COMPOSITION FACTORS FOR THE GRADED HECKE ALGEBRA

Let G be a connected simple complex Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra, $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ a choice of positive roots corresponding to the Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let W denote the Weyl group.

We recall the algorithm in [Lu1] and [Lu2] in a combinatorial language. The abbreviation (Lu) stands for the definitions in [Lu2], while (Co) for the combinatorics. We are concerned for now only with the case of the graded Hecke algebra with equal parameters.

1. Ingredients

We fix a semisimple element $\chi \in \mathfrak{h}$ such that χ is the middle element of a Lie triple in $\mathfrak{g}$. We can choose χ such that $w_0\chi = -\chi$. (For example, by writing χ from the weighted Dynkin diagram.) The goal is to compute the matrix of multiplicities for the modules of the Hecke algebra with central ("infinitesimal") character χ. Let υ denote an indeterminate, which in the end will be specialized to $\upsilon = 1$.

1.1. The element χ induces a grading on the Lie algebra $\mathfrak{g}$, by its ad-action:

$$\mathfrak{g}_n = \{x \in \mathfrak{g} : [\chi, x] = nx\}. \tag{1.1.1}$$

If $\mathfrak{p}$ is any subalgebra of $\mathfrak{g}$, we denote similarly $\mathfrak{p}_n = \mathfrak{g}_n \cap \mathfrak{p}$. We also define

$$\begin{aligned} &r_n(\mathcal{R}) = \{\alpha \in \mathcal{R} : \langle \alpha, \chi \rangle = n\}, \text{ for any subset of roots } \mathcal{R}, and \\ &r_n(w) = r_n(\Delta) \cap (w^{-1} \cdot \Delta^+), \text{ for any element } w \in W. \end{aligned} \tag{1.1.2}$$

1.2. Define

$$\begin{aligned} &G(\chi) = \{g \in G : Ad(g)\chi = \chi\}, \quad W(\chi) = \{w \in W : w\chi = \chi\}, \\ &\mathcal{B}(\chi) = G(\chi)\text{-orbits in } \{\mathfrak{b}' : \mathfrak{b}' \text{ Borel subalgebra with } \chi \in \mathfrak{b}'\}. \end{aligned} \tag{1.2.1}$$

The space in which the constructions take place is $\mathcal{K}(\chi)$, defined as

$$\begin{aligned} &(Lu) : \mathcal{K}(\chi) = \mathbb{Q}(\upsilon)\text{-vector space with basis } \mathcal{B}(\chi), \\ &(Co) : \mathcal{K}(\chi) = \mathbb{Q}(\upsilon)\text{-vector space with basis } W/W(\chi). \end{aligned} \tag{1.2.2}$$

The space $\mathcal{K}(\chi)$ is has an involution

$$\beta : \mathcal{K}(\chi) \to \mathcal{K}(\chi), \text{ determined by } \beta(\upsilon) = \upsilon^{-1}. \tag{1.2.3}$$

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1.3. One defines the space $(\mathcal{B}\times\mathcal{B})(\chi)$:

$$(Lu) : (\mathcal{B}\times\mathcal{B})(\chi) = G(\chi)\text{-orbits on } \{(\mathfrak{b}',\mathfrak{b}'') : \chi\in\mathfrak{b}'\cap\mathfrak{b}''\} \text{ (under the diagonal action)}, \tag{1.3.1}$$
$$(Co) : (\mathcal{B}\times\mathcal{B})(\chi) = (W\times W)/W(\chi) \text{ (where } W(\chi) \text{ is regarded as the diagonal subgroup)},$$

and a function $\tau : (\mathcal{B}\times\mathcal{B})(\chi)\to\mathbb{Z}$ as in the following. For a Borel subalgebra $\mathfrak{b}'$, let $\mathfrak{u}'$ denote the unipotent radical. Then

$$\begin{aligned}(Lu) &: \tau((\mathfrak{b}',\mathfrak{b}'')) = -\dim\frac{\mathfrak{u}'_0+\mathfrak{u}''_0}{\mathfrak{u}'_0\cap\mathfrak{u}''_0} + \dim\frac{\mathfrak{u}'_2+\mathfrak{u}''_2}{\mathfrak{u}'_2\cap\mathfrak{u}''_2},\\ (Co) &: \tau((w_1,w_2)) = \#(r_2(w_1)\vee r_2(w_2)) - \#(r_0(w_1)\vee r_0(w_2)),\end{aligned} \tag{1.3.2}$$

where $\vee$ denotes the "union minus intersection" set operator.

The space $\mathcal{K}(\chi)$ has a second involution which associates to each Borel subalgebra, the opposite Borel subalgebra. On cosets, this is induced by

$$\sigma(w) = w_0\cdot w. \tag{1.3.3}$$

Set

$$c = \#r_2(\Delta) - \#r_0(\Delta). \tag{1.3.4}$$

Lemma.
(a) The map τ in definition (Co) of (1.3.2) is well-defined, i.e.

$$\tau((w_1w,w_2w)) = \tau((w_1,w_2)),\ \textit{for any } w\in W(\chi).$$

(b) Since $w_0\chi=-\chi$, $\tau((w_1,w_2))=\tau((w_1w_0,w_2w_0))$, for every (w_1,w_2).
(c) For any $w_1,w_2\in W$,

$$\tau((w_1,w_2)) + \tau((\sigma(w_1),w_2)) = c.$$

(So it is independent of w_1,w_2).

Proof. Part (a) is immediate. For part (b), we note that $r_n(w)=r_{-n}(w_0w)$. Then part (b) follows from [Lu2] by the fact that the definition (Lu) is the same if one uses 2 or -2. Part (c) follows from the fact that $r_n(w_1)\cap r_n(w_0w_1)=\emptyset$, and $r_n(w_1)\cup r_n(w_0w_1)=r_n(\Delta)$. □

1.4. Let $pr_j : (\mathcal{B}\times\mathcal{B})(\chi)\to\mathcal{B}(\chi)$, $j=1,2$ denote the projection onto the j-th coordinate. One defines a symmetric bilinear form

$$(\ :\ ) : \mathcal{K}(\chi)\times\mathcal{K}(\chi)\to\mathbb{Q}(v)$$

by

$$(Lu):\ e_\chi^{-1}\cdot([\mathfrak{b}']:[\mathfrak{b}'']) = \sum_{\substack{\Omega\in(\mathcal{B}\times\mathcal{B})(\chi)\\ pr_1\Omega=[\mathfrak{b}'],\, pr_2\Omega=[\mathfrak{b}'']}} (-v)^{\tau(\Omega)}, \tag{1.4.1}$$

$$(Co):\ e_\chi^{-1}\cdot([w_1]:[w_2]) = \sum_{[(w',w'')]\in pr_1^{-1}([w_1])\cap pr_2^{-1}([w_2])} (-v)^{\tau((w',w''))} \tag{1.4.2}$$

$$= \sum_{w\in[w_1]}(-v)^{\tau((w,w_2))} = \sum_{w\in[w_2]}(-v)^{\tau((w_1,w))}.$$

In these formulas, $[\bullet]$ denotes the class of an element $\bullet$ in $\mathcal{B}(\chi)$, and similarly in $(\mathcal{B}\times\mathcal{B})(\chi)$.

The factor $e_\chi\in\mathbb{Q}(v)$ is a normalization factor, and it depends only on χ. If the choice is $e_\chi=(1-v^2)^{-\operatorname{rk}\mathfrak{g}}$, then we have the identity

$$\beta((\beta(\xi),\beta(\xi'))) = (-1)^{\operatorname{rk}\mathfrak{g}}(-v)^{2\operatorname{rk}\mathfrak{g}-c}(\sigma(\xi):\xi'),\ \text{ for all } \xi,\xi'\in\mathcal{K}(\chi). \tag{1.4.3}$$

In the following tables, for simplicity, we will take $e_\chi=1$.

Example $\mathbf{A_1}$. Let α denote the simple root, $\check\alpha$ the simple coroot, and s the simple reflection, and consider $\chi=\check\rho=\check\alpha$. Then $W(\chi)=W=\{1,s\}$. The bilinear form is

	1	s
1	1	$-v$
s	$-v$	1

.

In general, the bilinear form $(\ :\ )$ is (very) degenerate. Let Rad denote its radical.

Example $\mathbf{A_2}$. The simple roots are $\{\epsilon_1-\epsilon_2,\epsilon_2-\epsilon_3\}$, and let the simple reflections be denoted by s_1 and s_2. Then $W=\{1,s_1,s_2,s_1s_2,s_2s_1,s_1s_2s_1\}$. We look at $\chi=\check\rho=(2,0,-2)$. The bilinear form is

	1	$s_1s_2s_1$	s_1s_2	s_2s_1	s_1	s_2
1	1	v^2	$-v$	$-v$	$-v$	$-v$
$s_1s_2s_1$	v^2	1	$-v$	$-v$	$-v$	$-v$
s_1s_2	$-v$	$-v$	1	v^2	v^2	1
s_2s_1	$-v$	$-v$	v^2	1	1	v^2
s_1	$-v$	$-v$	v^2	1	1	v^2
s_2	$-v$	$-v$	1	v^2	v^2	1

.

In this case $\dim(Rad)=2$, and a basis is given by $\{s_1-s_2s_1,\ s_2-s_1s_2\}$.

Example $\mathbf{C_2}$, $\chi=(\mathbf{1},\mathbf{1})$. The simple roots are $\{\epsilon_1-\epsilon_2,2\epsilon_2\}$. We look at $\chi=(1,1)$, the middle nilpotent element of the nilpotent orbit (22) in $\mathfrak{sp}(4,\mathbb{C})$. Then $W(\chi)=\{1,s_1\}$, and $W/W(\chi)=\{[1],[s_2],[s_1s_2],[s_2s_1s_2]\}$. The bilinear form is

	$[1]$	$[s_2]$	$[s_1s_2]$	$[s_2s_1s_2]$
$[1]$	$1+v^{-2}$	$-v^{-1}-v$	$1+v^2$	$-v-v^3$
$[s_2]$	$-v^{-1}-v$	2	$-2v$	$1+v^2$
$[s_1s_2]$	$1+v^2$	$-2v$	2	$-v^{-1}-v$
$[s_2s_1s_2]$	$-v-v^3$	$1+v^2$	$-v^{-1}-v$	$1+v^{-2}$

.

In this example, the form is nondegenerate.

Example $\mathbf{C_2}$, $\chi = (\mathbf{1},\mathbf{0})$. In this case, χ is the middle element of the nilpotent orbit (211) in $\mathfrak{sp}(4,\mathbb{C})$. Then $W(\chi) = \{1, s_2\}$, and $W/W(\chi) = \{[1],[s_1s_2s_1],[s_2s_1],[s_1]\}$. The bilinear form is

	$[1]$	$[s_1s_2s_1]$	$[s_2s_1]$	$[s_1]$
$[1]$	$1+v^{-2}$	$-v^{-1}-v$	$-v^{-1}-v$	$1+v^{-2}$
$[s_1s_2s_1]$	$-v^{-1}-v$	$1+v^{-2}$	$1+v^{-2}$	$-v^{-1}-v$
$[s_2s_1]$	$-v^{-1}-v$	$1+v^{-2}$	$1+v^{-2}$	$-v^{-1}-v$
$[s_1]$	$1+v^{-2}$	$-v^{-1}-v$	$-v^{-1}-v$	$1+v^{-2}$

.

In this case, the radical has a basis $\{-[1]+[s_1],\ -[s_1s_2s_1]+[s_2s_1]\}$.

Questions.

(1) What is the rank of $(\ :\ )$? (The number of subquotients of the principal series at χ counted with multiplicity maybe?)
(2) Is it possible to describe a basis of the radical *Rad* a priori? (From [Lu2], *Rad* is preserved by the involution β.)

1.5. The space $\mathcal{K}(\chi)$ has an obvious involution defined on cosets as follows. If w_0 denote the longest Weyl group element, then the involution is

$$\gamma : \mathcal{K}(\chi) \to \mathcal{K}(\chi),\ \gamma([w]) = [w\cdot w_0],\ \text{for all } [w] \in W/W(\chi). \tag{1.5.1}$$

By lemma 1.3, $([w_1] : [w_2]) = (\gamma[w_1] : \gamma[w_2])$. An immediate consequence is that $\gamma(Rad) = Rad$.

1.6. Another basic ingredient of the algorithm is an *induction* map. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, such that $\chi \in \mathfrak{p}$. Let $\overline{\mathfrak{p}}$ denote the Levi component of $\mathfrak{p}$. One can define $\mathcal{K}_{\overline{\mathfrak{p}}}(\chi)$, $W_{\overline{\mathfrak{p}}}(\chi)$ *etc.* similarly to the definitions for $\mathfrak{g}$ in the previous sections. Let $proj : \mathfrak{p} \to \overline{\mathfrak{p}}$ denote the projection. The parabolic $\mathfrak{p}$ is not necessarily standard with respect to the fixed Borel $\mathfrak{b}$ (and choice of positive roots Δ^+).

The induction is defined as

$$(Lu):\ \mathrm{ind}_{\overline{\mathfrak{p}}}^{\mathfrak{g}} : \mathcal{B}_{\overline{\mathfrak{p}}}(\chi) \to \mathcal{B}(\chi),\ \mathrm{ind}_{\overline{\mathfrak{p}}}^{\mathfrak{g}}(\mathfrak{b}') = proj^{-1}(\mathfrak{b}'). \tag{1.6.1}$$

To define the same map combinatorially, let $\mathfrak{u}_p$ be the unipotent radical of $\mathfrak{p}$. Then the roots in $\Delta(\mathfrak{u}_p)$ are a subset of $\Delta(\mathfrak{g})$, but not necessarily of Δ^+. Let w_p be a Weyl group of minimal length such that $w_p(\Delta(\mathfrak{u}_p)) \subset \Delta^+$. Then

$$(Co):\ \mathrm{ind}_{\overline{\mathfrak{p}}}^{\mathfrak{g}} : \mathcal{B}_{\overline{\mathfrak{p}}}(\chi) \to \mathcal{B}(\chi),\ \mathrm{ind}_{\overline{\mathfrak{p}}}^{\mathfrak{g}}([w]) = [w\cdot w_p^{-1}]. \tag{1.6.2}$$

2. Bases

The goal is to construct two pairs of bases $(\mathcal{Z}_+,\mathcal{U}_+)$ and $(\mathcal{Z}_-,\mathcal{U}_-)$ for the Grothendieck group of the graded Hecke algebra $\mathbb{H}$ at central character χ.

The definition is inductive. These four sets will be constructed in the space $\mathcal{K}(\chi)$. In the end, the change of bases matrix for the pair $(\mathcal{Z}_+,\mathcal{U}_+)$ (equivalently for $(\mathcal{Z}_-,\mathcal{U}_-)$) is the desired *multiplicity matrix*.

2.1. The standard modules with central character χ (and so the sets $\mathcal{Z}$ and $\mathcal{U}$) are parametrized by pairs $(\mathcal{O},\phi)$, where $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$ and $\phi \in \widehat{A}_{Ad(\mathfrak{g})}(\mathcal{O})$ is a representation which appears in Springer's correspondence. (Here $Ad(\mathfrak{g})$ denotes the adjoint group with Lie algebra $\mathfrak{g}$, and $A_{Ad(\mathfrak{g})}(\mathcal{O})$ is the group of components of the centralizer in $Ad(\mathfrak{g})$ of an element e of $\mathcal{O}$.)

We need a list of the nilpotent orbits $\mathcal{O}$ in $\mathfrak{g}$ which restrict to $G(\chi)$-orbits in $\mathfrak{g}_2$ (notation as in sections 1.1 and 1.2). Denote this list by $\mathcal{E}(\chi)$.

The following procedure is not as in [Lu2], but although not elegant, computationally may be equivalent. Let $\mathcal{O}$ be a nilpotent orbit, and fix a Lie triple $\{e,h,f\}\in\mathfrak{g}$ for it, with $h\in\mathfrak{h}$. Let $\mathfrak{z}(e,h,f)$ denote the centralizer of $\{e,h,f\}$ in $\mathfrak{g}$. Consider a most general element $\nu\in\mathfrak{h}\cap\mathfrak{z}(e,h,f)$. If

$$(Co): \text{there exists } w\in W \text{ such that } w(h+\nu_0)=\chi \text{ for some } \nu_0 \text{ as above,} \tag{2.1.1}$$

then we add $\mathcal{O}$ to $\mathcal{E}(\chi)$. In this case, we set $s=w\cdot h$, and define a Levi suabalgebra and two parabolic subalgebras as in [Lu2]. Let $\mathfrak{g}^n$, $r^n(\Delta)$, $n\in\mathbb{Z}$, denote the similar eigenspaces for the $ad(s)$-action. The Levi subalgebra is

$$\mathfrak{l}_s=\bigoplus_i(\mathfrak{g}_i\cap\mathfrak{g}^i),\ \text{ with roots } \Delta(\mathfrak{l}_s)=\bigoplus_i(r_i(\Delta)\cap r^i(\Delta)). \tag{2.1.2}$$

The two parabolic subalgebras are

$$\mathfrak{p}_{s,+}=\bigoplus_{i\le j}(\mathfrak{g}_i\cap\mathfrak{g}^j),\ \text{ with roots } \Delta(\mathfrak{u}_{s,+})=\bigoplus_{i<j}(r_i(\Delta)\cap r^j(\Delta)), \tag{2.1.3}$$
$$\mathfrak{p}_{s,-}=\bigoplus_{i\ge j}(\mathfrak{g}_i\cap\mathfrak{g}^j),\ \text{ with roots } \Delta(\mathfrak{u}_{s,-})=\bigoplus_{i>j}(r_i(\Delta)\cap r^j(\Delta)).$$

Clearly $\mathfrak{p}_{s,+}\cap\mathfrak{p}_{s,-}=\mathfrak{l}_s$.

Example $\mathbf{A_2}$. For $\chi=(2,0,-2)$, $\mathcal{E}(\chi)=\{(111),(21),(3)\}$. We write:

$\mathcal{O}$	$h+\nu$	ν_0	s
(111)	$(0,0,0)+(\nu_1,\nu_2,-\nu_1-\nu_2)$	$(2,0)$	$(0,0,0)$
(21)	$(1,-1,0)+(\nu,\nu,-2\nu)$	1	$(1,-1,0)$
(3)	$(2,0,-2)$		$(2,0,-2)$

The parabolic subalgebras are given in this case by:

$\mathcal{O}$	s	$\Delta(\mathfrak{l}_s)$	$\Delta(\mathfrak{u}_{s,+})$	$\Delta(\mathfrak{u}_{s,-})$
(111)	$(0,0,0)$	$\emptyset$	$\Delta^-(A_2)$	$\Delta^+(A_2)$
(21)	$(1,-1,0)$	$\{\pm(\epsilon_1-\epsilon_2)\}$	$\{-\epsilon_2+\epsilon_3,-\epsilon_1+\epsilon_3\}$	$\{\epsilon_2-\epsilon_3,\epsilon_1-\epsilon_3\}$
(3)	$(2,0,-2)$	$\Delta(A_2)$	$\emptyset$	$\emptyset$

2.2. We retain the notation from the previous subsection. Consider $\mathcal{O} \in \mathcal{E}(\chi)$, and let $\mathfrak{l}_s$, $\mathfrak{p}_{s,+}$, $\mathfrak{p}_{s,-}$ be the subalgebras defined in (2.1.2) and (2.1.3). The bases $\mathcal{Z}_\pm$ and $\mathcal{U}_\pm$ are partitioned

$$\mathcal{Z}_\pm = \sqcup_{\mathcal{O}\in\mathcal{E}(\chi)}\mathcal{Z}_\pm(\mathcal{O}), \quad \mathcal{U}_\pm = \sqcup_{\mathcal{O}\in\mathcal{E}(\chi)}\mathcal{U}_\pm(\mathcal{O}). \tag{2.2.1}$$

If $\mathfrak{l}_s = \mathfrak{g}$, then $\mathcal{O} = \mathcal{O}_m$ is necessarily the unique maximal nilpotent orbit in $\mathcal{E}(\chi)$.

Assume that $\mathcal{O} \neq \mathcal{O}_m$. Then $\mathfrak{l}_s$ is a proper Levi subalgebra of $\mathfrak{g}$. By construction, there exists a Lie triple (e', s, f') of $\mathcal{O}$, such that $(e', s, f') \subset \mathfrak{l}_s$. Let $\mathcal{O}_{\mathfrak{l}_s}$ denote the nilpotent orbit of e' in $\mathfrak{l}_s$. By induction, we can assume that the bases $\mathcal{Z}^{\mathfrak{l}_s}_\pm(\mathcal{O}_{\mathfrak{l}_s})$ corresponding to central character s are constructed for $\mathfrak{l}_s$.

Then

$$\mathcal{Z}_+(\mathcal{O}) = \mathrm{ind}^{\mathfrak{g}}_{\overline{\mathfrak{p}}_{s,+}}(\mathcal{Z}^{\mathfrak{l}_s}_+(\mathcal{O}_{\mathfrak{l}_s})), \quad \mathcal{Z}_-(\mathcal{O}) = \mathrm{ind}^{\mathfrak{g}}_{\overline{\mathfrak{p}}_{s,-}}(\mathcal{Z}^{\mathfrak{l}_s}_-(\mathcal{O}_{\mathfrak{l}_s})), \tag{2.2.2}$$

where we identify $\overline{\mathfrak{p}}_{s,+} \cong \mathfrak{l}_s$, and $\overline{\mathfrak{p}}_{s,-} \cong \mathfrak{l}_s$.

We note that the elements in each set $Z_\pm(\mathcal{O})$ are parametrized by certain representations of the group of components $\widehat{A}_{Ad(\mathfrak{g})}(\chi, e)$.

Proposition ([Lu2],2.17). *If $\mathcal{O}, \mathcal{O}' \in \mathcal{E}(\chi)$ with $\mathcal{O} \neq \mathcal{O}'$, and if $(\xi, \xi') \in \mathcal{Z}_+(\mathcal{O}) \times \mathcal{Z}_+(\mathcal{O}')$ or $(\xi, \xi') \in \mathcal{Z}_-(\mathcal{O}) \times \mathcal{Z}_-(\mathcal{O}')$, then*

$$(\xi : \xi') = 0. \tag{2.2.3}$$

2.3. The construction of the bases $\mathcal{Z}_+$,$\mathcal{U}_+$, respectively $\mathcal{Z}_-$,$\mathcal{U}_-$ is done in parallel, so we will use the subscript $\pm$ for simplicity.

Let us denote

$$\mathcal{Z}'_\pm = \mathcal{Z}_\pm \setminus \mathcal{Z}_\pm(\mathcal{O}_m), \quad \mathcal{U}'_\pm = \mathcal{U}_\pm \setminus \mathcal{U}_\pm(\mathcal{O}_m). \tag{2.3.1}$$

The multiplicity matrix computed by the algorithm is matrix with coefficients in $\mathbb{Z}[v]$,

$$\mathcal{N} = \left(\begin{array}{c|c} \mathcal{N}_{1,1} & \mathcal{N}_{1,2} \\ \hline \mathcal{N}_{2,1} & \mathcal{N}_{2,2} \end{array}\right), \tag{2.3.2}$$

where

(1) $\mathcal{N}_{1,1}$ is an upper triangular matrix of size $\#\mathcal{Z}'_\pm \times \#\mathcal{Z}'_\pm$ with one on the diagonal computed in equation (2.3.7),
(2) $\mathcal{N}_{1,2}$ is a matrix of size $\#\mathcal{Z}'_\pm \times \#\mathcal{Z}_\pm(\mathcal{O}_m)$ computed in equation (2.4.4),
(3) $\mathcal{N}_{2,1}$ is the zero matrix of size $\#\mathcal{Z}_\pm(\mathcal{O}_m) \times \#\mathcal{Z}'_\pm$,
(4) $\mathcal{N}_{2,2}$ is the identity matrix of size $\#\mathcal{Z}_\pm(\mathcal{O}_m) \times \#\mathcal{Z}_\pm(\mathcal{O}_m)$.

The sets $\mathcal{Z}'_\pm$ are thus constructed by induction. One sets a partial ordering $\leq$ on $\mathcal{Z}'_\pm$ coming from the closure ordering for nilpotent orbits. In this order, the unique element in $\mathcal{Z}_\pm(0)$ is the minimal element.

Now we explain the construction of $\mathcal{U}'_\pm$.

Define the matrices

$$\mathcal{M}_\pm = ((\xi : \xi'))_{\xi,\xi' \in \mathcal{Z}'_\pm}. \tag{2.3.3}$$

By proposition 2.2, these matrices are block-diagonal, with blocks of sizes $\#\mathcal{Z}_\pm(\mathcal{O})$.

Lemma ([Lu2],1.11,3.7)**.** *The two matrices $\mathcal{M}_+,\ \mathcal{M}_-$ are invertible.*

For every $\xi \in Z'_\pm$, we find the vector

$$V_\xi = (a'_{\xi,\xi'})_{\xi' \in Z'_\pm} = \mathcal{M}_\pm^{-1} \cdot ((\beta(\xi), \xi')_{\xi' \in Z'_\pm}. \tag{2.3.4}$$

By lemma 1.13 in [Lu2], $a'_{\xi,\xi} = 1$, and $a'_{\xi,\xi'} = 0$ unless $\xi' \leq \xi$. Moreover, from [Lu2], 1.14,

$$\beta(V_\xi^T) \cdot V_{\xi'} = \begin{cases} 1, & \text{if } \xi = \xi' \\ 0, & \text{if } \xi \neq \xi' \end{cases}, \tag{2.3.5}$$

where V^T denotes the transpose of V.

Proposition ([Lu2])**.** *There exists a unique family $\{c_{\xi,\xi'} :\ \ \xi,\xi' \in Z'_\pm\}\}$ such that*

(i) $c_{\xi,\xi} = 1$, $c_{\xi,\xi'} = 0$ *if* $\xi' \not\leq \xi$, *and* $c_{\xi,\xi'} \in v\mathbb{Z}[v]$ *if* $\xi' < \xi$;

(ii) $c_{\xi,\xi'} = \sum_{\xi'' \in \mathcal{Z}'_\pm} \beta(c_{\xi,\xi''})\ a'_{\xi'',\xi'}$.

Set

$$\mu_\xi = \sum_{\xi' \in \mathcal{Z}'_\pm} c_{\xi,\xi'}\xi'. \tag{2.3.6}$$

Then $\mathcal{U}'_\pm = \{\mu\xi : \xi \in \mathcal{Z}'_\pm\}$.

In other words, in the multiplicity matrix,

$$\mathcal{N}_{1,1} = (c_{\xi,\xi'})_{\xi,\xi' \in \mathcal{Z}'_\pm}. \tag{2.3.7}$$

2.4. It remains to explain the computation of the sets $\mathcal{Z}_+(\mathcal{O}_m)$ and $\mathcal{U}_+(\mathcal{O}_m)$. (The other pair is computed in the obvious analogue way.)

Since $\mathcal{K}(\chi)$ has a symmetric bilinear form, for every subspace $\mathcal{W} \subset \mathcal{K}(\chi)$, we can define the orthogonal complement $\mathcal{W}^\perp$. Clearly, $Rad \subset \mathcal{W}^\perp$.

Let $\mathcal{W}_+$ be the subspace spanned by $\mathcal{Z}'_+$. In fact, $\mathcal{Z}_+$ is a basis of $\mathcal{W}_+$. Define the projections Y_+, respectively $Y_+^\perp$ of $\mathcal{K}(\chi)$ onto $\mathcal{W}_+$, respectively $\mathcal{W}_+^\perp$. Explicitly, $Y_+^\perp(x) = x - Y_+(x)$, where

$$Y_+(x) = \sum_{\xi \in \mathcal{Z}'_+} a_{x,\xi}\xi, \quad \text{and } (a_{x,\xi})_{\xi \in \mathcal{Z}'_+} = \mathcal{M}_+^{-1} \cdot ((x : \xi'))_{\xi' \in \mathcal{Z}'_+}. \tag{2.4.1}$$

Proposition ([Lu2])**.** *Let J_- be defined by*

$$J_- = \{\xi_0 \in \mathcal{Z}'_- : \ Y_+^{\perp}(\mu_{\xi_0}) \notin Rad\}. \tag{2.4.2}$$

The sets $\mathcal{Z}_+(\mathcal{O}_m)$ and $\mathcal{U}_+(\mathcal{O}_m)$ are then obtained as follows:

$$\mathcal{Z}_+(\mathcal{O}_m) = \{\xi = Y_+^{\perp}(\mu_{\xi_0}) : \ \xi_0 \in J_-\}, \quad \mathcal{U}_+(\mathcal{O}_m) = \{\mu_\xi = \mu_{\xi_0} : \ \xi_0 \in J_-\}. \tag{2.4.3}$$

This concludes the construction of the bases. To complete the matrix of multiplicities, one finds

$$\mathcal{N}_{1,2} = (c_{\xi,\xi'})_{\xi\in\mathcal{Z}_+(\mathcal{O}_m),\xi'\in\mathcal{Z}'_+} = \mathcal{M}_+^{-1} \cdot ((\mu_\xi : \xi''))_{\xi\in\mathcal{Z}_+(\mathcal{O}_m),\xi''\in\mathcal{Z}'_+}. \tag{2.4.4}$$

2.5. In the example of A_1, with $\chi = \check{\alpha}$, $\mathcal{E}(\chi) = \{(11),(2)\}$, $\mathcal{O}_m = (2)$, and the bases are

$$\begin{aligned} &\mathcal{Z}_+ = \{s, 1 + \upsilon \cdot s\} \qquad && \mathcal{Z}_- = \{1, s + \upsilon \cdot 1\}, \\ &\mathcal{U}_+ = \{s, 1\} && \mathcal{U}_- = \{1, s\}. \end{aligned} \tag{2.5.1}$$

The matrix of multiplicities is

$$\mathcal{N} = \left(\begin{array}{c|c} 1 & -\upsilon \\ \hline 0 & 1 \end{array}\right). \tag{2.5.2}$$

In this case, the involution γ defined in section 1.5 preserves $\mathcal{U}_+$ and $\mathcal{U}_-$, but reverses the order, and takes $\mathcal{Z}_+$ to $\mathcal{Z}_-$ preserving the order.

Question. Can the Iwahori-Matsumoto involution be described combinatorially?

Also, in view of the real case, a natural question is if the Jantzen filtration conjecture holds in this setting as well. (One would need to consider $\mathcal{N}^{-1}$ instead, to define *Lusztig polynomials*). It is satisfied for A_1: the principal series $X(\nu)$, is reducible at $\nu = 1$, (this corresponds to our χ) and the intertwining operator is $\mathcal{A}((2),\nu) = +1$, $\mathcal{A}((11),\nu) = \frac{1-\nu}{1+\nu}$, so the filtration has two levels, 0, 1, with multiplicity one on each level.

3. Examples

3.1. An example in $\mathfrak{sp}(6)$. The central character is $\chi = (3,1,1)$, the middle element of the nilpotent $(4,2)$. There are 10 orbits of $G(\chi)$ on $\mathfrak{P}_2$. We list the parametrization of these orbits, the dimensions, the corresponding Levi subalgebras and the basis elements $\mathcal{Z}_-$ and $\mathcal{U}_-$. The bases $\mathcal{Z}_+$ and $\mathcal{U}_+$ are obtained by multiplication by w_0.

We encode the cosets $W/W(\chi)$ by the of W action on $(3,1,1)$.

s	Levi $\mathfrak{l}_s$	Dim	$\mathcal{Z}_-$
$(0,0,0)$	$\pm\{\epsilon_2-\epsilon_3\}$	0	$\frac{1}{v+v^{-1}}[3,1,1]$
$(1,-1,0)$	$\pm\{\epsilon_1-\epsilon_2\}$	2	$[1,3,1]+v[3,1,1]$
$(0,0,1)$	$\pm\{2\epsilon_3\}$	2	$[3,1,-1]+v[3,1,1]$
$(1,-1,1)$	$\pm\{\epsilon_1-\epsilon_2,2\epsilon_3\}$	3	$[1,3,-1]+v[1,3,1]+v[3,1,-1]+v^2[3,1,1]$
$(0,1,1)$	$\pm\{\epsilon_2\pm\epsilon_3,2\epsilon_2,2\epsilon_3\}$	3	$\frac{1}{v+v^{-1}}[3,-1,-1]-v[3,1,-1]-\frac{v}{v+v^{-1}}[3,1,1]$
$(0,1,1)$		3	$[3,-1,1]+\frac{1}{v+v^{-1}}[3,-1,-1]-\frac{v}{v+v^{-1}}[3,1,1]$
$(2,0,2)$	$\pm\{\epsilon_1-\epsilon_2,\epsilon_2+\epsilon_3,\epsilon_1+\epsilon_3\}$	4	$[-1,1,3]+v[1,3,-1]+v[3,-1,1]+v^2[3,1,-1]$
$(3,1,0)$	$\pm\{\epsilon_1\pm\epsilon_2,2\epsilon_2,2\epsilon_1\}$	4	$[1,-3,-1]+v[1,1,3]+v[1,3,-1]+v^2[1,3,1]$
$(3,1,1)$	Δ	5	$\frac{1}{v+v^{-1}}[-3,-1,-1]-v[-1,1,3]-v[1,-3,-1]$ $-\frac{v^2}{v+v^{-1}}[3,-1,-1]-v^2[1,1,3]-v^2[1,3,-1]$ $-v^2[3,-1,1]-v^3[1,3,1]$
$(3,1,1)$		5	$[-3,-1,1]+\frac{1}{v+v^{-1}}[-3,-1,-1]+v[-1,1,3]+v[1,3,1]$ $-\frac{v^2}{v+v^{-1}}[3,-1,-1]+v^2[1,3,-1]+v^2[3,1,1]+v^3[3,1,-1]$

s	$\mathcal{U}_-$
$(0,0,0)$	$\frac{1}{v+v^{-1}}[3,1,1]$
$(1,-1,0)$	$[1,3,1]+\frac{1}{v+v^{-1}}[3,1,1]$
$(0,0,1)$	$[3,1,-1]+\frac{1}{v+v^{-1}}[3,1,1]$
$(1,-1,1)$	$[1,3,-1]$
$(0,1,1)$	$\frac{1}{v+v^{-1}}[3,-1,-1]$
$(0,1,1)$	$[3,-1,1]+\frac{1}{v+v^{-1}}[3,-1,-1]$
$(2,0,2)$	$[-1,1,3]$
$(3,1,0)$	$[1,-3,-1]+v[1,1,3]+v^2[1,3,-1]+\frac{v^2}{v+v^{-1}}[1,3,1]$
$(3,1,1)$	$\frac{1}{v+v^{-1}}[-3,-1,-1]$
$(3,1,1)$	$[-3,-1,1]+\frac{1}{v+v^{-1}}[-3,-1,-1]$

The basis elements satisfy the conditions:

1. $(\xi:\xi')=0$, when $\xi,\xi'\in\mathcal{Z}$ correspond to distinct orbits.
2. $(\xi:\xi')=1+v\mathbb{Z}[v]$, when $\xi,\xi'\in\mathcal{Z}$ correspond to the same orbit.
3. the change of basis matrix is upper triangular (see below).
4. $\overline{\mu}-\mu$ is an element of the radical of the bilinear form.
5. $\mathcal{Z}$ (similarly $\mathcal{U}$) is a basis for $\mathcal{K}/Rad$.
6. The elements in $\mathcal{U}_+$ which give the two tempered basis elements in $\mathcal{U}_-$ correspond to the orbits $(0,0,0)$ and $(0,0,1)$.

The change of basis matrix from $\mathcal{Z}$ to $\mathcal{U}$ (this is the one computed by the algorithm) is:

$$\left(\begin{array}{cccccccc|cc}
1 & -v^2 & -v^2 & (v+v^3) & -v^3 & v & -(v^2+v^4) & -v^4 & -v^5 & v^3 \\
0 & 1 & 0 & -v & 0 & 0 & v^2 & v^2 & v^3 & -v \\
0 & 0 & 1 & -v & v & 0 & v^2 & v^2 & v^3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -v & -v & -v^2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & v & 0 & v^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -v & 0 & 0 & v^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & v & -v \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

Just for the record, the change of basis matrix from $\mathcal{U}$ to $\mathcal{Z}$ is

$$\left(\begin{array}{cccccccc|cc}
1 & v^2 & v^2 & -v+v^3 & 0 & -v & -v^2 & -v^2 & v^3 & 0 \\
0 & 1 & 0 & v & 0 & 0 & 0 & 0 & 0 & v \\
0 & 0 & 1 & v & -v & 0 & v^2 & 0 & 0 & v^3 \\
0 & 0 & 0 & 1 & 0 & 0 & v & v & -v^2 & v^2 \\
0 & 0 & 0 & 0 & 1 & 0 & -v & 0 & 0 & -v^2 \\
0 & 0 & 0 & 0 & 0 & 1 & v & 0 & -v^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -v & v \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -v & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

3.2. An example in $\mathfrak{sp}(4)$. Consider $\chi = (1,1)$, the middle element of the nilpotent orbit $(2,2)$ in $\mathfrak{sp}(4)$. There are 3 orbits:

s	Levi	Dim	$\mathcal{Z}_-$
$(0,0)$	$\pm\{\epsilon_1 - \epsilon_2\}$	0	$\frac{v}{1+v^2}[1,1]$
$(0,1)$	$\pm\{2\epsilon_2\}$	2	$[1,-1] + v[1,1]$
$(1,1)$	$\mathfrak{g}$	3	$\frac{v}{1+v^2}[-1,-1] - v[1,-1] - \frac{v^2}{1+v^2}[1,1]$
$(1,1)$		3	$[-1,1] + \frac{v}{1+v^2}[-1,-1] - \frac{v^2}{1+v^2}[1,1]$

s	$\mathcal{U}_-$
$(0,0)$	$\frac{v}{1+v^2}[1,1]$
$(0,1)$	$[1,-1] + \frac{v}{1+v^2}[1,1]$
$(1,1)$	$\frac{v}{1+v^2}[-1,-1]$
$(1,1)$	$[-1,1] + \frac{v}{1+v^2}[-1,-1]$

The change of basis matrix is

$$\left(\begin{array}{cc|cc}
1 & -v^2 & -v^3 & v \\
0 & 1 & v & 0 \\ \hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right).$$

3.3. An example in F_4. The following represents the matrix for $\chi = (3,1,1,1)$, the middle element of the nilpotent $F_4(a_3)$ (component group S_4 in F_4) (19×19 upper uni-triangular matrix). There are 12 orbits and 19 local systems. The last 4 columns, $15-19$ correspond to 4 local systems on the dense orbit (that is to the tempered modules). The dimensions of the orbits corresponding to the columns are (the columns corresponding to the same orbit are grouped together):

$$(0,4,\underbrace{6,6},7,7,\underbrace{8,8},8,9,10,\underbrace{10,10},\underbrace{11,11},\underbrace{12,12,12,12}).$$

Columns $1-8$:

$$\begin{pmatrix}
1 & -v^2-v^4 & -v^2-v^4-v^6 & v^2+v^4 & -v-2v^3-2v^5-v^7 & v^5+v^7 & v^4+v^6+v^8 & v^6 \\
 & 1 & v^2 & & v+v^3 & -v^3 & -v^2-v^4 & -v^2 \\
 & & 1 & & v & -v & -v^2 & \\
 & & & 1 & -v & & & \\
 & & & & 1 & & -v & \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix}$$

Columns $9-13$**:**

$$\begin{pmatrix}
v^4+v^6+v^8 & -v^3-v^5-v^7-v^9 & v^2+v^4+2v^6+v^8+v^{10} & -v^8-v^{10} & v^4+2v^6+v^8 \\
-v^4 & v+v^3+v^5 & -v^2-v^4-v^6 & v^4+v^6 & -v^2-v^4 \\
-v^2 & v^3 & -v^2-v^4 & v^4 & -v^2 \\
v^2 & & v^2 & & v^2 \\
-v & v^2 & -v-v^3 & v^3 & -v \\
 & -v^2 & v+v^3 & -v^3 & v \\
 & -v & v^2 & -v^2 & \\
 & -v & & -v^2 & \\
1 & & v^2 & & \\
 & 1 & -v & v & \\
 & & 1 & & \\
 & & & 1 & \\
 & & & & 1
\end{pmatrix}$$

Columns $14-19$**:**

$$\begin{pmatrix}
v^5+2v^7+2v^9+v^{11} & -v^3-v^5-v^7-v^9 & v^{12} & -v^8-v^{10} & -v^6-v^8-v^{10} & v^6 \\
-v^3-2v^5-v^7 & v^5 & -v^8 & v^6 & v^4+v^6 & \\
-2v^3-v^5 & v^3 & -v^6 & v^4 & v^2+v^4 & \\
v^3 & -v-v^3 & & -v^4 & & v^2 \\
-2v^2-v^4 & v^2 & -v^5 & v^3 & v^3 & \\
v^2+v^4 & -v^2 & v^5 & -v^3 & -v & \\
v+v^3 & & v^4 & & -v^2 & \\
 & & & & & \\
v+v^3 & -v-v^3 & v^4 & -v^2-v^4 & -v^2 & v^2 \\
-v^2 & & -v^3 & & & \\
v & -v & v^2 & -v^2 & & \\
-v & & -v^2 & & v^2 & \\
v & & & -v^2 & & \\
1 & & v & -v & -v & \\
 & 1 & & v & & -v \\
 & & 1 & & & \\
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1
\end{pmatrix}.$$

References

[Lu1] G. Lusztig, *Study of perverse sheaves arising from graded Lie algebras*, Adv. Math. 112 (1995), 147–217.

[Lu2] ———, *Graded Lie algebras and intersection cohomology*, preprint.