Stability

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Let G be a connected reductive linear algebraic group defined over $\mathbb{R}$. We denote by $G(\mathbb{R})$ its $\mathbb{R}$-points.

1 Definitions

Definition 1. *A semisimple element g in $G(\mathbb{R})$ is said to be strongly regular if the centralizer $Z_{G(\mathbb{R})}(g)$ is a Cartan subgroup.*

This is a stronger notion than that of regular elements for which only the Lie algebra $\mathfrak{z}_{G(\mathbb{R})}(g)$ is required to be a Cartan subalgebra. Let us denote by $G(\mathbb{R})_{SR}$ the set of strongly regular elements. This is an open dense subset of $G(\mathbb{R})$.

Definition 2. *Two strongly regular semisimple elements g, g' of G are called stably conjugate if there exists $h \in G(\mathbb{C})$ such that $hgh^{-1} = g'$.*

Stable conjugacy is a weaker notion than usual conjugacy. The canonical example is the rotations $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ in $SL(2,\mathbb{R})$ which are not conjugate in $SL(2,\mathbb{R})$, but are stably conjugate by the element $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ of $SL(2,\mathbb{C})$. Any stable conjugacy class is a finite disjoint union of (usual) conjugacy classes.

If π is an irreducible representation of $G(\mathbb{R})$, the character Θ_π is the distribution

$$\Theta_\pi(f) = \mathrm{tr}(\pi(f)), \quad f \in C_c^\infty(G(\mathbb{R})), \tag{1}$$

where

$$\pi(f) = \int_{G(\mathbb{R})} f(g)\pi(g)\, dg. \tag{2}$$

Clearly, the definition can be extended to any finite-length representation, and we can also consider virtual representation π. Since Θ_π is an invariant distribution on $G(\mathbb{R})$, it is determined by Harish-Chandra's theorem by its restriction to $G(\mathbb{R})_{SR}$.

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Definition 3. *A virtual representation* π *(or character* Θ_π*) is said to be stable if* $\Theta_\pi(g) = \Theta_\pi(g')$*, whenever* g *and* g' *are stably conjugate strongly regular semisimple elements.*

If $G(\mathbb{R})$ has equal rank, for every infinitesimal character λ and every central character χ, denote

$$\Psi_{\lambda,\chi} = \{\pi : \pi \text{ discrete series with infinitesimal character } \lambda \text{ and central character } \chi\}. \tag{3}$$

They form an L-packet.

Theorem 1 (Shelstad)**.** *Assume* $G(\mathbb{R})$ *has equal rank. Then*

$$\sum_{\pi \in \Psi_{\lambda,\chi}} \pi \tag{4}$$

is stable.

The definitions above make sense for any local field $\mathbb{F}$ of characteristic 0, by replacing $G(\mathbb{C})$ with $G(\overline{\mathbb{F}})$.

Theorem 2 (Waldspurger)**.** *Let* $\mathbb{F}$ *be a local field of characteristic* 0, *and let* $P = MN$ *be an* $\mathbb{F}$*-rational parabolic subgroup. For every* π_M *a stable virtual representation of* $M(\mathbb{F})$*, the parabolically induced virtual representation* $\mathrm{Ind}_{P(\mathbb{F})}^{G(\mathbb{F})}(\pi_M)$ *is stable as well.*

Over $\mathbb{R}$ more is known to be true.

Theorem 3 (Shelstad)**.** *The lattice of stable virtual representations of* $G(\mathbb{R})$ *is spanned over* $\mathbb{Z}$ *by the set*

$$\{\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_M) : \pi_M \textit{ stable combination of discrete series of the form } (4)\}, \tag{5}$$

where $P(\mathbb{R})$ *ranges over all* $G(\mathbb{R})$ *conjugacy classes of cuspidal (rational) parabolic subgroups.*

One of the points of theorem 3 is that one does not need to include limits of discrete series in the basis. For example, in $SL(2,\mathbb{R})$, the stable combination of discrete series are $\pi_k \oplus \pi_{-k}$, for infinitesimal character $k-1$, $k \in \mathbb{Z}_{\geq 2}$, while the stable combination of limits of discrete series is $\pi_1 \oplus \pi_{-1}$ at infinitesimal character 0. But the stable combination of limits of discrete series can be regarded as parabolically induced from the Borel subgroup, $Ind_{B(\mathbb{R})}^{SL(2,\mathbb{R}}(\mathsf{sgn} \otimes 1)$, i.e., the nonspherical principal series at infinitesimal character 0.

Question. It is natural to ask if theorem 3 has a counterpart for $\mathbb{F}$ a p-adic field. Of course, in this case, the combinations of discrete series (4) need to be taken with the appropriate multiplicities, since, unlike the case of $\mathbb{F} = \mathbb{R}$, the component groups parameterizing the members of an L-packet are not always abelian.

2 Stability in Atlas

We consider $\mathbb{F} = \mathbb{R}$ and the question of producing stable combinations of characters for $G = G(\mathbb{R})$. Let K be a maximal compact corresponding to the Cartan involution θ. Let W denote the (abstract) Weyl group.

For every H a θ-stable Cartan subgroup, recall that we have the notion of regular characters $\widehat{H}_\rho$ (we assume the infinitesimal character is ρ). We denote by $\pi(\gamma)$ and $\overline{\pi}(\gamma)$, the standard module and the irreducible Langlands subrepresentation, respectively, attached to (the K-conjugacy class of) γ. The block equivalence on regular characters is generated by the following relation between $\gamma^1 \in \widehat{H}^1_\rho$ and $\gamma^2 \in \widehat{H}^2_\rho$:

$$\gamma^1 \sim \gamma^2 \text{ if and only if } \overline{\pi}(\gamma^2) \text{ appears as a subquotient of } \pi(\gamma^1). \tag{6}$$

Equivalently, a block is the smallest subset of regular characters which is closed under conjugation, cross actions, and Cayley transforms.

We identify $W(\mathfrak{g}, \mathfrak{h})$ with the abstract W. The cross action $w \times \gamma \in \widehat{H}_\rho$, $w \in W$, $\gamma \in \widehat{H}_\rho$, gives a way to produce stable virtual characters. Let $\gamma \in \widehat{H}_\rho$ be fixed. Let W^{im} denote the imaginary Weyl group, and let $[\gamma]$ be the K-conjugacy class of γ.

Definition 4. *The set*

$$cp(\gamma) = \{\gamma' : \gamma' \in W^{im} \times [\gamma]\} \tag{7}$$

is called a pseudo L-packet.

Theorem 4 (Vogan)**.** *Every block $\mathcal{B}$ partitions into pseudo L-packets.*

Let $w\gamma$ denote the usual conjugation by $w \in W$. Define the cross stabilizer of γ:

$$W_1(\gamma) = \{w \in W(G, H) : w \times \gamma = w\gamma\}. \tag{8}$$

Every pseudo L-packet gives rise to a stable virtual character:

$$\sum_{w \in W^{im}/W^{im} \cap W_1(\gamma)} \pi(w \times \gamma). \tag{9}$$

In fact, these virtual characters form a basis for the lattice of stable virtual representations.

The indexing set may be given more precisely. Decomposes $H = TA$ into the compact and vector parts, and set $M = Z_G(A)$ to be the centralizer of A in G. This is a Levi subgroup. Then

$$W^{im} \cap W_1(\gamma) = W(M, H). \tag{10}$$

Example. Assume $G(\mathbb{R})$ is equal rank and one chooses $H \subset K$. A particular example in this case is when $\pi(\gamma) = \overline{\pi}(\gamma)$ is a discrete series. Then $cp(\gamma)$ is the

L-packet consisting of all discrete series with the same infinitesimal character and central character as $\pi(\gamma)$, and so (9) is the same as (4).

More generally, a virtual character of the form (9) equals an induced $\mathrm{Ind}_P^G(\sum \pi_M)$, for $P = MN$ (N is chosen so that γ is "antidominant"), where $\sum \pi_M$ is a stable L-packet sum of discrete series for M, so its stability follows from theorems 1 and 2. The identification with theorem 3 is now clear.

Example. Consider $G(\mathbb{R}) = Sp(4,\mathbb{R})$, and the large block at infinitesimal character ρ. There are 12 representations labeled $0, 1, \ldots, 11$. The block structure is as follows:

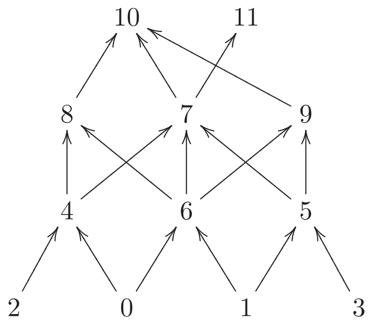
empty: type
Lie type: C2 sc s
main: realform
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
real: block
possible (weak) dual real forms are:
0: so(5)
1: so(4,1)
2: so(2,3)
enter your choice: 2
Name an output file (return for stdout, ? to abandon):

0(0,6):	0	0	[i1,i1]	1	2	(6, *)	(4, *)	
1(1,6):	0	0	[i1,i1]	0	3	(6, *)	(5, *)	
2(2,6):	0	0	[ic,i1]	2	0	(*, *)	(4, *)	
3(3,6):	0	0	[ic,i1]	3	1	(*, *)	(5, *)	
4(4,4):	1	2	[C+,r1]	8	4	(*, *)	(0, 2)	2
5(5,4):	1	2	[C+,r1]	9	5	(*, *)	(1, 3)	2
6(6,5):	1	1	[r1,C+]	6	7	(0, 1)	(*, *)	1
7(7,2):	2	1	[i2,C-]	7	6	(10,11)	(*, *)	2,1,2
8(8,3):	2	2	[C-,i1]	4	9	(*, *)	(10, *)	1,2,1
9(9,3):	2	2	[C-,i1]	5	8	(*, *)	(10, *)	1,2,1
10(10,0):	3	3	[r2,r1]	11	10	(7, *)	(8, 9)	1,2,1,2
11(10,1):	3	3	[r2,rn]	10	11	(7, *)	(*, *)	1,2,1,2

The order in the block is obtained using blockorder as in figure 11 below.

Every block element is parameterized by a pair (x, y) (the ones after the numbering in the table above). A pseudo L-packets consists of block elements with the same y:

- $\{0, 1, 2, 3\}$ (these are the discrete series);
- $\{4, 5\}$;
- $\{6\}$;



(11)

Figure 1: The large block for $Sp(4,\mathbb{R})$.

- $\{7\}$;
- $\{8,9\}$;
- $\{10\}$;
- $\{11\}$.

For a block $\mathcal{B}$, let $\check{\mathcal{B}}$ denote the dual block in the sense of Vogan. There is a pairing $\langle\ ,\ \rangle : \mathbb{Z}[\mathcal{B}] \times \mathbb{Z}[\check{\mathcal{B}}] \to \mathbb{Z}$ defined on irreducibles and extanded by linearity. More precisely, the blocks $\mathcal{B}$ and $\check{\mathcal{B}}$ have the same parameter set S, and for every $\gamma, \mu \in S$ one sets

$$\langle \overline{\pi}(\gamma), \overline{\check{\pi}}(\mu) \rangle = \epsilon_{\gamma,\mu}\delta_{\gamma,\mu}, \tag{12}$$

where $\epsilon_{\gamma,\mu} \in \{+1,-1\}$ is specified precisely. Recall that for example, the discrete series (if they exist) in $\mathcal{B}$ are dual to the principal series representations.

Vogan's duality says that

$$\langle \pi(\gamma), \check{\pi}(\mu) \rangle = \epsilon_{\gamma,\mu}\delta_{\gamma,\mu}. \tag{13}$$

An important criterion for stability is the following.

Theorem 5 (Vogan)**.** *Suppose $\pi = \sum a_i \pi_i$ is a virtual representation, where π_i are irreducible representations belonging to the same block $\mathcal{B}$. Then π is stable if and only if for every virtual representation $\check{\sigma} \in \check{\mathcal{B}}$ such that $\Theta_{\check{\sigma}}$ vanishes near zero, one has $\langle \pi, \check{\sigma} \rangle = 0$.*

In order to use this criterion, one needs to produce virtual representations whose characters vanish near zero. A basis of these characters is given by virtual differences of principal series

$$\mathrm{Ind}_P^G(\sigma_M) - \mathrm{Ind}_P^G(\sigma_M \otimes \chi), \tag{14}$$

where χ is a character if the component group M/M^0. The simplest such example is in $SL(2,\mathbb{R})$, for the minimal principal series, where $P = B$, $M \cong \mathbb{R}^\times$, and so a character vanishing near zero is $\mathrm{Ind}_{MN}^G(\mathsf{triv}) - \mathrm{Ind}_{MN}^G(\mathsf{sgn})$.

References

[ABV] J. Adams, D. Barbasch, D.A. Vogan, Jr., *Langlands classification and irreducible characters for real reductive groups*, Birhauser, 1992.

[AV] J. Adams, D.A. Vogan, Jr., *Harish-Chandra's method of descent*, Amer. J. Math., vol. 114, no. 6, 1992, 1243–1256.

[Ar] J. Arthur, *Unipotent automorphic representations: Conjectures*, Orbites unipotentes et répresentations II, Astérisque 171-172 (1989), 13–71.

[S] D. Shelstad, *L-indistinguishability for real groups*, Math. Ann. 259 (1982), 385–430.

[V] D.A. Vogan, Jr., *Irreducible characters of semisimple Lie groups IV*, Duke Math. J., vol. 49, no. 4, 1982, 943–1073.