SOFTWARE FOR COMPUTING STANDARD REPRESENTATIONS

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ABSTRACT. Let $G_{\mathbb{R}}$ be the real points of a complex connected reductive algebraic group G. Let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$. We describe an algorithm for computing restrictions of standard representations of $G_{\mathbb{R}}$ to $K_{\mathbb{R}}$. We are currently implementing the algorithm as a package of the *Atlas of Lie Groups and Representations* software developed by Fokko du Cloux.

1. Introduction

Let θ be a Cartan involution for $\mathfrak{g}_{\mathbb{R}} = Lie(G_{\mathbb{R}})$ then θ extends to an involution of $\mathfrak{g} = Lie(G)$ giving $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the usual Cartan decomposition with $+1$-eigenspace $\mathfrak{k}$ and -1eigenspace $\mathfrak{p}$. Furthermore $Lie(K) = \mathfrak{k}$. Let B be a Borel subgroup of G and H a Cartan subgroup of G such that $H \subseteq B$. Then K acts on the flag variety G/B. Let H_1 be a θ-stable Cartan subgroup of G defined by the positive root system Δ_1^+ with B_1, the Borel subgroup defined by Δ_1^+. The orbit $K.B_1 = \mathfrak{O}_1$ in G/B determines H_1 up to K-conjugacy. The underlying philosophy is that irreducible $G_{\mathbb{R}}$-representations are to be understood via the algebraic structure of their $(\mathfrak{g}, K_{\mathbb{R}})$ modules.

In Atlas K-orbits on G/B correspond to

$$\coprod_{\text{disjoint union over } K\text{-classes of } \theta\text{-stable } H_i} W(G, H_i)/W(K, H_i).$$

A continued standard representation $(\Delta_{1,im}^+, \lambda_1)$ is given by an orbit $K.B_1$ and a Harish-Chandra module $(Lie(H_1), H_1 \cap K)$.

Theorem 1.1. *(Hecht, Milicic, Schmid, Wolf)*
(i) Continued standard representations depend only on the imaginary positive roots $\Delta_{1,im}^+$.
(2) If λ_1 *is positive on* $\Delta_{1,im}^+$ *then* $(\Delta_{1,im}^+, \lambda_1)$ *is an actual standard representation.*
(3) Restriction to K *depends only on* $\Delta_{1,im}^+$ *and* $\lambda_1|_{H_1 \cap K}$.

1.1. **Example.** Let $G_{\mathbb{R}} = U(2,1)$ and $K_{\mathbb{R}} = U(2) \times U(1)$. Then $G = GL(3, \mathbb{C})$ and $K = GL(2, \mathbb{C}) \times GL(1, \mathbb{C})$.

There are two K-conjugacy classes of θ-stable Cartan subgroups:

$H_1 =$ diagonal matrices that is $(\mathbb{C}^*)^3$

Key words and phrases.

and

$$H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} . \begin{pmatrix} e^{\phi} & 0 & 0 \\ 0 & e^{\gamma} & 0 \\ 0 & 0 & e^{\gamma} \end{pmatrix}$$

In the H_1 case $\Delta_1^+ = \Delta_1^+ im$ and $W(G, H_1) = S_3$. Let $\rho = (1, 0, -1)$. If we want to compute standard representations with infinitesimal character ρ then each λ will correspond to a permutation of ρ with the first two coordinates in decreasing order.

So λ_1 takes the following values; $(1, 0, -1)$, $(1, -1, 0)$ and $(0, -1, 1)$ accounting for 3 discrete series of $U(2, 1)$.

In the case of H_2 there are no imaginary roots and $[W(K, H_{2,\mathbb{R})} = S_1 \times S_2]$. Here λ_2 takes the following values; $(1, 0, -1)$, $(0, 1, -1)$ and $(-1, 1, 0)$. When restricted to K we obtain $\phi - \gamma, 0$ and $-\phi + \gamma$, which account for 3 principal series of $U(2, 1)$.

1.2. **How to describe restriction to K.** First we will work on the fundamental Cartan in order to obtain a simpler formula. We note that $K.B_1 \simeq K/(B_1 \cap K)$ which is a complete flag variety for K and up to a ρ-shift λ_1 corresponds to a line bundle $\mathfrak{L}_{\lambda_1}$ on $K/(B_1 \cap K)$. This is data for discrete series of $G_{\mathbb{R}}$. The lowest K-type is $H^{top}(K/(B_1 \cap K), \mathfrak{L}_{\lambda_1})$ which corresponds to an irreducible representation of K with highest weight λ_1 (up to a $\rho_G - 2\rho_K$)-shift. Here $top = \dim_{\mathbb{C}} K/(B_1 \cap K)$

Ideas from D-module theory require that one " adds formal derivatives away from K-orbits. The final formula is:

$$(\Delta_{im}^+, \lambda_1)|_k = \sum_{m \geq 0} \sum_i (-1)^i H^{top-i}(K/(B_1 \cap k), \mathfrak{L}\lambda_1 \otimes S^m(\mathfrak{n} \cap p)).$$

Here $\mathfrak{b}_1 = \mathfrak{h}_1 \oplus \mathfrak{n}$ with $Lie(B_1) = \mathfrak{b}_1$. (The lowest K-type is buried in the above formula for $m = 0$.)

To extract the lowest K-type one has to invert $\sum_{m \geq 0} S^m(\mathfrak{n} \cap p)$ to get $\sum_{j=0}^{\dim(\mathfrak{n} \cap \mathfrak{p})} (-1)^j \wedge^j (\mathfrak{n} \cap \mathfrak{p})$.

Use Koszul Theorem here to set

$$\sum_{j=0}^{\dim(\mathfrak{n} \cap p)} \sum_{m \geq 0} (-1)^j (\wedge^j \otimes S^m)(\mathfrak{n} \cap p) = \mathbb{C}.$$

Finally we obtained,

$$H^{top}(K/(B_1 \cap k), \mathfrak{L}\lambda_1) = \sum_{j=0}^{\dim(\mathfrak{n} \cap p)} (-1)^j \text{standardrep } (\lambda_1 \otimes \wedge^j (\mathfrak{n} \cap p))|_K.$$

2. Formula for general Cartan subgroups

Let H_i be a Cartan subgroup of G and fix a set of positive imaginary roots $\Delta^+_{i,im}$. We can extend $\Delta^+_{i,im}$ to a positive root system Δ^+_i which is as θ-stable as possible, that is if α is positive then either α is real, $\theta(\alpha) = -\alpha$ or α is not real and $\theta(\alpha)$ is positive. This gives some θ-stable parabolic subalgebra $\mathfrak{q}_i = \mathfrak{l}_i \oplus \mathfrak{u}_i$ such that the roots in $\mathfrak{l}_i$ are all real and the ones in $\mathfrak{u}_i$ are not real. Let $\mathfrak{b}_i$ be the Borel subalgebra corresponding to Δ^+_i. Let Q be the unique conjugate of Q_i containing B. Denote by $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ the parabolic subalgebra of $\mathfrak{g}$ associated to Q. Then the natural map $G/B \to G/Q$ carries $K.\mathfrak{b}_i \to K.\mathfrak{q}_i \simeq K/(Q_i \cap K)$, a closed orbit in G/Q. This says that the fiber over $\mathfrak{q}_i$ is $(L_i \cap K)/(H_i \cap K)$ which is open (orbit) in $L/(L \cap B)$.

The conclusion is that K-orbit of $\mathfrak{b}_i$ is a fiber bundle over closed $K/(Q_i \cap K)$. There is a standard representation of $L_{\mathbb{R}}$ related to the fiber $(L_i \cap K)/(H_i \cap K)$. It is the sections of the bundle defined by a character $\lambda_i|_{(H_i \cap K)}$:

$$Ind^{L_i \cap K}_{H_i \cap K} \lambda_i|_{(H_i \cap K)} = \bigoplus_{\tau\,:\,\text{irreducible rep of } L_i \cap K} mult(\lambda_i|_{(H_i \cap K)} \text{ in } \tau|_{(H_i \cap K)})$$

2.1. **Example.** Let $G_{\mathbb{R}} = GL(3, \mathbb{R})$ and $K_{\mathbb{R}} = O(3, \mathbb{R})$ then $G = GL(3, \mathbb{C})$ and $K = O(3, \mathbb{C})$. G/B is the variety of flags in $\mathbb{C}^3$.

The closed orbits of K consist of flags $(L \subset L^{\perp})$ where L is a line of length zero . For example a line generated by $(1, i, 0)$ included in the plane generated by $(1, i, 0)$ and $(0, 0, i)$. According to Wiit's theorem any two lines of length zero are conjugate by K.

The open orbits of K consist of flags $(L \subset P)$ such that L and $P^{\perp}$ are both of non zero length. An example would be the line generated by $(1, 0, 0)$ in plane P generated by $(1, 0, 0)$ and $(0, 1, 0)$. A counter example would be the line generated by $(1, 0, 0)$ in plane P generated by $(1, 0, 0)$ and $(0, 1, i)$ since in this case $P^{\perp}$ is generated by $(0, 1, i)$ which is of length zero. (Witt's theorem says such pairs (line, plane) are a single orbit.

The stabilizer of $(1, 0, 0) \subset \langle (1, 0, 0), (0, 1, 0) \rangle$ is the set of upper triangular matrices in K that is $(\pm 1, \pm 1, \pm 1)$.

The complete flag in $\mathbb{C}^3$ contains $O(3, \mathbb{C})/O(1, \mathbb{C})^3$ (open)
λ is a character of $O(1, \mathbb{C})^3 = \{\epsilon_1, \epsilon_2, \epsilon_3\}$
Sections of this λ-bundle are

$Ind^{O(3,\mathbb{C})}_{O(1,\mathbb{C})^3} \lambda|_{(H_i \cap K)}$ = sum of all $O(3, \mathbb{C})$ irreducible representations τ with multiplicities = mult of λ in $\tau|_{O(1,\mathbb{C})^3}$

2.2. **Zuckerman Formula.** Trivial rep of $K =$
$\sum_{KGBorbits:\mathfrak{O}} (-1)^{\text{codim } \mathfrak{O}} \text{ stdrep}(\mathfrak{O}, triv\lambda)|_K$
One of the terms in the sum is the standard representation for this open orbit restricted to K + terms for lower dimensional orbits.

List of all orbits for $O(3, \mathbb{C})$ on complete flags:
dim 3 open non-zero length $L \subset P$ with $P^{\perp}$ non-zero length
dim 2 zero length $L \subset P \neq L^{\perp}$
dim 2 non-zero length $L \subset P$ with $P^{\perp}$ zero length

dim 1 closed zero length $L \subset L^{\perp}$

Standard representations data includes λ_i, character of $H_i \cap K$. So we have this algebraic bundle over K orbits and we are interested in sections. Since the L_i orbit is open in the fiber one can differentiate in those directions (sections of bundle on K orbit). We need to add derivatives that are transverse corresponding to $\mathfrak{u}_i \cap \mathfrak{p}$. (This is away from K-orbit directions in which you can't differentiate)

Theorem 2.1. *StdReps*$|_K = \sum_p (-1)^p H^{top-p}(K/(Q_i \cap K), \text{stdrep for } L_i|_{L_i \cap K} \otimes S(\mathfrak{u}_i \cap \mathfrak{p}))$ *with top* $= \dim_{\mathbb{C}} K/(Q_i \cap K)$.

Hiding inside of the above formula is the lowest K-type

$$H^{top}(K/(Q_i \cap K), \text{lowest } (L_i \cap K) - \text{type of strep for} L_i) \otimes S^0(\mathfrak{u}_i \cap \mathfrak{p}).$$

How do you write lowest K-type as combination of standard representations?

Use Zuckerman formula:

Lowest $L_i \cap K$-type $= \sum_{\mathfrak{O}} (-1)^{codim \mathfrak{O}} \text{stdrep } |_{L_i \cap K}$

where $\mathfrak{O}$ is an orbit of $L_i \cap K$ on $L_i/(L_i \cap B)$

Lowest K-type $= \sum_{\mathfrak{O}} (-1)^{codim \mathfrak{O}} H^{top}(K/(Q_i \cap K), \text{stdrep for} L_i|_{L_i \cap K}, \mathfrak{O})$

We need to put the transverse derivatives in. Using Kozul identity

$$S(\mathfrak{u} \cap \mathfrak{p}) \otimes \sum_{j=0}^{\dim(\mathfrak{u} \cap p)} (-1)^j (\wedge^j (\mathfrak{u} \cap p)) = \mathbb{C}.$$

Lowest K-type=

$$\sum_{j=0}^{\dim(\mathfrak{u} \cap p)} \sum_{\mathfrak{O}} (-1)^j H^{top}(K/(Q_i \cap K), (\text{stdrep for } L_i|_{L_i \cap K}, \mathfrak{O}, \lambda_i) \otimes \wedge^j (\mathfrak{u} \cap p) \otimes S(\mathfrak{u} \cap \mathfrak{p}))$$

$$= \sum_{\text{subset of roots of size j in } \mathfrak{u} \cap \mathfrak{p}} \sum_{\mathfrak{O}} (-1)^{codim \mathfrak{O}} \text{stdrep for } G_{\mathbb{R}}.$$

To compute the Zuckerman terms one proceeds as follows:

1. For each Cartan subgroup H_i construct a θ-stable parabolic subalagebra $\mathfrak{q}_i = \mathfrak{l}_i \oplus \mathfrak{u}_i$ such that L_i is split (all roots are real).

2. Call (cartan (L_i)) to obtain normalized involutions $\{\theta_i^j\}$. Zuckerman formula for L_i is indexed by the KGB orbits for $L_i \cap K$ on $L_i/(B \cap L_i)$. These orbits are indexed by Cartan subgroups $\{\theta_i^j\}$ and correspond to

$$\coprod_{j \leftrightarrow H_i^j} W(L_i, H_i)/W(L_i \cap K, H_i^j).$$

What emerge are cosets $w.W(L_i, H_i)/W(L_i \cap K, H_i^j)$ corresponding to the characters $(w.\rho_{L_i} + \rho_{L_i})|_{(H_i^j \cap K)}$ contributing to the Zuckerman formula with some sign $(-1)^{\text{ length(KGBELT)}}$.

3. For each j set a pair (m, μ) with $m \in \mathbb{Z}$ and μ a character of $H_i^j \cap K$.

4. Compute $\sum_{[j:\#\text{ of Cartans in } L_i]} \sum_{(m,\mu)} m.\text{stdrep}(H_i^j, \lambda_i^j + \mu)$

In general H_i^J will be more compact than H_i because of Cayley transform in real roots in L_i. Roughly $H_i \cap K \subseteq H_i^j \cap K$. So $\lambda_i \to \lambda_i^j$. The roots of $\alpha_1 \ldots \alpha_l$ in H_i^j are orthogonal to the real roots in L_i and $H_i^j = (H_i \cap K).(SO_2)^l$. We define $\lambda_i^j = \lambda_i$ and trivial on the SO_2 factors.

For the outer sum list all the roots of $H_i^j \cap K$ in $\mathfrak{u} \cap \mathfrak{p}$.

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