{i,\rho}$ when restricted to $\mathcal{W}_L$, the Weyl group of a Levi subalgebra $L \subseteq \mathfrak{g}$, the Lie algebra of G. Of course $\mathcal{W}_L$ is a subgroup of $\mathcal{G}$.

Recall that in this case all Levi subalgebras can be represented by a Dykin diagram of the form:

where the roots represented by filled circles are not part of the subsystem that generates the Levi subalgebra. Consequently the Weyl group $\mathcal{W}_L$ is ether of the form:

$$\mathcal{W}_L = \mathcal{S}_{\pi_1} \times \mathcal{S}_{\pi_2} \times \cdots \times \mathcal{S}_{\pi_{j-1}} \times \mathcal{W}_{\pi_j}$$

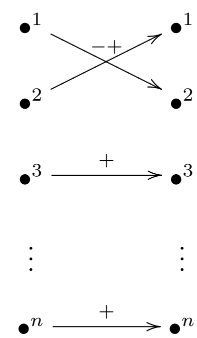
or

$$\mathcal{W}_L = \mathcal{S}_{\pi_1} \times \mathcal{S}_{\pi_2} \times \cdots \times \mathcal{S}_{\pi_{j-1}} \times \mathcal{S}_{\pi_j}$$

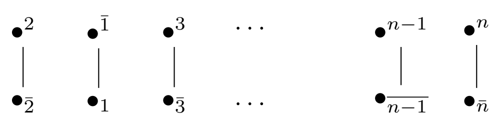
with $\sum_{i=1}^{j} \pi_i = n$ and $\mathcal{W}_{\pi_j} = (\mathbb{Z}/2\mathbb{Z})^{\pi_j} \rtimes \mathcal{S}_{\pi_j}$.

Let $\mathcal{G} = \mathcal{W}_n$, $H = \mathcal{W}_i \times \mathcal{W}_{n-i}$ and $K = \mathcal{W}_L$. Maintaining the above notations, we would like to understand $\mathrm{Res}_K \mathrm{Ind}_H^{\mathcal{G}} \chi_i \otimes \tilde{\rho}$.

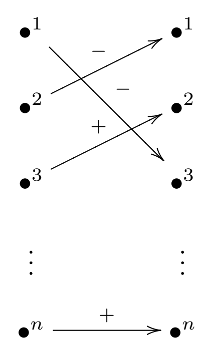
In order to proceed we will need a set of representatives S, of the double cosets $H\backslash\mathcal{G}/K$. We describe an element of $\mathcal{G}$ as a signed directed bipartite graph as follows:



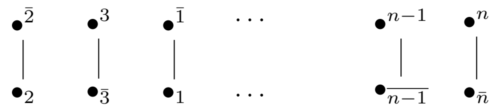
If an edge (x, y) is labeled with a minus sign then x and $\bar{x}$ are switched before the permutation occurs otherwise no switching takes place. For example under the action of the above permutation the original graph will be transformed into:



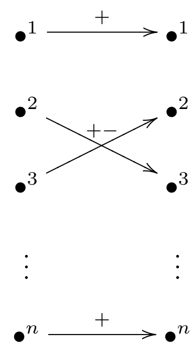
If we acted by



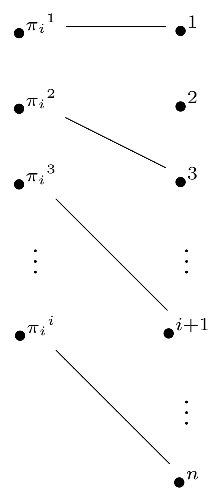
then the image would be:



if we denote by g_1 the first signed permutation and let g_2 be the second one then g_2g_1 is given by:



The domain of each permutation is partitioned into j subsets : $E_{\pi_1}, E_{\pi_2}, \dots E_{\pi_j}$ such that the edges in $|E_{\pi_i}| = \pi_i$ and the edges in E_{π_i} are permuted by $\mathcal{S}_{\pi_i}$ or $\mathcal{W}_{\pi_i}$ while its domain is partitioned into two subsets E_k with $|E_k| = i$ and E_{n-k} with $|E_{n-k}| = n-k$ such that the edges in E_k are permuted by $\mathcal{W}_k$ and those in E_{n-k} are permuted by $\mathcal{W}_{n-k}$. Given a permutation we can make all the sign + by multiplying on the left by element of H using the appropriate permutations and flipping back if necessary. So that for each E_{π_i} we obtain the following diagram, neglecting the signs for now on:



Here is a rough but useful interpretation of the above picture: for each $E_{\mathfrak{p}_i}$ we arranged to have no crossed edges and to have the first q_i edges permuted to edges in E_k while the remaining $\pi_i - q_i$ are permuted to edges in E_{n-k}. So at the end we see that every permutation is conjugate to a permutation of the class describe above. Within each double coset there is only one permutation that can be used to do this. Therefore the normal form obtained in this fashion is unique.

To specify each normal form we only need to know how many arrows from each E_{π_i} go to E_k. More precisely the double cosets are indexed by j-tuples

$$Q = \{\mathfrak{q}_1, q_2, \dots, q_j\} \text{ with } 0 \leq q_i \leq \pi_i \text{ and } \sum_{i=1}^{j} q_i = |E_k|.$$

Let S_Q be the set of representatives defined by Q. For $s \in S_Q$, let $H_s = sHs^{-1} \cap K$, which is a subgroup of K. If ρ is any finite dimensional representation of H then we can define a representation of ρ^s of H_s by setting

$$\rho^s(x) = \rho(s^{-1}xs), \text{ for } x \in H_s.$$

Since H_s is a subgroup of K the induced representation $\mathrm{Ind}_{H_s}^{K} \rho^s$ is well defined. The following proposition gives the desired decomposition.

Proposition 3.3. $\mathrm{Res}_K \mathrm{Ind}_H^{\mathcal{G}} \chi_i \otimes \tilde{\rho} = \bigoplus_{s \in S_Q} \mathrm{Ind}_{H_s}^{K} \rho^s.$

Proof. See Serre [5] Proposition 22. □

We wish to compute the multiplicity of the signed representation which is in fact the dimension of

$$\mathrm{Hom}(sign(W_L), \bigoplus_{s \in S_Q} \mathrm{Ind}_{H_s}^{W_L} \rho^s).$$

Since Hom is an additive functor this is equal to

$$\bigoplus_{s \in S_Q} \mathrm{Hom}(sign(W_L), \mathrm{Ind}_{H_s}^{W_L} \rho^s).$$

But Ind and Res are adjoint functors. So the preceding expression is equivalent to

$$\bigoplus_{s \in S_Q} \mathrm{Hom}(\mathrm{Res}_{H_s}^{W_L} sign(W_L), \rho^s) = \bigoplus_{s \in S_Q} \mathrm{Hom}(sign(H_s), \rho^s).$$

Therefore the multiplicity that we are looking for is the sum over Q of the multiplicitie of $sign(H_s$ in ρ^s for $s \in Q$. So we need to understand the group H_s and the representation ρ^s for all $s \in Q$.

Observe that $H = \mathcal{W}_i \times \mathcal{W}_{n-i} = \mathrm{Stab}\{e_1, e_2, \dots e_i\}$, in other words it stabilizes the edges that mapped to edges in E_i. Moreover

$$s^{-1}Hs = \mathrm{Stab}\{s^{-1}e_1, s^{-1}e_2, \dots s^{-1}e_i\}.$$

Consequently in each E_{π_i} H_s permutes the edges that mapped to E_i between themselves and does the same thing to those that are mapped to E_{n-i}. Hence we must have either

$$H_s = \mathcal{S}_{q_1} \times \mathcal{S}_{\pi_1 - q_1} \times \mathcal{S}_{q_2} \times \mathcal{S}_{\pi_2 - q_2} \times \cdots \times \mathcal{S}_{q_j} \times \mathcal{S}_{\pi_j - q_j}$$

or

$$H_s = \mathcal{S}_{q_1} \times \mathcal{S}_{\pi_1 - q_1} \times \mathcal{S}_{q_2} \times \mathcal{S}_{\pi_2 - q_2} \times \cdots \times \mathcal{W}_{q_j} \times \mathcal{W}_{\pi_j - q_j}.$$

To obtain ρ^s, we conjugate H_s back to H. As before we must have either

$$sH_s s^{-1} = \mathcal{S}_{q_1} \times \mathcal{S}_{q_2} \times \cdots \times \mathcal{S}_{q_j} \times \mathcal{S}_{\pi_1 - q_1} \times \mathcal{S}_{\pi_2 - q_2} \times \cdots \times \mathcal{S}_{\pi_j - q_j}$$

or

$$sH_s s^{-1} = \mathcal{S}_{q_1} \times \mathcal{S}_{q_2} \times \cdots \times \mathcal{W}_{q_j} \times \mathcal{S}_{\pi_1 - q_1} \times \mathcal{S}_{\pi_2 - q_2} \times \cdots \times \mathcal{W}_{\pi_j - q_j}$$

Conjugation preserves parity. Hence:

$$\mathrm{mult}(sign(H_s), \rho^s) = \mathrm{mult}(sign(sH_s s^{-1}), \rho).$$

We may take $\rho = \chi_i \otimes \rho_i^D \otimes \rho_{n-i}^E$ as a representation of $\mathcal{W}_i \times \mathcal{W}_{n-i}$. Hence we have two possible cases:

• The multiplicity of $sign(\mathcal{S}_{q_1} \times \mathcal{S}_{q_2} \times \cdots \times \mathcal{S}_{q_j})$ in ρ_i^D is the # of semi-standard tableaux of shape $D^\perp$ and content $(q_1, q_2, \ldots, q_j)$ that is $K_{D^\perp Q}$. The multiplicity of $sign(\mathcal{S}_{\pi_1 - q_1} \times \mathcal{S}_{\pi_2 - q_2} \times \cdots \times \mathcal{S}_{\pi_j - q_j})$ in ρ_{n-i}^E is $K_{E^\perp(\Pi - Q)}$. The factor χ_i is one dimensional. It restricts to the trivial representation. Here $\Pi = \{\pi_1, \pi_2, \ldots, \pi_j\}$.

• Since $\mathcal{W}_{q_j} = (\mathbb{Z}/2\mathbb{Z})^{q_j} \rtimes \mathcal{S}_{q_j}$ and $\mathcal{W}_{\pi_j - q_j} = (\mathbb{Z}/2\mathbb{Z})^{\pi_j - q_j} \rtimes \mathcal{S}_{\pi_j - q_j}$ we only have to deal with the extra information coming from χ_i on $(\mathbb{Z}/2\mathbb{Z})^{q_j}$ and $(\mathbb{Z}/2\mathbb{Z})^{\pi_j - q_j} \rtimes \mathcal{S}_{\pi_j - q_j}$. We want it to be a sign representation on both. The only way that could happen is when $\pi_j = q_j$. Hence we obtain the same formula as above except that it will be taken over Q with $\pi_j = q_j$.

These two cases lead to the following formula:

$$\mathrm{mult}(sign(H_s), \rho^s) = K_{D^\perp Q} K_{E^\perp(\Pi - Q)}.$$

The desired multiplicity is obtained is obtained by summing over the $H \backslash \mathcal{G} / K$ double cosets.

3.1. **Algorithm.** Up to conjugacy the Levi factors in type B or C are all of the form:

$$L_\Pi = \mathcal{S}_{\pi_1} \times \cdots \times \mathcal{S}_{\pi_{k-1}} \times \mathcal{S}_{\pi_k} \text{ with } \sum_i \pi_i = n$$

or

$$L_\Pi^k = \mathcal{S}_{\pi_1} \times \cdots \times \mathcal{S}_{\pi_{k-1}} \times \mathcal{W}_{\pi_k} \text{ with } \sum_i \pi_i = n$$

Define $L_\Pi^i = \mathcal{S}_{\pi_1} \times \cdots \times \mathcal{W}_{\pi_i} \times \cdots \times \mathcal{S}_{\pi_{k-1}} \times \mathcal{S}_{\pi_k}$. Observe that $L_\Pi^i \simeq L_{\Pi'}^k$ for some Π' and the two subgroups admit the same branching multiplicities.

Let D and E be Young diagrams such that $|D| + |E| = n$. Denote by V the irreducible of $\mathcal{W}_n$ corresponding to (D, E). Finally, let $m(\Pi, D, E)$ be the multiplicity of the $sgn(L_\Pi)$ in V and $m(\Pi, i, D, E)$ the multiplicity of the $sgn(L_\Pi^i)$ in V. Then from the above theorem we have:

$$m(\Pi, D, E) = \sum_{\lambda+\mu=\pi} K_{D_\lambda} K_{E_\mu)}.$$

This is the number of ordered pairs of semi-standard Young tableaux such that the first has shape D, the second has shape E and the sum of the two contents is Π. Moreover L_Π is in $L(V)$ if and only if there exists a pair of semi-standard tableaux with shapes D and E and content sum Π. Similary

$$m(\Pi, i, D, E) = \sum_{\lambda+\mu=\Pi, \lambda_i=0} K_{D_\lambda} K_{E_\mu}.$$

L^i_Π is in $L(V)$ if and only if there exists an ordered pair of semi-standard tableaux with shapes D and E and content sum Π with all the i's appearing in the second tableau.

Given Ł(V) we would like to recover the pair of partitions (D, E). This is done using the following algorithm which computes the rows of (D, E). This algorithm is similar to the type A algorithm. Let d_i and e_i denote the ith rows $D^\perp$ and $E^\perp$ respectively. The algorithm proceeds as follows:

$$\begin{aligned}
d_1 + e_1 &= \max\ \{\pi_1 | L_\pi \in L(V)\} \\
e_1 &= \max\ \{\pi_1 | L^1_\pi \in L(V)\} \\
d_2 + e_2 &= \max\ \{\pi_2 | L_{(d_1+e_1,\pi_2,\dots)} \in L(V)\} \\
e_2 &= \max\ \{\pi_2 | L^2_{(d_1+e_1,\pi_2,\dots)} \in L(V)\} \\
&\vdots \qquad \vdots \\
d_i + e_i &= \max\ \{\pi_i | L_{(d_1+e_1,\dots,d_{i-1}+e_{i-1},\pi_i,\dots)} \in L(V)\} \\
e_i &= \max\ \{\pi_2 | L^i_{(d_1+e_1,\dots,d_{i-1}+e_{i-1},\pi_i,\dots)} \in L(V)\}
\end{aligned}$$

4. Type D

If $\mathfrak{g}$ is of type D then its Weyl group $\mathcal{W}_n$ is a subgroup of of a Weyl group of type B or C of order 2. Furthermore the set of irreducible representations of $\mathcal{W}_n$ is in one to one correspondence with the set of unordered pairs $\{D, E\}$ of partitions such that $|D|+|E| = n$ except that each pair $\{D, D\}$ corresponds to two irreducible representations denoted by $\{D, D\}_+$ and $\{D, D\}_-$.

We say that a subgroup $\mathcal{L}$ of $\mathcal{W}_n$ a Levi subgroup if $\mathcal{L}$ is the Weyl group of a Levi subalgebra of $\mathfrak{g}$. Let α_{n-1} and α_n be the last two non connected nodes in the Dynkin diagram of type D_n with $n \geq 4$. We have three kinds of Levi subgroups:

- If α_n is not selected then $\mathcal{L} = L_\Pi = \mathcal{S}_{\pi_1} \times \cdots \times \mathcal{S}_{\pi_{k-1}} \times \mathcal{S}_{\pi_k}$ with $\sum_i \pi_i = n$.
- If α_n and α_{n-1} are both selected then $\mathcal{L} = L^k_\Pi = \mathcal{S}_{\pi_1} \times \cdots \times \mathcal{S}_{\pi_{k-1}} \times \mathcal{W}_{\pi_k} \times \mathcal{S}_{\pi_{k+1}} \times \cdots \times \mathcal{S}_{\pi_j}$ with $\sum_i \pi_i = n$..
- Otherwise $\mathcal{L} = \tilde{L}_\Pi$: image of L_Π under the outer automorphism.

Using the Wigner-Makey little groups method described above, we have the following theorem:

Theorem 4.1. *Maintaining the above notations and letting $m(\tilde{\Pi}, \{D, E\})$ be the multiplicity of the $sgn(\tilde{L}_\Pi)$ in V :*

If $D \neq E$ then

$$m(\Pi, \{D, E\}) = m(\tilde{\Pi}, \{D, E\}) = \sum_{\lambda+\mu=\Pi} K_{D_\lambda} K_{E_\mu}$$

and

$$m(\Pi, k, \{D, E\}) = \sum_{\lambda+\mu=\Pi, \lambda_k \mu_k = 0} K_{D_\lambda} K_{E_\mu}.$$

Moreover,

$$m(\Pi, \{D, D\}_+) = 1/2 \sum_{\lambda+\mu=\Pi, \lambda\neq\mu} K_{D_\lambda} K_{D_\mu} + \sum_{2\lambda=\Pi} K^2_{D_\lambda},$$

$$m(\tilde{\Pi}, \{D, D\}_+) = 1/2 \sum_{\lambda+\mu=\Pi, \lambda\neq\mu} K_{D_\lambda} K_{D_\mu},$$

$$m(\Pi, k, \{D, D\}_\pm) = 1/2 \sum_{\lambda+\mu=\Pi, \lambda\neq\mu, \lambda_k\mu_k=0} K_{D_\lambda} K_{D_\mu},$$

$$m(\Pi, \{D, D\}_-) = 1/2 \sum_{\lambda+\mu=\Pi, \lambda\neq\mu} K_{D_\lambda} K_{D_\mu},$$

$$m(\tilde{\Pi}, \{D, D\}_-) = 1/2 \sum_{\lambda+\mu=\Pi, \lambda\neq\mu} K_{D_\lambda} K_{D_\mu} + \sum_{2\lambda=\Pi} K^2_{D_\lambda},$$

4.1. **Algorithm.** Again we define $L(V)$ to be the collection of all Levi subgroups of G admitting *sgn* representations on V. We can now give an algorithm for recovering the pair of Young tableaux $\{D, E\}$.

1. Compute

$$\begin{aligned}
\alpha_1 =&\ \max\ \{\pi_1 | L_\pi \in L(V)\} \\
\alpha_2 =&\ \max\ \{\pi_2 | L_{(\alpha_1, \pi_2, \dots)} \in L(V)\} \\
\vdots &\qquad \vdots \\
\alpha_i =&\ \max\ \{\pi_i | L_{(\alpha_1, \dots, \alpha_{i-1}, \pi_i, \dots)} \in L(V)\}
\end{aligned}$$

Stop when $\sum \alpha_k = n$.

2. Compute

$$\begin{aligned}
\beta_1 =&\ \max\ \{\pi_1 | \tilde{L}_\pi \in L(V)\} \\
\beta_2 =&\ \max\ \{\pi_2 | \tilde{L}_{(\beta_1, \pi_2, \dots)} \in L(V)\} \\
\vdots &\qquad \vdots \\
\beta_i =&\ \max\ \{\pi_i | \tilde{L}_{(\beta_1, \dots, \beta_{i-1}, \pi_i, \dots)} \in L(V)\}
\end{aligned}$$

Stop when $\sum \beta_k = n$.

3. If $\alpha_i \neq \beta_i$ for some i then either all α_i's are even and one (or more) of the β_i's is (are) odd, or the other way around.

If all the α_i's are even then set $\alpha_i := \alpha_i/2$.

Stop: the representation V is $\{D, D\}_+$ for $D = \{\alpha_1, \alpha_2 \ldots \alpha_i\}$.

If all the β_i's are even then set $\beta_i := \beta_i/2$.

Stop: the representation V is $\{D, D\}_-$ for $D = \{\beta_1, \beta_2, \ldots \beta_i\}$.

4. If $\alpha_i = \beta_i$ for all i then

compute:

$$
\begin{aligned}
d_1 &= \max \{\pi_1 | L^1_\pi \in L(V)\} \\
d_2 &= \max \{\pi_2 | L^2_{(\alpha_1, \pi_2, \ldots)} \in L(V)\} \\
&\vdots \qquad \vdots \\
d_i &= \max \{\pi_i | L^i_{(\alpha_1, \ldots, \alpha_{i-1}, \pi_i, \ldots)} \in L(V)\}
\end{aligned}
$$

Stop when when some $d_k > \alpha_k/2$.(This guaranteed to happen as a consequence of the above multiplicity theorem.)

compute:

$$
\begin{aligned}
f_{k+1} &= \max \{\pi_{k+1} | L^k_{(\alpha_1, \ldots, \alpha_{i-1}, d_k, \pi_{k+1} \ldots)} \in L(V)\} \\
d_{k+1} &= f_{k+1} + d_k - \alpha_k \\
f_{k+2} &= \max \{\pi_{k+2} | L^{k+1}_{(\alpha_1, \ldots, \alpha_{k-1}, d_k, f_{k+1}, \pi_{k+2} \ldots)} \in L(V)\} \\
d_{k+2} &= f_{k+2} + d_{k+1} - \alpha_{k+1} \\
&\vdots \qquad \vdots
\end{aligned}
$$

Stop when all d_j's are obtained and then set $e_j = \alpha_j - d_j$ for all j.

Stop: the representation V is $\{D, E\}$ for $D = \{d_1, d_2, \ldots d_j\}$ and $E = \{e_1, e_2, \ldots e_j\}$

4.2. **Sketch of a proof of correctness of the above algorithm.** Observe that if α_i is different from β_i for any i then the representation is either $\{D, D\}_+$ or $\{D, D\}_-$ because the multiplicity formulas for L_Π and $\tilde{L}_P i$ are the same for distinct diagrams. In this case let k be the number of parts of D; then using a tableau-counting argument similar to that given in types A and B above, we see that $\alpha_i = \beta_i = 2d_i$ for $1 \leq i \leq k-1$ but $\alpha_k \neq \beta_k$ since the unique tableau-pair of content sum $(2d_1, \ldots, 2d_k)$ is counted by only one of the formulas above for $m(\Pi, \{D, D\}_\pm)$ and $m(\tilde{\Pi}, \{D, D\}_\pm)$. In fact this argument shows that α_k and β_k differ by one, so that exactly one of them is even and this one is consequently equal to $2d_k$.

Similarly, if $\alpha_i = \beta_i$ for all i then their common value is $d_i + e_i$, and D and E are distinct. Let k be the least integer for which $d_k \neq e_k$. Relabeling D and E if necessary, we can assume that $d_k > e_k$ whence $d_k > \alpha_k/2$ and $d_i = e_i = \alpha_i/2$ for $i < k$. The algorithm now follows immediately by tableau-counting.

5. Examples Using the Atlas Data

We shall use the data provided by the Atlas software in order to give some examples using the algorithms described above. In all cases the actual nilpotent orbit is obtained by mapping the results given by the algorithms through the Springer correspondence. Procedures for carrying out such mappings are explicitly available in section 13.3 of [2]. Therefore we shall use them without further explanation and trust that the interested readers will consult the suggested reference for the higher rank cases. For each type we will use a cell from the Atlas output. For the reader's convenience we shall give the specific atlas commands that generate the cells.

5.1. **Type A.** Let $G = A_5$.

```
[alfred-gerard-noels-computer:/usr/local/atlas] anoel% ./atlas.exe
This is the Atlas of Reductive Lie Groups Software Package version 0.3.
Build date: Mar 16 2008 at 17:13:20.
Enter "help" if you need assistance.

empty: wcell
Lie type: A5
elements of finite order in the center of the simply connected group:
Z/6
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
(weak) real forms are:
0: sl(3,H)
1: sl(6,R)
enter your choice: 1
possible (weak) dual real forms are:
0: su(6)
1: su(5,1)
2: su(4,2)
3: su(3,3)
enter your choice: 3
```

The output consist of eighteen cells. We shall use cell number 13.

```
// cell #13:
0[70]: {1,3,5} --> 1,2,4
1[90]: {1,4} --> 0,3
2[95]: {2,5} --> 0,3
3[113]: {2,4} --> 1,2,4
4[126]: {3} --> 3
```

The data of interest are within the brackets. Each set $\{\alpha_i, \alpha_k, \dots \alpha_l\}$ consists of simple roots and therefore defines a Levi factor. The Atlas output for each cell is a collection of Levi factors for which $\mathcal{W}$, in this case $\mathcal{S}_6$, admits a sign representation. The list $\{1, 3, 5\}$ tells us that $\mathcal{W}$ admits a sign representation when restricted to the Weyl group of the Levi defined by the simple roots $\alpha_1, \alpha_3,$ and α_5 which is of type $A_1 \times A_1 \times A_1$.

The algorithm computes the first row by finding the longest string of consecutive simple roots and add 1 to it. In this case we have $\pi_1 = 2$. Next we find that $\pi_2 = 2$ and finally we end up with the list $\{1, 3, 5\}$ which gives $\pi_3 = 2$. So the nilpotent orbit is parametrized by the partition $[3, 3]$.

5.2. **Type C.** Let $G = C_3$.

```
[alfred-gerard-noels-computer:/usr/local/atlas] anoel% !!
./atlas.exe
This is the Atlas of Reductive Lie Groups Software Package version 0.3.
Build date: Mar 16 2008 at 17:13:20.
Enter "help" if you need assistance.

empty: wcell
Lie type: B3
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
(weak) real forms are:
0: so(7)
1: so(6,1)
2: so(5,2)
3: so(4,3)
enter your choice: 3
possible (weak) dual real forms are:
0: sp(3)
1: sp(2,1)
2: sp(6,R)
enter your choice: 2
Name an output file (return for stdout, ? to abandon):



// cell #10:
0[16]: {2,3} --> 1
1[23]: {1,3} --> 0,2
2[26]: {1,2} --> 1,3
3[34]: {1,3} --> 2,4
4[38]: {2,3} --> 3
```

We see that $d_1, +e_1 = 3$. This comes from the list $\{1, 2\}$. The list $\{2, 3\}$ tells us that $e_1 = 2$. So $d_1 = 1$. So the pair of partitions is $([1], [1, 1])$ which is associated to the nilpotent $[2, 2, 1, 1]$ by the Springer correspondence.

5.3. **Type C.** Let $G = D_4$.

```
[alfred-gerard-noels-computer:/usr/local/atlas] anoel% !!
./atlas.exe
This is the Atlas of Reductive Lie Groups Software Package version 0.3.
Build date: Mar 16 2008 at 17:13:20.
Enter "help" if you need assistance.
```

```
empty: wcell
Lie type: D4
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
(weak) real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 4
possible (weak) dual real forms are:
0: so(8)
1: so(6,2)
2: so*(8)[0,1]
3: so*(8)[1,0]
4: so(4,4)
enter your choice: 4
Name an output file (return for stdout, ? to abandon):


// cell #17:
0[10]: {1,3,4} --> 1,5,6,7
1[17]: {2} --> 0
2[37]: {2,4} --> 1,5,6,9
3[41]: {2,3} --> 1,5,7,9
4[43]: {1,2} --> 1,6,7,9
5[53]: {3,4} --> 2,3
6[58]: {1,4} --> 2,4
7[63]: {1,3} --> 3,4
8[81]: {1,3,4} --> 5,6,7,9
9[94]: {2} --> 8
```

The algorithm first computes the α_i's and the $\beta_i' s$ as follows:

To find α_1 we look for the longest consecutive string which does not contain roots 4. This is given by the set $\{1,2\}$. Hence we conclude that $\alpha_1 = 3$.

To find β_1 we look for the longest consecutive string which does not contain roots 3. This is given by the set $\{1,2\}$. Hence we conclude that $\beta_1 = 3$.

The next step gives us $\alpha_2 = 1$ and $\beta_2 = 1$. Since $\alpha_i = \beta_i$ for all i's, we find that $d_1 = 2$ from the set $\{\{1,3,4\},\{3,4\}\}$. Since $d_1 > \alpha_1/2$ we move to compute f_2 which turns out to be 2. So $d_2 = 2+2-3 = 1$, $e_1 = 3-2 = 1$ and $e_2 = 1-1 = 0$. We obtain the pairs of partitions $([2,1],[1])$ which corresponds to the nilpotent $[3,1,1]$.

5.4. **Some more examples in higher ranks.** Let $G = B_5$.

```
[alfred-gerard-noels-computer:/usr/local/atlas] anoel% ./atlas.exe
This is the Atlas of Reductive Lie Groups Software Package version 0.3.
Build date: Mar 16 2008 at 17:13:20.
Enter "help" if you need assistance.

empty: wcell
Lie type: B5
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
(weak) real forms are:
0: so(11)
1: so(10,1)
2: so(9,2)
3: so(8,3)
4: so(7,4)
5: so(6,5)
enter your choice: 5
possible (weak) dual real forms are:
0: sp(5)
1: sp(4,1)
2: sp(3,2)
3: sp(10,R)
enter your choice: 3
Name an output file (return for stdout, ? to abandon):


// cell #39:
0[420]: {1,2,4,5} --> 2,8,10,21,28
1[469]: {1,2,3,5} --> 5,6,13,16
2[500]: {1,3,5} --> 0,4,6,29
3[542]: {1,2,4,5} --> 7,10,18,20,21
4[552]: {1,3,4} --> 2,8,11,21,27
5[553]: {1,2,4} --> 1,8,9,10,18,21
6[592]: {2,3,5} --> 2,8
7[624]: {1,3,5} --> 3,12,13,23
8[636]: {2,4} --> 4,6,17
9[639]: {1,3,4} --> 5,15,16
10[663]: {1,2,5} --> 5,16,17,24
11[668]: {1,3,5} --> 4,14,17,23
12[691]: {1,3,4} --> 7,18,19,21,27
13[697]: {2,3,5} --> 7,18
14[708]: {1,4,5} --> 11,20,21
```

```
15[710]: {2,3,4} --> 8,9,18,22,27
16[728]: {1,3,5} --> 9,10,21,22
17[730]: {2,5} --> 8,11
18[748]: {2,4} --> 12,13,24
19[752]: {1,3,5} --> 12,24,25,29
20[759]: {2,4,5} --> 14,17,23,26
21[778]: {1,4} --> 16,26
22[786]: {2,3,5} --> 15,16,17,23,24,26,29
23[797]: {3,5} --> 20,27
24[808]: {2,5} --> 18,19
25[810]: {1,4,5} --> 19,21,28
26[818]: {2,4} --> 21,22,27
27[829]: {3,4} --> 23,26,29
28[838]: {2,4,5} --> 24,25,26,29
29[851]: {3,5} --> 27,28
```

We have $d_1 + e_1 = 4$ given by $\{1, 2, 3, 5\}$ and $e_1 = 2$ given by the set of all the sets that end with $\{4, 5\}$. We see then that $d_2 + e_2 = 1$ from the set $\{1, 2, 3, 5\}$ and $e_2 = 1$ from the set $\{4, 5\}$. It follows that the corresponding pairs of nilpotent for this representation is $([1, 1], [2, 1])$.

Let $G = D_6$.

```
[alfred-gerard-noels-computer:/usr/local/atlas] anoel% !!
./atlas.exe
This is the Atlas of Reductive Lie Groups Software Package version 0.3.
Build date: Mar 16 2008 at 17:13:20.
Enter "help" if you need assistance.

empty: wcell
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): s
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 5
possible (weak) dual real forms are:
0: so(12)
1: so(10,2)
```

```
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 5
Name an output file (return for stdout, ? to abandon):

// cell #57:
0[1100]: {1,3,4,6} --> 3,4,17,18,30
1[1104]: {1,3,4,5} --> 3,5,17,19,30
2[1107]: {1,4,5,6} --> 3,6,30,44
3[1297]: {1,3,5,6} --> 0,1,2,7,12
4[1306]: {2,4,6} --> 0,7,8,20
5[1310]: {2,4,5} --> 1,7,10,20
6[1314]: {2,4,5,6} --> 2,7,12,20,40,43
7[1489]: {2,5,6} --> 3,4,5
8[1495]: {2,3,6} --> 4,16,18
9[1499]: {1,2,4,6} --> 4,14,18,30,34
10[1501]: {2,3,5} --> 5,15,19
11[1505]: {1,2,4,5} --> 5,14,19,30,33
12[1507]: {3,5,6} --> 6,17
13[1660]: {2,3,5,6} --> 7,8,10,12,20,21,25,28,40
14[1663]: {1,2,5,6} --> 7,9,11,21,41,42
15[1666]: {2,3,4} --> 10,20,22,25
16[1668]: {2,3,4} --> 8,20,23,28
17[1670]: {3,4} --> 12,20
18[1678]: {1,3,6} --> 8,9,23
19[1683]: {1,3,5} --> 10,11,22
20[1803]: {2,4} --> 13,17,30
21[1806]: {1,3,5,6} --> 13,14,18,19,30,35,36,44
22[1809]: {1,3,4} --> 15,19,30,33,35
23[1810]: {1,3,4} --> 16,18,30,34,36
24[1820]: {1,2,3,6} --> 18,34
25[1822]: {2,3,6} --> 15,31,35
26[1826]: {3,4,6} --> 17,31,37
27[1828]: {1,2,3,5} --> 19,33
28[1832]: {2,3,5} --> 16,32,36
29[1834]: {3,4,5} --> 17,32,37
30[1922]: {1,4} --> 20,21
31[1924]: {2,4,6} --> 20,25,26,38,40
32[1927]: {2,4,5} --> 20,28,29,39,40
33[1934]: {1,2,4} --> 22,27,41
34[1935]: {1,2,4} --> 23,24,42
35[1944]: {1,3,6} --> 22,25,38,41
36[1950]: {1,3,5} --> 23,28,39,42
37[1959]: {3,5,6} --> 26,29,40,43
38[2014]: {1,4,6} --> 30,31,35,44
39[2017]: {1,4,5} --> 30,32,36,44
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40[2020]: {2,5,6} --> 31,32,37,44
41[2031]: {1,2,6} --> 33,35
42[2038]: {1,2,5} --> 34,36
43[2054]: {4,5,6} --> 37
44[2087]: {1,5,6} --> 38,39,40
```

The algorithm first computes the α_i's and the $\beta_i's$ as follows:

To find α_1 we look for the longest consecutive string which does not contain roots #6. This is given by the set $\{\{1,2,3,5\},\{1,2,3,6\}\}$. Hence we conclude that $\alpha_1 = 4$.

To find β_1 we look for the longest consecutive string which does not contain roots #5. This is given by the set $\{\{1,2,3,5\},\{1,2,3,6\}\}$. Hence we conclude that $\beta_1 = 4$.

From $\{1,2,3,5\}$ and $\{1,2,3,6\}$ we obtain $\alpha_2 = 2$ and $\beta_2 = 2$ respectively.

Next we find $d_1 = 3$ from $\{\{4,5,6\},\{1,4,5,6\}\}$. since $d_1 > \alpha_1/2$ we proceed to compute $f_2 = 2$ from $\{1,4,5,6\}$. It follows that $d_2 = 2+3-4 = 1, e_1 = 4-3 = 1$ and $e_2 = 2-1 = 1$. Hence, this representation is $[(2,1,1),(2)]$.

References

[1] F. du Cloux, M. van Leeuwen, *The Atlas of Lie Groups and Representations*, http://www.liegroups.org/

[2] R. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters*, John Wiley & Sons, London, (1985).

[3] W. Fulton and J. Harris, *Representation Theory: A First Course*, (Graduate Texts in Mathematics / Readings in Mathematics), Spriger-Verlag, New York, (1991).

[4] G. Lusztig, *Left cells in Weyl groups*, Lecture Notes in Mathematics, **1024**, (1983).

[5] J.-P. Serre, *Linear Representations of Finite Groups*, (Graduate Texts in Mathematics / Readings in Mathematics), Spriger-Verlag, New York, (1977).

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